

## Exact solutions of $(3 + 1)$ -dimensional nonlinear evolution equations

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**ABSTRACT.** In this paper, the kudryashov method has been used for finding the general exact solutions of nonlinear evolution equations that namely the  $(3 + 1)$ -dimensional Jimbo-Miwa equation and the  $(3 + 1)$ -dimensional potential YTSE equation, when the simplest equation is the equation of Riccati.

**Keywords:** kudryashov method, Jimbo-Miwa equation, Potential YTSE equation, Riccati equation.

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### 1. INTRODUCTION

The research area of nonlinear equations has been very active for the past few decades. There are various kinds of nonlinear equations that appear in various areas of physical and mathematical sciences. Much effort has been made on the construction of exact solutions of nonlinear equations, for their important role in the study of nonlinear physical phenomena [2, 9]. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fiber, biology, oceanology [12], solid state physics, chemical physics and geometry. In recent years, the powerful and efficient methods to find analytic solutions of nonlinear equation have drawn a lot of interest by a diverse group of scientists, such as Tanh-function method, extended

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Tanh-function method [3, 18, 19], Sine-cosine method [6, 20, 21],  $(\frac{G'}{G})$ -expansion method [1, 7, 13].

In this paper we obtain the exact solutions of Jimbo-Miwa and Potential YTSF equations by using the kudryashov method. The Jimbo-Miwa equation is the second equation in the well known KP hierarchy of integrable systems, which is used to describe certain interesting (3+1)-dimensional waves in physics but not pass any of the conventional integrability tests.

The kudryashov method was developed by Kudryashov [10, 11] on the basis of a procedure analogous to the first step of the test for the Painleve property [4].

The paper is organized as follows:

In Section 2, we explain the main steps of the kudryashov method. In Section 3, we apply this method to the (3 + 1)-dimensional Jimbo-Miwa equation. In Section 4, we use the method to the (3 + 1)-dimensional potential-YTSF equation. concluding remarks are summarized in Section 5.

## 2. DESCRIPTION OF THE KUDRYASHOV METHOD

In this section we recall the basic idea of the kudryashov method [8]. Let we have a partial differential equation and by means of an appropriate transformation this equation is reduced to a nonlinear ordinary differential equation as follow:

$$P(u, u', u'', u''', \dots) = 0. \quad (2.1)$$

Exact solution of this equation can be constructed as finite series

$$u(x) = \sum_{i=0}^n a_i (G(x))^i \quad (2.2)$$

Where  $G(x)$  is a solution of some ordinary differential equation referred to as the simplest equation, and  $A_0, A_1, A_2, \dots, A_M$  are parameters to be determined.

In this paper we use the equation of Riccati, as the simplest equation

$$G'(x) = cG(x) + dG(x)^2 \quad (2.3)$$

This equation is well-known nonlinear ordinary differential equation which process exact solution constructed by elementary function. In this paper we work with the following solutions of the Riccati equation

$$G(x) = \frac{c \exp [c(x + x_0)]}{1 - d \exp [c(x + x_0)]} \quad (2.4)$$

for case  $d < 0$ ,  $c > 0$ , here  $x_0$  is a constant of integration. and

$$G(x) = -\frac{c \exp [c(x + x_0)]}{1 + d \exp [c(x + x_0)]} \quad (2.5)$$

for case  $d > 0$ ,  $c < 0$ , similar above  $x_0$  is a constant of integration. the simplest equation has two properties: (i) The order of simplest equation is lesser than equation (2.1), (ii) we know the general solution of the simplest equation or we know at least exact analytical particular solution(s) of the simplest Eq(2.3).

Now  $u(x)$  can be determined explicitly by using the following three steps:

- Step (1). By considering the homogeneous balance between the highest nonlinear terms and the highest order derivatives of  $u(x)$  in Eq.(2.1), the positive integer  $n$  in (2.2) is determined.
- Step (2). By substituting Eq.(2.2) with Eq.(2.3) into Eq.(2.1) and collecting all terms with the same powers of  $G$  together, the left hand side of Eq.(2.1) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of  $A_i$  ( $i = 0, 1, 2, \dots, n$ ),  $c$ ,  $d$ .
- Step (3). Solving the system of algebraic equations and then substituting the results and the general solutions of (2.4) or (2.5) into (2.2) gives solutions of (2.1).

Now, we will demonstrate the kudryashov method on two of nonlinear equations, Jimbo-Miwa equation and Potential YTSF equation.

### 3. THE JIMBO-MIWA EQUATION

The Jimbo-Miwa equation is used to describe certain interesting (3+1)-dimensional waves in physics. This equation is

$$u_{xxx}y + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \quad (3.1)$$

Where  $u : R_x \times R_y \times R_z \times R_t^+ \rightarrow R$ .

There are many efforts to solve Eq.(3.1). Tang and Liang applied the multi-linear variable separation scheme to Eq.(3.1) [15]. In [5] the Hirota's bilinear formalism was employed to obtain three-soliton solutions. some exact solutions of Eq.(3.1) obtained by an extended rational expansion method and symbolic computation[17]. The traveling wave solutions for the equation presented using the  $\frac{G'}{G}$ -expansion method[14]. In this paper we obtain exact solutions of this equation using kudryashov method. By the transformation  $\xi = x + y + z - \nu t$ , Eq.(3.1) becomes:

$$u^{(4)} + 3[(u')^2]' - (2\nu + 3)u'' = 0 \quad (3.2)$$

Integrating Eq.(3.2) once with respect to  $\xi$  and setting the integration constant as zero yields

$$u''' + 3(u')^2 - (2\nu + 3)u' = 0 \quad (3.3)$$

Setting  $u' = v$ , Eq.(3.3) becomes

$$v'' + 3v^2 - (2\nu + 3)v = 0 \quad (3.4)$$

With balancing according procedure that be described, we get  $n = 2$ , therefore the solution of Eq.(3.4) can be expressed as follows:

$$v(\xi) = \sum_{i=0}^2 A_i(G(\xi))^i \quad (3.5)$$

where  $G(\xi)$  satisfies the Riccati equation and  $A_0, A_1, A_2$  are parameters to be determined. solutions of Riccati equation are given in (2.4), (2.5). With substituting (3.5) into (3.4) and use of (2.3) and then equating all coefficients of the functions  $G^i$  to zero, we obtain  $A_0, A_1$  and  $A_2$ :

$$\text{Case 1 : } A_0 = -\frac{c^2}{3}, \quad A_1 = -2cd, \quad A_2 = -2d^2; \quad 3cd + c^3d \neq 0 \quad (3.6)$$

$$\text{Case 2 : } A_0 = 0, \quad A_1 = -2cd, \quad A_2 = -2d^2; \quad -3cd + c^3d \neq 0 \quad (3.7)$$

Recall  $u'(x, y, z, t) = v(x, y, z, t)$ , therefore when  $d < 0$  and  $c > 0$  the solution of Eq.(3.1) with using Case 1 (3.6) is given by

$$u_1(x, y, z, t) = -\frac{c^2(x + y + z - \nu t)}{3} - \frac{2c}{1 - d \exp[c(x + y + z - \nu t + x_1)]}, \quad (3.8)$$

where  $\nu = -\frac{1}{2}(c^2 + 3)$  and  $x_1$  is a constant of integration. also solution of Eq.(3.1) with using Case 2 (3.7) is given by

$$u_2(\xi) = -\frac{2c}{1 - d \exp[c(x + y + z - \nu t + x_2)]} \quad (3.9)$$

where  $\nu = \frac{1}{2}(c^2 - 3)$  and  $x_2$  is a constant of integration.

And when  $d > 0$  and  $c < 0$  the solution of Eq.(3.1) with using Case 1 (3.6) is given by

$$u_3(x, y, z, t) = -\frac{c^2(x + y + z - \nu t)}{3} - \frac{2c}{1 + d \exp[c(x + y + z - \nu t + x_3)]}, \quad (3.10)$$

where  $\nu = -\frac{1}{2}(c^2 + 3)$  and  $x_3$  is a constant of integration. also solution of Eq.(3.1) with using Case 2 (3.7) is given by

$$u_4(x, y, z, t) = -\frac{2c}{1 + d \exp[c(x + y + z - \nu t + x_4)]} \quad (3.11)$$

where  $\nu = \frac{1}{2}(c^2 - 3)$  and  $x_4$  is a constant of integration.

#### 4. THE POTENTIAL-YTSF EQUATION

Now we would like to use this method to obtain the exact solutions of the potential-YTSF equation read:

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + 3u_{yy} = 0, \quad (4.1)$$

Eq.(4.1) is called the potential-YTSF equation which was firstly introduced by Yu, Toda, Sasa and Fukuyama (YTSF) [23]. Recently, this equation was studied and some single soliton and periodic solitary solutions were obtained [24, 22]. [16] obtained exact solutions of (4.1) by Exp-function method. By the transformation  $\xi = x + y + z - \nu t$ , Eq.(4.1) becomes:

$$u^{(4)} + 3[(u')^2]' + (4\nu + 3)u'' = 0 \quad (4.2)$$

Integrating Eq.(4.2) once with respect to  $\xi$  and setting the integration constant as zero yields

$$u''' + 3(u')^2 + (4\nu + 3)u' = 0 \quad (4.3)$$

Setting  $u' = v$ , Eq.(4.3) becomes

$$V'' + 3V^2 + (4\nu + 3)V = 0 \quad (4.4)$$

With balancing according procedure that be described, we get  $n = 2$ , therefore the solution of Eq.(4.1) can be expressed as follows:

$$v(\xi) = \sum_{i=0}^2 A_i(G(\xi))^i \quad (4.5)$$

where  $G(\xi)$  satisfies the Riccati equation and  $A_0, A_1, A_2$  are parameters to be determined. solutions of Riccati equation are given in (2.4), (2.5). With substituting (4.5) into (4.4) and use of (2.3) and then equating all coefficients of the functions  $G^i$  to zero, we obtain  $A_0, A_1$  and  $A_2$ :

$$\text{Case 1 : } A_0 = -\frac{c^2}{3}, \quad A_1 = -2cd, \quad A_2 = -2d^2; \quad acd \neq 0 \quad (4.6)$$

$$\text{Case 2 : } A_0 = 0, \quad A_1 = -2cd, \quad A_2 = -2d^2; \quad acd \neq 0 \quad (4.7)$$

Recall  $u'(x, y, z, t) = v(x, y, z, t)$ , therefore when  $d < 0$  and  $c > 0$  the solutions of Eq.(4.1) with using Case 1 (4.6) is given by

$$u_{1,2}(x, y, z, t) = -\frac{c^2(x + y + z - \nu_{1,2}t)}{3} - \frac{2c}{1 - d \exp[c(x + y + z - \nu_{1,2}t + x_1)]}, \quad (4.8)$$

where  $\nu_{1,2} = \frac{1}{8}(-3 \pm \sqrt{16c^2 + 9})$  and  $x_1$  is a constant of integration. also solutions of Eq.(4.1) with using Case 2 (4.7) is given by

$$u_{3,4}(x, y, z, t) = -\frac{2c}{1 - d \exp[c(x + y + z - \nu_{3,4}t + x_2)]} \quad (4.9)$$

where  $\nu_{3,4} = \frac{1}{8}(-3 \pm \sqrt{9 - 16c^2})$  and  $x_2$  is a constant of integration.

And when  $d > 0$  and  $c < 0$  the solutions of Eq.(4.1) with using Case 1 (4.6) is given by

$$u_{5,6}(x, y, z, t) = -\frac{c^2(x + y + z - \nu_{1,2}t)}{3} - \frac{2c}{1 + d \exp[c(x + y + z - \nu_{1,2}t + x_3)]}, \quad (4.10)$$

where  $x_3$  is a constant of integration.

also solutions of Eq.(4.1) with using Case 2 (4.7) is given by

$$u_{7,8}(x, y, z, t) = -\frac{2c}{1 + d \exp[c(x + y + z - \nu_{3,4}t + x_4)]} \quad (4.11)$$

where  $x_4$  is a constant of integration.

## 5. CONCLUDING REMARKS

In this paper, the simplest equation method has been successfully used to obtain exact solutions of the Jimbo-Miwa equation and Potential YTSE equation. As the simplest equations, we have used the equation of Riccati . For this simplest equation, we have obtained a balance equation. By means of balance equations, we obtained exact solutions of the studied class nonlinear PDEs. We have also verified that these solutions we have found are indeed solutions to the original equations.

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