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On the c_0-solvability of a class of infinite systems of functional-integral equations

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Abstract. In this paper, an existence result for a class of infinite systems of functional-integral equations in the Banach sequence space c_0 is established via the well-known Schauder fixed-point theorem together with a criterion of compactness in the space c_0. Furthermore, we include some remarks to show the vastity of the class of infinite systems which can be covered by our result. The applicability of the main result is demonstrated by means of an example as a model of neural nets.

Keywords: Infinite system of functional-integral equations, Schauder fixed-point theorem, Sequence spaces.

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1. Introduction

The theory of infinite systems of functional-integral equations has been a major source of research work since so many natural phenomena are modeled by these systems. Similar to such systems, the infinite systems of differential equations are motivated by numerous world real problems which can be encountered in the theory of stochastic processes, the perturbation theory and quantum mechanics, the theory of neural nets, the

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theory of dissociation of polymers and so on (see also [6],[8],[10]-[12]). For instance, we observe that in the study of the heat conduction problem via the method of semi discretization we deal with an infinite system of ordinary differential equations with initial conditions.

Recently, finite and infinite systems of equations have been considered in certain spaces in several monographs and research papers. For example, the first author and Jalilian [1] investigated nondecreasing positive solutions for a system of singular integral equations using the monotonicity property of the superposition operators together with the concept of measure of noncompactness introduced by Banaś and Goebel in 1980 (see also [2]-[5],[7],[9]).

Throughout this work, we study the existence of solution for the infinite system of functional-integral equations as form of

$$u^i(t) = f_i\left(t, u^i(t), \int_0^t g_i(t,s,u^1(s),u^2(s),...)ds\right) \tag{1.1}$$

in Banach sequence space c_0, where $i = 1, 2, ...$ and $t \in I = [0, T]$, f and g are the functions under some certain conditions and specified later.

In this paper, we utilize the fixed-point method together with a generalization of Arzéla theorem to obtain some sufficient conditions for verifying the existence of solutions for Eq. (1.1). Next, we present some remarks to show that the class of systems as form of (1.1) contains some comparable systems of differential equations. At the end, to show the practicability of obtained result we provide an illustrative example including a model of neural nets and investigate the existence of solutions for the corresponding systems of equations under some imposed conditions.

2. Notation and auxiliary facts

Let E be a real Banach space equipped with the norm $\|\cdot\|_E$. For interval $I = [0, T]$, the notation $C(I, E)$ indicates the family of all continuous functions with values in E defined on I. As we know that this space with the usual norm $\|u\|_C = \max\{\|u\|_E : t \in I\}$ is a Banach space, a generalization of Arzéla (see [2]) will be helpfully applied in this work while it provides a criterion of compactness in the space $C(I, E)$.

Theorem 2.1. *A bounded subset X of the space $C(I, E)$ is relatively compact if and only if all functions belonging to X are equicontinuous on I and the set $X(t) := \{x(t) : x \in X\}$ is relatively compact in E for each $t \in I$.*

In what follows, we present our result in the Banach space c_0 including of all real sequences converging to zero with the norm $\|u\|_{c_0} = \max\{|u_i| :$

$i = 1, 2, ...\}$ for $u = (u_1, u_2, ...)$. We notice that a bounded subset X of c_0 is relatively compact if and only if

$$\lim_{i\to\infty}\left[\sup_{u\in X}[\max\{|u_k| : k \geq i\}]\right] = 0.$$

3. Main results

In order to investigate the solutions of problem (1.1) we consider the following hypotheses. Note that for the convenience of the readers, we will write $g_i(t, s, u(s))$ instead of $g_i(t, s, u^1(s), u^2(s), ...)$.

(H_1): The functions $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous for $i = 1, 2, ...$ while the sequence $(|f_i(t, 0, 0)|)$ is a nonincreasing sequence on I.

(H_2): There exist continuous functions $m_i : I \to [0, 1)$ and $k_i : I \to \mathbb{R}_+$ such that $m_i(t)$ and $k_i(t)$ are uniformly bounded on I and we have

$$|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)| \leq m_i(t)|u_1 - u_2| + k_i(t)|v_1 - v_2|, \quad i = 1, 2, ...$$

for all $u_j, v_j \in \mathbb{R}$, $j = 1, 2$ and $t \in I$. Besides, $m_1(t) \leq M$ for some $M \in (0, 1)$ and $(m_i(t))$ is a nonincreasing sequence at each point $t \in I$.

(H_3): The functions $g_i : I \to \mathbb{R}$ are uniformly continuous with respect to the usual norm. In addition, the functions

$$\overline{g}_i : t \mapsto \sup\{|g_i(t, s, u^1, u^2, ...u^j, ...)| : 0 \leq s \leq t, u^j \in \mathbb{R}\}$$

are bounded on I for each $i = 1, 2, ...$ such that $(k_j(t)\overline{g}_j(t))$ is a nonincreasing sequence on I.

Now we are in a position to formulate our main result.

Theorem 3.1. *Under the hypotheses (H_1)-(H_3), the infinite system of functional-integral equations* (1.1) *has at least one solution $u(t) = (u^i(t))$ such that $u(t) \in c_0$ for each $t \in I$ and $u^i \in C(I, \mathbb{R})$ for $i = 1, 2, ...$.*

Proof. Consider the set U_r as a subset of $C(I, c_0)$ including of all functions $u(t) = (u^i(t))$ such that

$$\sup\{|u^j(t)| : j \geq i\} \leq r_i(t), \quad \text{for } i = 1, 2, ... \text{ and } t \in I$$

where $r_i(t)$ are the nonnegative functions which are given by the following equality

$$r_i(t) := \frac{tk_i(t)\overline{g}_i(t) + |f_i(t, 0, 0)|}{1 - m_i(t)}, \quad \text{for } i = 1, 2, ... \text{ and } t \in I \tag{3.1}$$

and

$$\overline{g}_i(t) = \sup\{|g_i(t, s, u)| : 0 \leq s \leq t, u \in \mathbb{R}^\infty\}$$

as given by (H_3). Now, we define the operator $\mathcal{F}$ on the space $C(I, c_0)$ in the following way

$$(\mathcal{F}u)(t) = ((\mathcal{F}u)_i(t)) = \left(f_i(t, u^i(t), \int_0^t g_i(t, s, u(s))ds) \right).$$

We easily see that U_r is $\mathcal{F}$-invariant. In fact, by taking the index i as fixed and $u \in U_r$ together with condition (H_1) we get

$$\begin{aligned}
|(\mathcal{F}u)_j(t)| &= |f_j(t, u^j(t), \int_0^t g_j(t, s, u(s))ds)| \\
&\leq |f_j(t, u^j(t), \int_0^t g_j(t, s, u(s))ds) - f_j(t, 0, 0)| + |f_j(t, 0, 0)| \\
&\leq m_j(t)|u^j(t)| + k_j(t) \int_0^t |g_j(t, s, u(s))|ds + |f_j(t, 0, 0)| \\
&\leq m_i(t) \sup\{|u^j(t)| : j \geq i\} + tk_j(t)\overline{g}_j(t) + |f_i(t, 0, 0)| \\
&\leq m_i(t)r_i(t) + tk_i(t)\overline{g}_i(t) + |f_i(t, 0, 0)| \\
&= r_i(t)
\end{aligned}$$

for all $j \geq i$. This shows that $\mathcal{F}$ is a self-mapping of the set U_r. To prove the continuity of $\mathcal{F}$ on the closed set U_r let $u \in U_r$ and (u_n) be a sequence in U_r such that $u_n \to u$. Then we obtain

$$\begin{aligned}
|(\mathcal{F}u)_i(t) - (\mathcal{F}u_n)_i(t)| &\leq |f_i(t, u^i(t), \int_0^t g_i(t, s, u(s))ds) \\
&\quad - f_i(t, u_n^i(t), \int_0^t g_i(t, s, u_n(s))ds)| \\
&\leq m_i(t)|u^i(t) - u_n^i(t)| \\
&\quad + k_i(t) \int_0^t |g_i(t, s, u(s)) - g_i(t, s, u_n(s))ds|.
\end{aligned}$$

Now, applying the condition (H_3) we conclude that $(\mathcal{F}u_n)_i$ is uniformly convergent to $(\mathcal{F}u)_i$ on $t \in I$ for each $i \in \mathbb{N}$. This implies that $\mathcal{F}u_n$ is convergent to $\mathcal{F}u$ in $U_r \subseteq C(I, c_0)$ and the desired assertion is obtained. We also can show that $\mathcal{F}U_r$ is equicontinuous on I. To do this, let $u_0 \in U_r$ and $\epsilon > 0$ is given. Then there exists $\delta > 0$ such that for each

$u \in U_r$ we have

$$|(\mathcal{F}u)_i(t) - (\mathcal{F}u_0)_i(t)| \leq |f_i(t, u^i(t), \int_0^t g_i(t,s,u(s))ds) - f_i(t, u_0^i(t), \int_0^t g_i(t,s,u_0(s))ds)| \leq M\|u - u_0\|_{c_0} + k_i(t)\int_0^t |g_i(t,s,u(s)) - g_i(t,s,u_0(s))ds|$$

which uniform boundedness of k_i together with uniform continuity of g_i imply that

$$|(\mathcal{F}u)(t) - (\mathcal{F}u_0)(t)| \leq \epsilon \quad \text{for all } \ t \in I$$

whenever $\|u - u_0\|_{c_0} < \delta$ for some $\delta > 0$ and the claim is obtained. Now considering the set $V_r = Conv\mathcal{F}U_r$ (i.e. the closed convex hull of the set $\mathcal{F}U_r$) we easily see that V_r is closed, bounded and equicontinuous on I. In addition to the recent fact we also have $\mathcal{F}V_r \subseteq V_r \subseteq U_r$. On the other hand, for $u \in U_r$, we have

$$|(\mathcal{F}u)_i(t)| \leq r_i(t), \quad \text{for } i = 1, 2, ... \text{ and } t \in I.$$

Taking into account the hypothesis (H_2), (H_3) and (3.1) we see that $(r_i(t))$ converges uniformly on I to the function vanishing identically on I and hence for each $\epsilon > 0$ there exists a positive integer n_0 such that

$$|(\mathcal{F}u)_i(t)| \leq \epsilon, \quad \text{for } i \geq n_0 \text{ and } t \in I.$$

Now by the virtue of the criterion of compactness in the space c_0 we deduce that for any $t \in I$, the set $\mathcal{F}V_r(t)$ is relatively compact in c_0 (cf. Section 2). Following the statement as above we can be allowed to infer that V_r is relatively compact in $C(I, c_0)$ (cf. Theorem 2.1). Moreover, the closedness of V_r yields that it is compact. Therefore, considering the fact that the operator $\mathcal{F}$ maps continuously the set V_r into itself, we conclude by Schauder fixed-point principle that $\mathcal{F}$ has a fixed point in the set V_r and hence our problem has a solution in c_0. This completes the proof. □

4. Concluding remarks

Here we give some remarks which are obtained by the main result as consequence:

(1) The infinite system of integral equations of the form

$$u^i(t) = a_i(t) + \int_0^t g_i(t, s, u^1(s), u^2(s), ...)ds \tag{4.1}$$

which have been investigated in several papers can be deduced by Eq. (1.1).

(2) It is worthwhile mentioning that the result obtained in the former section can be applied to the infinite system of differential equations of the form

$$u_i' = g_i(t, u_1, u_2, ...)$$

with the initial conditions $u_i(0) = u_i^0$ for $i = 1, 2, ...$ and $t \in I$. Also, the perturbed diagonal infinite system of differential equations

$$u_i' = a_i(t)u_i + g_i(t, u_1, u_2, ...) \tag{4.2}$$

with the same initial conditions can be investigated by formulating our result.

In the following we consider a natural phenomenon which can be modeled as a infinite systems of differential equations.

Example 4.1. Consider the neural nets model given by the scalar equations

$$z_n'(t) = -\alpha_n z_n(t) + f_n(z(t)), \;\; z_n(0) = c_n, \;\; t \in I = [0, T] \tag{4.3}$$

where $\alpha_n > 0$, $0 < c_n < 1$,

$$f_n(z) = [1 + \exp(-\gamma_n - \sum_{m=1}^{\infty} \beta_{mn} z_m)]^{-1}, \quad \gamma_n \geq 0, \; \sum_{m=1}^{\infty} |\beta_{mn}| \leq b < \infty,$$

for each $n \geq 1$ and $z = (z_1, z_2, z_3, ...)$.
Obviously, system (4.3) is a member of class of systems as form of (4.2) which also can be included in (4.1). Suppose that (α_n) and (γ_n) are nondecreasing and nonincreasing sequences in real line, respectively. Also, consider $(\beta_{mn} z_m)$ is a nonincreasing sequence of functions on I for $m = 1, 2, ...$, then the assumptions in previous section implies that condition (H_1) and (H_2) are easily satisfied. It only remains to show that (H_3) holds. To do this, let first note that

$$\overline{g}_n(s) = \sup\left\{[1 + \exp(-\gamma_n - \sum_{m=1}^{\infty} \beta_{mn} z_m(s))]^{-1} : z_m \in C(I, \mathbb{R})\right\}.$$

is bounded on I for each $n = 1, 2, ...$ since we have

$$
\begin{aligned}
|f_n(z)(s)| &\le 1 + \inf \left\{ \exp\,(-\gamma_n - \sum_{m=1}^{\infty} \beta_{mn} z_m(s)) : z_m \in C(I, \mathbb{R}) \right\} \\
&\le 1 + \gamma_1 \sup \left\{ \exp\,(\sum_{m=1}^{\infty} \beta_{mn} z_m(s)) : z_m \in C(I, \mathbb{R}) \right\} \\
&\le 1 + \gamma_1 \sup_{z \in \mathbb{R}^\infty} \exp(b\|z\|_{c_0})
\end{aligned}
$$

where $z = (z_1, z_2, ...) \in C(I, c_0)$. Now by imposing the condition uniformly boundedness for $r_n(t)$ (as in the proof of the result) one can obtain the assertion.

We also remark that $\overline{g}_i(s)$ is nonincreasing sequence on I, because the sequences (γ_n) and $(\beta_{mn} z_m)$ (for $m = 1, 2, ...$) are so.

Remark 4.2. Note that in the previous example the initial condition as mentioned above is redundant and based on the conditions of our result we don't need it to prove the existence of solutions.

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