

## Some Results for the Jacobi-Dunkl Transform in the Space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$

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**ABSTRACT.** In this paper, using a generalized Jacobi-Dunkl translation operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the  $(\phi, p)$ -Lipschitz Jacobi-Dunkl condition in the space  $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ ,  $\alpha \geq \beta \geq \frac{-1}{2}$ ,  $\alpha \neq \frac{-1}{2}$ .

**Keywords:** Jacobi-Dunkl operator, Jacobi-Dunkl transform, generalized Jacobi-Dunkl translation.

*2000 Mathematics subject classification:* 42B37.

### 1. INTRODUCTION AND PRELIMINARIES

Titchmarsh's ([8], Theorem 85) characterized the set of functions in  $L^2(\mathbb{R})$  satisfying the Cauchy-Lipschitz Condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

**Theorem 1.1.** [8] *Let  $\alpha \in (0, 1)$  and assume that  $f \in L^2(\mathbb{R})$ . Then the following are equivalent:*

(a)  $\|f(x+h) - f(x)\| = O(h^\alpha)$ , as  $h \rightarrow 0$ ,

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Received: 01 July 2015

Revised: 24 July 2016

Accepted: 06 August 2016

$$(b) \int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}), \quad \text{as } r \rightarrow \infty,$$

where  $\widehat{f}$  stands for the Fourier transform of  $f$ .

A similar result of Theorem 1.1 has been established for the Jacobi transform in the space  $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$  (see [11]). In this paper, we prove a generalization of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the  $(\phi, p)$ -Lipschitz Jacobi-Dunkl condition in the space  $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ ,  $1 < p \leq 2$ . For this purpose, we use the generalized Jacobi-Dunkl translation operator.

In this section, we recapitulate from ([1]-[6]) some results related to the harmonic analysis associated with Jacobi-Dunkl operator  $\Lambda_{\alpha,\beta}$ .

The Jacobi-Dunkl function with parameters  $(\alpha, \beta)$ ,  $\alpha \geq \beta \geq \frac{-1}{2}$ ,  $\alpha \neq \frac{-1}{2}$ , is defined by the formula:

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda} \frac{d}{dx} \varphi_\mu^{\alpha,\beta}(x), & \text{if } \lambda \in \mathbb{C} \setminus \{0\}, \\ 1, & \text{if } \lambda = 0, \end{cases}$$

with  $\lambda^2 = \mu^2 + \rho^2$ ,  $\rho = \alpha + \beta + 1$  and  $\varphi_\mu^{\alpha,\beta}$  is the Jacobi function given by:

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh(x))^2\right),$$

$F$  is the Gauss hypergeometric function (see [1],[7]).

$\psi_\lambda^{\alpha,\beta}$  is the unique  $C^\infty$ -solution on  $\mathbb{R}$  of the differential-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta} \mathcal{U} = i\lambda \mathcal{U}, & \lambda \in \mathbb{C}, \\ \mathcal{U}(0) = 1, \end{cases}$$

(see [5]), where  $\Lambda_{\alpha,\beta}$  is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta} \mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'_{\alpha,\beta}(x)}{A_{\alpha,\beta}(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

with

$$A_{\alpha,\beta}(x) = 2^\rho (\sinh |x|)^{2\alpha+1} (\cosh |x|)^{2\beta+1},$$

i.e.,

$$\Lambda_{\alpha,\beta} \mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha + 1) \coth x + (2\beta + 1) \tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}.$$

Using the relation

$$\frac{d}{dx} \varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2 + \rho^2}{4(\alpha + 1)} \sinh(2x) \varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function  $\psi_\lambda^{\alpha,\beta}$  can be written in the form above (see [2])

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i \frac{\lambda}{4(\alpha+1)} \sinh(2x) \varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R},$$

where  $\lambda^2 = \mu^2 + \rho^2$ ,  $\rho = \alpha + \beta + 1$ .

Denote  $L_{\alpha,\beta}^p(\mathbb{R}) = L_{\alpha,\beta}^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ ,  $1 < p \leq 2$  the space of measurable functions  $f$  on  $\mathbb{R}$  such that

$$\|f\|_{p,\alpha,\beta} = \left( \int_{\mathbb{R}} |f(x)|^p A_{\alpha,\beta}(x) dx \right)^{1/p} < +\infty.$$

Using the eigenfunctions  $\psi_\lambda^{\alpha,\beta}$  of the operator  $\Lambda_{\alpha,\beta}$  called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform by

$$\mathcal{F}_{\alpha,\beta} f(\lambda) = \int_{\mathbb{R}} f(x) \psi_\lambda^{\alpha,\beta}(x) A_{\alpha,\beta}(x) dx, \quad \lambda \in \mathbb{R},$$

and the inversion formula by

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta} f(\lambda) \psi_{-\lambda}^{\alpha,\beta}(t) d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2 - \rho^2} |C_{\alpha,\beta}(\sqrt{\lambda^2 - \rho^2})|} \mathbb{I}_{\mathbb{R} \setminus ]-\rho, \rho[}(\lambda) d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu} \Gamma(\alpha+1) \Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu)) \Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C} \setminus (i\mathbb{N}),$$

and  $\mathbb{I}_{\mathbb{R} \setminus ]-\rho, \rho[}$  is the characteristic function of  $\mathbb{R} \setminus ]-\rho, \rho[$ .

The Jacobi-Dunkl transform is a unitary isomorphism from  $L_{\alpha,\beta}^2(\mathbb{R})$  onto  $L^2(\mathbb{R}, d\sigma(\lambda))$ , i.e.,

$$\|f\|_{2,\alpha,\beta} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\mathbb{R}, d\sigma(\lambda))}. \quad (1.1)$$

Plancherel's theorem (1.1) and the Marcinkiewics interpolation theorem (see [8]) we get for  $f \in L_{\alpha,\beta}^p(\mathbb{R})$  with  $1 < p \leq 2$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}, d\sigma(\lambda))} \leq K \|f\|_{p,\alpha,\beta}, \quad (1.2)$$

where  $K$  is a positive constant (see [6]).

The operator of Jacobi-Dunkl translation is defined by :

$$T_x f(y) = \int_{\mathbb{R}} f(z) d\nu_{x,y}^{\alpha,\beta}(z), \quad \forall x, y \in \mathbb{R},$$

where  $\nu_{x,y}^{\alpha,\beta}(z)$ ,  $x, y \in \mathbb{R}$  are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z) = \begin{cases} K_{\alpha,\beta}(x, y, z)A_{\alpha,\beta}(z)dz, & \text{if } x, y \in \mathbb{R}^*, \\ \delta_x, & \text{if } y = 0, \\ \delta_y, & \text{if } x = 0. \end{cases}$$

Here,  $\delta_x$  is the Dirac measure at  $x$ . And,

$$K_{\alpha,\beta}(x, y, z) = M_{\alpha,\beta}(\sinh(|x|) \sinh(|y|) \sinh(|z|))^{-2\alpha} \mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x, y, z) \\ \times (g_\theta(x, y, z))_+^{\alpha-\beta-1} \sin^{2\beta} \theta d\theta,$$

where

$$I_{x,y} = [-|x| - |y|, -||x| - |y||] \cup [||x| - |y||, |x| + |y|] \\ \rho_\theta(x, y, z) = 1 - \sigma_{x,y,z}^\theta + \sigma_{z,x,y}^\theta + \sigma_{z,y,x}^\theta, \\ \forall z \in \mathbb{R}, \theta \in [0, \pi], \sigma_{x,y,z}^\theta = \begin{cases} \frac{\cosh(x) + \cosh(y) - \cosh(z) \cos(\theta)}{\sinh(x) \sinh(y)}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0, \end{cases} \\ g_\theta(x, y, z) = 1 - \cosh^2(x) - \cosh^2(y) - \cosh^2(z) + 2 \cosh(x) \cosh(y) \cosh(z) \cos \theta, \\ t_+ = \begin{cases} t, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and,

$$M_{\alpha,\beta} = \begin{cases} \frac{2^{-2\rho} \Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha-\beta) \Gamma(\beta+\frac{1}{2})}, & \text{if } \alpha > \beta, \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi_\lambda^{\alpha,\beta}(h) \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad (1.3)$$

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i\lambda \mathcal{F}_{\alpha,\beta}(f)(\lambda). \quad (1.4)$$

For  $\alpha \geq \frac{-1}{2}$ , we introduce the normalized Bessel function of first kind and order  $\alpha$  [10] defined by:

$$j_\alpha(x) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n + \alpha + 1)}, \quad x \in \mathbb{R}.$$

Moreover, we see that

$$\lim_{x \rightarrow 0} \frac{j_\alpha(x) - 1}{x^2} \neq 0,$$

by consequence, there exists  $C_1 > 0$  and  $\eta > 0$  satisfying

$$|x| \leq \eta \Rightarrow |j_\alpha(x) - 1| \geq C_1 |x|^2. \quad (1.5)$$

**Lemma 1.2.** *Let  $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$ . Then for  $|\nu| \leq \rho$ , there exists a positive constant  $C_2$  such that*

$$|1 - \varphi_{\mu+i\nu}^{\alpha,\beta}(x)| \geq C_2|1 - j_\alpha(\mu x)|.$$

*Proof.* (See [4], Lemma 9). □

Denote by  $L_p^m(\Lambda_{\alpha,\beta}), 1 < p \leq 2, m = 0, 1, 2, \dots$ , the class of functions  $f \in L_{\alpha,\beta}^p(\mathbb{R})$  that have on  $\mathbb{R}$  generalized derivatives  $f'(x), f''(x), \dots, f^{(2m)}(x)$  in the sense of Levi (see [9]) and belong to  $L_{\alpha,\beta}^p(\mathbb{R})$  with  $\Lambda_{\alpha,\beta}^m f \in L_{\alpha,\beta}^p(\mathbb{R})$ . i.e.,

$$L_p^m(\Lambda_{\alpha,\beta}) = \left\{ f \in L_{\alpha,\beta}^p(\mathbb{R}) / \Lambda_{\alpha,\beta}^m f \in L_{\alpha,\beta}^p(\mathbb{R}) \right\},$$

where  $\Lambda_{\alpha,\beta}^0 f = f, \Lambda_{\alpha,\beta}^m f = \Lambda_{\alpha,\beta}(\Lambda_{\alpha,\beta}^{m-1} f), m = 0, 1, 2, \dots$

## 2. MAIN RESULT

In this section we give the main result of this paper. We need first to define  $(\phi, p)$ -Lipschitz Jacobi-Dunkl class.

Denote  $N_h$  by

$$N_h = T_h + T_{-h} - 2I,$$

where  $I$  is the unit operator in the space  $L_{\alpha,\beta}^p(\mathbb{R})$ .

**Definition 2.1.** A function  $f \in L_p^m(\Lambda_{\alpha,\beta})$  is said to be in  $(\phi, p)$ -Lipschitz Jacobi-Dunkl class, denoted by  $Lip(\phi, p, \alpha, \beta)$ , if

$$\|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad \text{as } h \rightarrow 0,$$

where  $m = 0, 1, 2, \dots$  and  $\phi$  is a continuous increasing function on  $[0, \infty)$ , satisfying  $\phi(0) = 0$  and  $\phi(ts) = \phi(t)\phi(s)$  for all  $t, s \in [0, \infty)$ .

**Lemma 2.2.** *For  $f \in L_p^m(\Lambda_{\alpha,\beta})$ , then*

$$\left( \int_{\mathbb{R}} 2^q \lambda^{qm} |\varphi_{\mu}^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \right)^{\frac{1}{q}} \leq K \|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $m = 0, 1, 2, \dots$

*Proof.* From (1.4), we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}^m f)(\lambda) = i^m \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0, 1, 2, \dots \quad (2.1)$$

We use formulas (1.3) and (2.1), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = i^m (\psi_{\lambda}^{(\alpha,\beta)}(h) + \psi_{\lambda}^{(\alpha,\beta)}(-h) - 2) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_{\lambda}^{(\alpha,\beta)}(h) = \varphi_{\mu}^{\alpha,\beta}(h) + i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi_{\mu}^{\alpha+1,\beta+1}(h),$$

$$\psi_{\lambda}^{(\alpha,\beta)}(-h) = \varphi_{\mu}^{\alpha,\beta}(-h) - i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi_{\mu}^{\alpha+1,\beta+1}(-h),$$

and  $\varphi_{\mu}^{\alpha,\beta}$  is even, then

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = 2i^m (\varphi_{\mu}^{\alpha,\beta}(h) - 1) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By formula 1.2, we have the result.  $\square$

**Theorem 2.3.** *Let  $f$  belong to  $Lip(\phi, p, \alpha, \beta)$ . Then*

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad \text{as } r \rightarrow \infty,$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $m = 0, 1, 2, \dots$

*Proof.* Assume that  $f \in Lip(\phi, p, \alpha, \beta)$ , then we have

$$\|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad \text{as } h \rightarrow 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} \lambda^{qm} |\varphi_{\mu}^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q}{2^q} \|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta}^q.$$

By (1.5) and Lemma 1.2, we get:

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |1 - \varphi_{\mu}^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \geq C_1^q C_2^q \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} |\mu h|^{2q} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

From  $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$  we have

$$\begin{aligned} \left(\frac{\eta}{2h}\right)^2 - \rho^2 &\leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2 \\ \Rightarrow \mu^2 h^2 &\geq \frac{\eta^2}{4} - \rho^2 h^2. \end{aligned}$$

Take  $h \leq \frac{\eta}{3\rho}$ , then we have  $\mu^2 h^2 \geq C_3 = C_3(\eta)$ .

So,

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |1 - \varphi_{\mu}^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \geq C_1^q C_2^q C_3^q \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

There exists then a positives constants  $C$  and  $K_1$  such that

$$\begin{aligned} \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) &\leq C \int_{\mathbb{R}} \lambda^{qm} |1 - \varphi_{\mu}^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\leq K_1 \phi^q(h) = K_1 \phi(h^q). \end{aligned}$$

For all  $0 < h < \frac{\eta}{3\rho}$ . Then we have,

$$\int_{r \leq |\lambda| \leq 2r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq K_2 \phi(r^{-q}), \quad r \rightarrow \infty.$$

where  $K_2 = K_1\phi(\eta^q 2^{-q})$ .

Furthermore, we obtain

$$\begin{aligned} \int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) &= \left( \int_{r \leq |\lambda| \leq 2r} + \int_{2r \leq |\lambda| \leq 4r} + \int_{4r \leq |\lambda| \leq 8r} + \dots \right) \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\leq K_2\phi(r^{-q}) + K_2\phi((2r)^{-q}) + K_2\phi((4r)^{-q}) + \dots \\ &\leq K_2\phi(r^{-q}) + K_2\phi(2^{-q})\phi(r^{-q}) + K_2\phi((2^{-q})^2)\phi(r^{-q}) + \dots \\ &\leq K_2\phi(r^{-q})(1 + \phi(2^{-q}) + \phi((2^{-q})^2) + \dots). \end{aligned}$$

We have  $\phi(2^{-q}) < 1$ , then

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq K_3\phi(r^{-q}),$$

where  $K_3 = K_2(1 - \phi(2^{-q}))^{-1}$ .

Finally, we get

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad \text{as } r \rightarrow \infty.$$

Thus, the proof is finished.  $\square$

**Corollary 2.4.** *Let  $f \in L_p^m(\Lambda_{\alpha,\beta})$ , and let*

$$\|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad \text{as } h \rightarrow 0.$$

Then

$$\int_{|\lambda| \geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(r^{-qm} \phi(r^{-q})), \quad \text{as } r \rightarrow \infty,$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $m = 0, 1, 2, \dots$

### Acknowledgements

The authors would like to thank the referee for his valuable comments and suggestions.

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