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Some properties of Invertible Elements in Fuzzy Banach Algebras

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Abstract. In this paper, we introduce fuzzy Banach algebra and study the properties of invertible elements and its relation with open sets. We obtain some interesting results.

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1. Introduction

The concept of fuzzy set theory was first introduced by Zadeh[15] in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways by several authors. In 1984, Katsaras [8] defined a fuzzy norm on a linear space and at the same year Wu and Fang [14] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [4], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [3], [6], [7], [10], [12], [13]. In 1994, Cheng and Mordeson[5] introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [9]. In 2003, Bag and Samanta [1] modified

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the definition of Cheng and Mordeson by removing a regular condition. The Banach algebra was studied by Sadeqi and Amiripour[11].
In this paper, we introduce fuzzy Banach algebra and study the properties of invertible elements and its relation with open sets and we obtain some interesting results.

2. Preliminaries

In this section at first we define the fuzzy normed spaces and fuzzy Banach algebras and give several examples of these spaces.

Definition 2.1. [1] Let X be a complex vector space.
A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,
(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \in \mathbb{C} - \{0\}$;
(N4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
The pair (X,N) is called a fuzzy normed vector space.

As an example of fuzzy normed space,

Example 2.2. [1] Let $(X, \|\|)$ be a normed space. Define,

$$\mathrm{N}(x,t) = \begin{cases} 0 & t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

Then (X,N) is a fuzzy normed space.

Definition 2.3. [1] Let (X,N) be a fuzzy normed vector space.
(1) A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists a $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$.
(2) A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p = 1, 2, 3, ...$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 2.4. [1] Let (X,N) be a fuzzy normed space. The pair (X,N) is said to be complete if each cauchy sequence in (X,N) be a convergence sequence in (X,N).

Definition 2.5. [1] Let (X,N) be a fuzzy normed space and $x \in X, t \in \mathbb{R}^+, 0 < r < 1$. Then the set

$$B(x,r,t) = \{y \in X : N(x-y,t) > 1-r\}$$

is called an open ball in (X,N) with x as its center and r as its radius with respect to t.

Definition 2.6. [2] Let (X, N) be a fuzzy normed linear space and $G \subset X$. We say that G is open if for every $x \in G$ there exist $t > 0, 0 < r < 1$ such that $B(x, r, t) \subset G$.
then G is called an open set in (X, N).

Definition 2.7. [11] Let A be an algebra and (A, N) be complete fuzzy normed space, the pair (A, N) is said to be a fuzzy Banach algebra if for every $x, y \in A$ and $t, s \in \mathbb{R}$,

$$N(xy, ts) \geq min\{N(x, t), N(y, s)\}$$

Example 2.8. [11] Let $(A, \|\|)$ be a Banach algebra. Define,

$$\mathrm{N}(x, t) = \begin{cases} 0 & t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

Then (A, N) is a fuzzy Banach algebra.

Theorem 2.9. [1] *Let (X, N) be a fuzzy normed linear space satisfying (N6) that is for every $t > 0, N(x, t) > 0$ implies $x = 0$. We define $\|x\|_\alpha = \wedge\{t > 0 : N(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$.
then $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X and they are called α-norms on X corresponding to the fuzzy norm N on X.*

Proposition 2.10. [1] *Let (X, N) be a fuzzy normed space satisfying (N6)and $\{x_n\}$ be a sequence in X.
then $\lim_{n\to\infty} N(x_n - x, t) = 1, \forall t > 0$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$, for every $\alpha \in (0, 1)$.*

3. Main results

Let A be a fuzzy Banach algebra with the unit e. the set invertible elements of A is denoted by G.

Lemma 3.1. *Let (A, N) be a fuzzy Banach that satisfying (N6), then multiplication is continuous respect to $\|.\|_\alpha$.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be sequence in A and $x, y \in A$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ and $\lim_{n\to\infty} N(y_n - y, t) = 1$ for all $t > 0$. By proposition 2.10 $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ and $\lim_{n\to\infty} \|y_n - y\|_\alpha = 0$ for all $\alpha \in (0, 1)$. it follows that $\lim_{n\to\infty} \|x_n y_n - xy\|_\alpha = 0$, for every $\alpha \in (0, 1)$ and again by proposition 2.10 $\lim_{n\to\infty} N(x_n y_n - xy, t) = 1$ for all $t > 0$. □

Theorem 3.2. *Let (A, N) be a fuzzy Banach algebra satisfying (N6), let $x \in A$ be such that $x \in B(0, r, t)$, $0 < r < 1$ then $e - x$ is invertible and $(e - x)^{-1} = e + \sum_{i=1}^{\infty} x^i$.*

Proof. Since $x \in B(0,r,t)$ we have for any $t \in \mathbb{R}^+$, $N(x,t) > 1-r$. Let $s_n = \sum_{i=1}^{n} x^i$, we have

$$\begin{aligned}
&N(s_{n+p}-s_n,t) = N(x^{n+1}+x^{n+2}+...+x^{n+p},t) \\
&\geq \min\{N(x^{n+1},t_1), N(x^{n+2},t_2), ..., N(x^{n+p},t_p)\} \\
&\geq \min\{\min\{N(x,t_{11}), N(x,t_{12}), ..., N(x,t_{1n+1})\}, \\
&\quad \min\{N(x,t_{21}), N(x,t_{22}), ..., N(x,t_{2n+2})\}, ..., \\
&\quad \min\{N(x,t_{p1}), N(x,t_{p2}), ..., N(x,t_{pn+p})\}\} \\
&> \min\{(1-r),(1-r),...,(1-r)\} = (1-r),
\end{aligned}$$

where $t_1+t_2+...+t_p = t$ and $\prod_{j=1}^{n+i} t_{ij} = t_i$, $i = 1,2,...,p$.

Thus $\{s_n\}$ is Cauchy-sequence and since A is complete, series $\sum_{i=1}^{\infty} x^i$ is convergent. Let $s = e + \sum_{i=1}^{\infty} x^i$. Now $(e-x)(e+x+...+x^n) = (e+x+...+x^n)(e-x) = e-x^{n+1}$. On the other hand

$$\begin{aligned}
N(x^{n+1},t) &\geq \min\{N(x,t_1), N(x,t_2), ..., N(x,t_{n+1})\} \\
&> \min\{(1-r),(1-r),...,(1-r)\} = (1-r),
\end{aligned}$$

where $t_1+t_2+...+t_{n+1} = t$. So, $x^{n+1} \to 0$ as $n \to \infty$. Since multiplication on A is continuous, we obtain $(e-x)s = s(e-x) = e$. Hence $s = (e-x^{-1})$. □

Corollary 3.3. *Let (A,N) be a fuzzy Banach algebra satisfying (N6), let $x \in A$ be such that $e-x \in B(0,r,t)$, $0<r<1$ then x is invertible and* $x^{-1} = e + \sum_{i=1}^{\infty}(e-x)^i$.

Corollary 3.4. *Let (A,N) be a fuzzy Banach algebra satisfying (N6) and $x \in A$, $0 \neq \lambda$ be a scaleler such that $\frac{x}{\lambda} \in B(0,r,t)$, $0<r<1$. Then $\lambda e - x$ is invertible and* $(\lambda e - x)^{-1} = \sum_{i=1}^{\infty} \lambda^{-i} x^{i-1}$.

Proof. Since $\frac{x}{\lambda} \in B(0,r,t)$, for every $t \in \mathbb{R}^+$ we have $N(\frac{x}{\lambda},t) > 1-r$. Now $\lambda e - x = \lambda(e-\frac{x}{\lambda})$ and $N(e-(e-\frac{x}{\lambda}),t) = N(\frac{x}{\lambda},t) > 1-r$, by above corollary $(e-\frac{x}{\lambda})^{-1}$ exists and so $\lambda e - x$ is invertible.

$$(\lambda e - x)^{-1} = (\lambda(e-\frac{x}{\lambda}))^{-1} = \lambda^{-1}(e-\lambda^{-1}x)^{-1} =$$

$$\lambda^{-1}(e+\sum_{i=1}^{\infty}(e-(e-\lambda^{-1}x))^i = \lambda^{-1}(e+\sum_{i=1}^{\infty}\lambda^{-i}x^i) = \sum_{i=1}^{\infty}(\lambda^{-i}x^{i-1}$$

□

Theorem 3.5. *The set of all invertible elements of a fuzzy Banach algebra (A,N) that satisfying (N6), is an open subset of A.*

Proof. Let G be the set of all invertible elements of A. Let $x_0 \in G$ and $r \in (0,1)$ such that $N(x_0^{-1}, t) > 1 - r$. Let $x \in B(x_0, r, t)$, put $y = x_0^{-1}x$ and $z = e - y$.

$$N(z,t) = N(e-y,t) = N(y-e,t) = N(x_0^{-1}x - x_0^{-1}x_0, t) = N(x_0^{-1}(x - x_0), t) \geq \min\{N(x_0^{-1}, \sqrt{t}), N((x - x_0), \sqrt{t})\} \geq \min\{(1-r),(1-r)\} = 1 - r.$$

Thus $z \in B(0, r, t)$ and hence $e - z$ is invertible, that is, y is invertible, so $y \in G$. Since $x_0 \in G$ then $x_0 y = x_0 x_0^{-1} x = x$, hence $x \in G$. It follows that $B(x_0, r, t) \subset G$, so G is open. □

References

1. T. Bag, S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. 11 (2003) 687-705.
2. T. Bag, S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets Syst. 151 (20053) 513-547.
3. V. Balopoulos, A. G. Hatzimichailidis and Basil K. Papadopoulos, *Distance and similarity measures for fuzzy operators*, Inform. Sci. 177 (2007) 2336-2348.
4. R. Biswas, *Fuzzy inner product spaces and fuzzy norm functions*, Inform. Sci. 53 (1991) 185-190.
5. S. C. Cheng and J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. 86 (1994) 429-436.
6. C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets Syst. 48 (1992) 239-248.
7. O. Kaleva, S. Seikala *On fuzzy metric spaces*, Fuzzy Sets and Systems 12 (1984) 143-154.
8. A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets Syst. 12 (1984) 143-154.
9. I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica 11 (1975) 326- 334.
10. S.V. Krishna and K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets Syst. 63 (1994) 207-217.
11. I. Sadeqi and A. Amiripour, *Fuzzy Banach algebras*, First Joint Congress on Fuzzy and Intelligent Systems, Ferdowsi University of Mashhad, Iran, 29-31 Aug 2007.
12. B. Shieh, *Infinite fuzzy relation equations with continuous t-norms*, Inform. Sci. 178 (2008) 1961- 1967.
13. J. Z. Xiao and X.-H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets Syst. 133 (2003) 389-399.
14. Congxin Wu and Jinxuan Fang, *Fuzzy generalization of Klomogoroffs theorem*, J. Harbin Inst. Technol. 1 (1984) 1-7.
15. L.A. Zadeh, *Fuzzy sets, Information and control*, 8 (1965) 338-353.