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Special Bertrand Curves in semi-Euclidean space E_2^4 and Their Characterizations

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Abstract. In [14]; Matsuda and Yorozu proved that there is no special Bertrand curves in E^n $(n > 3)$ and they defined a new kind of Bertrand curves called $(N,\ B_2)$ -type Bertrand curves in 4-dimensional Euclidean space. In this paper , by using the similar methods of Matsuda and Yorozu , we define a quaternionic Bertrand curve in semi-Euclidean space E_2^4 and investigate its properties. Then we prove that the torsion and bitorsion of the quaternionic curve are not equal to zero in semi-Euclidean space E_2^4 and then we obtain $(N,\ B_2)$ -type quaternionic Bertrand curves by means of the $\{\kappa,\ \tau, (\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)\}$ functions of curve.

Keywords: Semi-Euclidean spaces, Quaternionic frame, Quaternionic Bertrand curves.

2000 Mathematics subject classification: 14H45, 53A04, 53A17.

1. Introduction

The geometry of curves has long captivated the interests of mathematicians, from the ancient Greeks to the era of Isaac Newton (1643-1727) and the invention of the calculus. It is a branch of geometry that deals with smooth curves in the plane and in the space by methods of differential and integral calculus. The theory of curves is simpler and narrower

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in scope because a regular curve in Euclidean space has no intrinsic geometry. One of the most important tools used to analyze a curve is the Frenet frame, a moving frame that provides a coordinate system at each point of curve that is "best adopted" to the curve near the point. Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve whose principal normal is the principal normal of another curve. It is characterized as curve whose curvature and torsion are in linear relation. There are many works related with Bertrand curves in the Euclidean space and Lorentzian space [1]-[7]. In 1845, Saint Venant [8] proposed the question upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of a curve. This question was answered by Bertrand in 1850 in a paper [9] in which he solved that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by κ and τ respectively, then for $\lambda, \mu \in R$ we have $\lambda\kappa+\mu\tau = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called Conjugate Bertrand Curves , or more commonly Bertrand Curves [10]. In 1888, C. Bioche give a new theorem in [11] to obtaining Bertrand curves by using the given two curves C_1 and C_2 in Euclidean 3-space. Later, in 1960, J. F. Burke in [12] give a theorem related with Bioche's thorem on Bertrand curves. In 1987, The Serret-Frenet formulae for a quaternionic curves in R^3 are introduced by K. Bharathi and M. Nagaraj. Moreover, they obtained the Serret-Frenet formulae for the quaternionic curves in R^4, [13]. Then, lots of studies have been published by using this studies. One of them is A. C. Çöken and A. Tuna's study [14]-[15] which they gave Serret-Frenet formulas, inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi- Euclidean spaces E_1^3 and E_2^4. In this paper , applying a similar method as the one given by Matsuda and Yorozu [16], we found that bitorsion of the quaternionic curve is not equal to zero in the semi-Euclidean space E_2^4, in order to obtain (N, B_2)-type quaternionic Bertrand curves based on $\kappa,\ \tau,\ \sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa$ functions of the curve in E_2^4.

2. PRELIMINARIES

Let Q_v be the four-dimensional vector space over a field v whose characteristic greater than 2. Let e_i $(1 \leq i \leq 4)$ be a basis for the vector space. Let the rule of multiplication on Q_v be defined on e_i and extended to the whole of the vector space distributivity as follows [17]: A semi-real quaternion is defined with $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + d$ or

($S_q = d$ and $\mathrm{V}_q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$). Then a quaternion q can now write as $q = S_q + V_q$, where S_q and V_q are the scalar part and vectorial part of q, respectively.) Such that
I. $e_i \times e_i = -\varepsilon(e_i), \quad 1 \leq i \leq 3$
II. $e_i \times e_j = -\varepsilon(e_i)\varepsilon(e_j)e_k$
where (ijk) is an even permutation of (123) in the semi-Euclidean space E_2^4. Notice here that we define the set of all semi-real quaternions by

Q_v, where v is an index $v = 1, 2$.

$$Q_v = \{q | q = ae_1 + be_2 + ce_3 + d; a, b, c, d \in R \quad \text{and} \quad e_1, e_2, e_3 \in R^3\}.$$

Using these basic products we can now expand the product of two quaternions to give

$$p \times q = S_p S_q + \langle\, V_p, V_q \,\rangle + S_p V_q + S_q V_p + V_p \wedge V_q \text{ for every } p, q \in Q_v$$

where we have used the quaternionic product contains all the products of semi-Euclidean space E_2^4 [13]. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows

$$\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} \text{for every } q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + d \in Q_v$$

which is called the Hamiltonian conjugation. This defines the symmetric non-degenerate valued bilinear form h as follows

$$h(p, q) = \frac{1}{2}\left[-\varepsilon(p)\varepsilon(\gamma q)(p \times \gamma q) - \varepsilon(q)\varepsilon(\gamma p)(q \times \gamma p)\right] \text{ for } E_2^4.$$

the norm of semi-real quaternion q is denoted by

$$\|q\|^2 = |h_v(q, q)| = |\varepsilon(q)(q \times \gamma q)| = \left|-a^2 - b^2 + c^2 + d^2\right|$$

for $p, q \in Q_v$ where if $h_v(p, q) = 0$ then p and q are called h -orthogonal. The concept of a spatial quaternion will be used throughout our work. q is called a spatial quaternion whenever $q + \gamma q = 0$ [14]-[15]. The Serret-Frenet formulae for quaternionic curves in semi- Euclidean space are given below:

Definition 2.1. Let

$$\alpha : I \subset R \ \rightarrow \ Q_v$$
$$s \rightarrow \alpha(s) = \sum_{i=1}^{4} \alpha_i(s)\overrightarrow{e_i}, 1 \leq i \leq 4, \ \overrightarrow{e_4} = 1$$

be a smooth curve in semi-Euclidean space E_2^4. Let the parameter s be chosen such that the tangent $T(s) = \alpha'(s)$ has unit magnitude.

Let $\{T, N, B_1, B_2\}$ be Frenet apparatus of the differentiable in semi-Euclidean space E_2^4. Then Frenet formulas are given by

$$\begin{aligned}
\frac{d}{ds}T(s) &= \varepsilon_N\ \kappa(s)N(s) \qquad (2.1)\\
\frac{d}{ds}N(s) &= \varepsilon_n\tau(s)B_1(s) - \varepsilon_t\varepsilon_N\kappa(s)T(s)\\
\frac{d}{ds}B_1(s) &= -\varepsilon_t\tau(s)N(s) + \varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)B_2(s)\\
\frac{d}{ds}B_2(s) &= -\varepsilon_{b_1}\tau(s)\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)B_1(s).
\end{aligned}$$

Where $\kappa = \varepsilon_N\|T(s)\|$ and $\|N(s)\|^2 = |\varepsilon_N|$.

3. (N, B_2)-Bertrand curves in semi-Euclidean space E_2^4

Definition 3.1. Let E_2^4 be the 4-dimensional semi-Euclidean space with the inner product $h(\alpha, \alpha^*)$. If there exists a corresponding relationship between the quaternionic space curves α and α^* such that at the corresponding points of the quaternionic curves, the principal normal lines of α and α^* are linearly dependent, then α is called a quaternionic Bertrand curve, and α^* a quaternionic Bertrand curve of α. The pair $\{\alpha,\ \alpha^*\}$ is said to be a quaternionic Bertrand pair.

Let $\alpha(s)$ be a quaternionic Bertrand curve in E_2^4 parameterized by its arc-length s and $\alpha^*(s)$ the quaternionic Bertrand partner curve with an arc-length parameter s^*, respectively, then by

$$\{T(s),\ N(s),\ B_1(s),\ B_2(s)\}$$

and

$$\{T^*(s^*),\ N^*(s^*),\ B_1^*(s^*),\ B_2^*(s^*)\}$$

the Frenet frames field along of α and α^*.

Definition 3.2. Let $\alpha(s)$ and $\alpha^*(s^*)$ be quaternionic curves in E_2^4. $\{T(s),\ N(s),\ B_1(s),\ B_2(s)\}$ and $\{T^*(s^*),\ N^*(s^*),\ B_1^*(s^*),\ B_2^*(s^*)\}$ are Frenet frames of α and α^* respectively, on this curves. And there exist a bijection. $\alpha(s)$ and $\alpha^*(s^*)$ are Bertrand curves if there exist a bijection

$$\begin{gathered}
\varphi : I \to I^*\\
s \to \varphi(s) = s* \ ,\ \tfrac{ds*}{ds} \neq 0
\end{gathered}$$

and $N(s)$, $N^*(s^*)$ are linearly dependent.

Theorem 3.3. *Let α be a quaternionic curve in the 4- dimensonal semi-Euclidean space . If $[\sigma - \varepsilon_T\varepsilon_t\varepsilon_N\kappa] \neq 0$ and $k(s) \neq 0$, then no quaternionic curve in E_2^4 is a Bertrand curve.*

Proof. Let α be a quaternionic Bertrand curve in E_2^4 and α^* be quaternionic Bertrand partner mate of α with an arc-length parameter s and s^*, respectively. Let the pair of $\alpha(s)$ and $\alpha^*(s^*) = \alpha^*(\varphi(s))$ be corresponding points of α and α^*. Then, the curve α^* is given by

$$\alpha^*(s) = \alpha^*(\varphi(s)) = \alpha(s) + \lambda(s)N(s) \tag{3.1}$$

where λ is a C^∞ function on I. Differentiating in equation (3.1) with respect to s and using the Frenet formulas given in (2.1), we get

$$\varphi'(s)T^*(\varphi(s)) = [1 - \varepsilon_t\varepsilon_N\lambda(s)\kappa(s)]\,T(s) + \lambda'(s)N(s) + \varepsilon_n\lambda(s)\tau(s)B_1(s).$$

$$\begin{aligned}\varphi'(s)h(T^*(\varphi(s)), N^*(\varphi(s)) &= [1 - \varepsilon_t\varepsilon_N\lambda(s)\kappa(s)]\,h(T(s), N^*(\varphi(s)))\\ &\quad +\lambda'(s)h(N(s),\ N^*(\varphi(s)))\\ &\quad +\varepsilon_n\lambda(s)\tau(s)h(B_1(s),\ N^*(\varphi(s))).\end{aligned}$$

Since

$$h(T^*(\varphi(s)), N^*(\varphi(s))) = 0,\ h(T(s), N^*(\varphi(s))) = 0$$

and

$$h(B_1(s),\ N^*(\varphi(s))) = 0,$$
$$N^*(\varphi(s)) = \pm N(s) \text{ and } \|N(s)\| = |\varepsilon_N| = 1,$$

we obtain that $\lambda'(s) = 0$, that is, λ is a non -zero constant. Thus, equation (3.1) can be written as

$$\alpha^*(s) = \alpha^*(\varphi(s)) = \alpha(s) + \lambda N(s)$$

and we obtain

$$\varphi'(s)T^*(\varphi(s)) = [1 - \varepsilon_t\varepsilon_N\lambda\kappa(s)]\,T(s) + \varepsilon_n\lambda\tau(s)B_1(s) \tag{3.2}$$

for all $s \in I$. By using equation (3.2), we get

$$T^*(\varphi(s)) = \frac{[1 - \varepsilon_t\varepsilon_N\lambda\kappa(s)]}{\varphi'(s)}T(s) + \frac{\varepsilon_n\lambda\tau(s)}{\varphi'(s)}B_1(s).$$

If we denote

$$a(s) = \frac{[1 - \varepsilon_t\varepsilon_N\lambda\kappa(s)]}{\varphi'(s)}, \quad b(s) = \frac{\varepsilon_n\lambda\tau(s)}{\varphi'(s)} \tag{3.3}$$

we can set

$$T^*(\varphi(s)) = a(s)T(s) + b(s)B_1(s). \tag{3.4}$$

Differentiating in equation (3.4) with respect to s and using the Frenet formulas given in equation (2.1), we obtain

$$\begin{aligned}\varphi'(s)\varepsilon_N\ \kappa^*(s)N^*(s) &= a'(s)T(s) + [a(s)\varepsilon_N\ \kappa(s) - b(s)\varepsilon_t\tau(s)]\,N(s)\\ &\quad +b'(s)B_1(s) + b(s)\varepsilon_n\,[\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa]\,(s)B_2(s)\end{aligned}$$

Since $N^*(\varphi(s)) = \pm N(s)$, we obtain

$$b(s)\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s) = 0.$$

By $\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa \neq 0$, we have $b(s) = 0$. From equation (3.3), we get $\frac{\varepsilon_n\lambda(s)\tau(s)}{\varphi'(s)} = 0$. Since $\tau(s) \neq 0$, we obtain that $\lambda = 0$. This completes the proof of theorem. □

Theorem 3.4. *Let α be a quaternionic curves in E_2^4 with curvature functions $\kappa,\ \tau, (\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)$ and $(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa) \neq 0$. Then α is a quaternionic $(N,\ B_2)$ -Bertrand curve if and only if there exist real numbers $\lambda,\ \mu,\ \gamma,\ \delta$ such that,*

(i) $\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s) \neq 0$

(ii) $\gamma\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right] + \lambda\varepsilon_t\varepsilon_N\kappa(s) = 1$

(iii) $\gamma\varepsilon_N\ \kappa(s) - \varepsilon_t\tau(s) = \delta\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)$

(iv) $\left[\gamma^2 - 1\right]\varepsilon_t\varepsilon_N\kappa(s)\tau(s) + \gamma\left\{\begin{array}{c}\varepsilon_N^2(\kappa(s))^2 - \varepsilon_t^2(\tau(s))^2 \\ -\varepsilon_n^2\left[\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]^2\end{array}\right\} \neq 0$

Proof. Let α be a quaternionic $(N,\ B_2)$ Bertrand curve with arc-length parameter s. The $(N,\ B_2)$ Bertrand mate α^* is given by

$$\alpha^*(s^*) = \alpha^*(\varphi(s)) = \alpha(s) + \lambda(s)N(s) + \mu(s)B_2(s) \text{ for all } s \in I. \quad (3.5)$$

where $\lambda(s)$ and $\mu(s)$ are C^∞-functions on I. Differentiating in equation (3.5) with respect to s and by using Frenet equations, we obtain

$$\varphi'(s)T^*(\varphi(s)) = \left\{\begin{array}{c}\left[1 - \lambda(s)\varepsilon_t\varepsilon_N\kappa(s)\right]T(s) + \lambda'(s)N(s) \\ +\left[\lambda(s)\varepsilon_n\tau(s) - \mu(s)\varepsilon_{b_1}\left[\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right](s)\right]B_1(s) \\ +\mu'(s)B_2(s)\end{array}\right\} \quad (3.6)$$

for all $s \in I$. Since $\text{span}\{N^*(\varphi(s),\ B_2^*(\varphi(s)\}$, $\text{span}\{N(s),\ B_2(s)\}$, we can put

$$N^*(\varphi(s)) = m(s)N(s) + n(s)B_2(s), \quad (3.7)$$

$$B_2^*(\varphi(s)) = p(s)N(s) + q(s)B_2(s), \quad (3.8)$$

and by using equations (3.7) and (3.8) we get

$$\begin{aligned} h(N^*(\varphi(s)),\ \varphi'(s)T^*(\varphi(s))) &= \lambda'(s)m(s) + \mu'(s)n(s) = 0 \\ h(B_2^*(\varphi(s)),\ \varphi'(s)T^*(\varphi(s))) &= \lambda'(s)p(s) + \mu'(s)q(s) = 0 \end{aligned}$$

where $\left|\begin{array}{cc} m(s) & n(s) \\ p(s) & q(s)\end{array}\right|$ is non-zero because $\{N^*(\varphi(s),\ B_2^*(\varphi(s)\}$ vector fieds must be linear independent. We obtain $\lambda'(s) = 0,\ \mu'(s) = 0$ that is , λ and μ are constant function on I . So, we can rewrite equation (3.5) , respectively as

$$\alpha^*(s^*) = \alpha^*(\varphi(s)) = \alpha(s) + \lambda N(s) + \mu B_2(s)$$

$$\varphi'(s)T^*(\varphi(s)) = \left\{ \begin{array}{c} [1-\lambda\varepsilon_t\varepsilon_N\kappa(s)]\,T(s) \\ +\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]B_1(s) \end{array} \right\} \quad (3.9)$$

where

$$\left(\varphi'(s)\right)^2 = [1-\lambda\varepsilon_t\varepsilon_N\kappa(s)]^2 + \left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]^2 \neq 0 \quad (3.10)$$

If we denote

$$a(s) = \frac{[1-\lambda\varepsilon_t\varepsilon_N\kappa(s)]}{\varphi'(s)}, \quad b(s) = \frac{\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]}{\varphi'(s)}. \quad (3.11)$$

It easy to obtain

$$T^*(\varphi(s)) = a(s)T(s) + b(s)B_1(s) \quad (3.12)$$

where $a(s)$ and $b(s)$ are C^∞- functions on I. Differentiating in equation (3.12) with respect to s and using the Frenet equations, we obtain

$$\left\{ \begin{array}{c} \varphi'(s)\varepsilon_N\ \kappa^*(s)N^*(s) = \\ a'(s)T(s) + b'(s)B_1(s) \\ +\left[b(s)\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]B_2(s) \end{array} \right\} \quad (3.13)$$

Since $N^*(\varphi(s)$ is expressed by linear combination of $N(s)$ and $B_2(s)$, we have $a'(s) = 0,\ b'(s) = 0$, that is, a and b are constant function on I. Thus we can rewrite equation (3.6) as

$$\begin{aligned} \varphi'(s)\varepsilon_N\ \kappa^*(s)N^*(s) &= \left[a\varepsilon_N\ \kappa(s) - b\varepsilon_t\tau(s)\right]N(s) \\ &+ b\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)B_2(s) \end{aligned} \quad (3.14)$$

for all $s \in I$. By using equation (3.11) we can easily show that

$$a\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right] = b\left[1-\lambda\varepsilon_t\varepsilon_N\kappa(s)\right] \quad (3.15)$$

where b must be a non-zero constant. If we take $b(s) = 0$, from equation (3.12) we get

$$\varphi'(s)\varepsilon_N\ \kappa^*(\varphi(s))N^*(\varphi(s)) = \varepsilon_N\ \kappa(s)N(s)$$

So we obtain $N^*(\varphi(s)) = \pm N(s)$ for all $s \in I$, and this is a contradiction. According to theorem 1, we obtain

$$\varphi'(s)\varepsilon_N\ \kappa^*(\varphi(s))N^*(\varphi(s)) = \varepsilon_N\ \kappa(s)N(s)$$

that is $N^*(\varphi(s)) = \pm N(s)$ for all $s \in I$. By theorem 1, this fact is a contradiction according to the theorem 1. Thus we must consider only the case of $b(s) \neq 0$. Then it can be easily seen that

$$\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s) \neq 0.$$

that is, we obtain the relation *(i)*. If we denote the constant γ by $\gamma = \frac{a}{b}$ and by using equation (3.15) we have

$$\gamma\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right] + \lambda\varepsilon_N\varepsilon_t\kappa(s) = 1 \text{ for all } s \in I.$$

Thus we obtain the relation *(ii)*. From equation (3.14), we have

$$h(\varphi'(s)\varepsilon_N\ \kappa^*(s)N^*(s),\ \varphi'(s)\varepsilon_N\ \kappa^*(s)N^*(s)) = \left[a(s)\varepsilon_N\ \kappa(s) - b(s)\varepsilon_t\tau(s)\right]^2 + \left[b(s)\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]^2$$

and then,

$$\left[\varphi'(s)\varepsilon_N\ \kappa^*(s)\right]^2 = \left\{\left[\gamma\varepsilon_N\ \kappa(s) - \varepsilon_t\tau(s)\right]^2 + \left[\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]^2\right\} \left\{\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right\}^2$$

for all $s \in I$. From *(ii)*, in equations (3.10) and (3.11)

$$\left[\varphi'(s)\varepsilon_N\kappa^*(s)\right]^2 = \frac{1}{\gamma^2+1}\left\{\left[\gamma\varepsilon_N\ \kappa(s) - \varepsilon_t\tau(s)\right]^2 + \left[\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]^2\right\}.$$

Since $\varphi'(s)\varepsilon_N\ \kappa^*(s) \neq 0$ by using equation (3.14), we have

$$N^*(\varphi(s)) = m(s)N(s) + n(s)B_2(s) \tag{3.16}$$

where

$$m(s) = \frac{\left[a(s)\varepsilon_N\ \kappa(s) - b(s)\varepsilon_t\tau(s)\right]}{\varphi'(s)\varepsilon_N\ \kappa^*(s)}\ ,$$

$$n(s) = \frac{\left[b(s)\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]}{\varphi'(s)\varepsilon_N\ \kappa^*(s)}$$

$$m(s) = \frac{\left[\gamma\varepsilon_N\ \kappa(s) - \varepsilon_t\tau(s)\right]\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]}{\varphi'(s)\varepsilon_N\ \kappa^*(s)} \tag{3.17}$$

$$n(s) = \frac{\left[\lambda\varepsilon_n\tau(s) - \mu\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]}{\varphi'(s)\varepsilon_N\ \kappa^*(s)} \tag{3.18}$$

we can rewrite by using equations (3.11), (3.13) and *(ii) as*

$$\frac{m(s)}{n(s)} = \frac{\gamma\varepsilon_N\ \kappa(s) - \varepsilon_t\tau(s)}{\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)}$$

for all $s \in I$. if we differentiate in equation (3.16) and using the Frenet equations, we have

$$\begin{gathered}\varepsilon_n\varphi'(s)\tau^*(\varphi(s))B_1^*(\varphi(s)) = \\ \varepsilon_t\varepsilon_N\varphi'(s)\kappa^*(\varphi(s))T^*(\varphi(s)) - m\varepsilon_t\varepsilon_N\kappa(s)T(s) \\ + \left(m\varepsilon_n\tau(s) - n\varepsilon_{b_1}\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right)B_1(s)\end{gathered} \tag{3.19}$$

for all $s \in I$. From equation (3.19), it holds

$$m'(s) = 0, \quad n'(s) = 0.$$

If we denote $\frac{m}{n} = \delta$, it is obvious that

$$\gamma\varepsilon_N\ \kappa(s) - \varepsilon_t\tau(s) = \delta\varepsilon_n\left(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)$$

Thus we prove *(iii)*. Now, by using equations (3.9),(3.17),(3.18) and (3.19)

$$\varepsilon_t\varepsilon_N\varphi'(s)\tau^*(\varphi(s))B_1^*(\varphi(s)) \neq 0$$

for all $s \in I$. We have

$$\left\{ \begin{array}{c} \left[\gamma^2 - 1\right] \varepsilon_t \varepsilon_N \kappa(s)\tau(s) + \\ \gamma \left[(\varepsilon_N \kappa(s))^2 - (\varepsilon_t \tau(s))^2 - (\varepsilon_n \left[(\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa)(s) \right])^2 \right] \end{array} \right\} \neq 0.$$

Thus we prove *(iv)*. Conversely, Let α be a quaternionic curve with curvature $\kappa,\ \tau,\ (\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa)$ satisfaying the relation (i), (ii), (iii) and (iv) for constant numbers $\lambda,\ \mu,\ \delta,\ \gamma$ and $\alpha^*(s)$ be a quaternionic mate of α curve such as

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)N(s) + \mu(s)B_2(s) \tag{3.20}$$

for all $s \in I$. Differentiating in equation (3.20) with respect to s and using the Frenet equations, we obtain

$$\frac{d\alpha^*(s^*)}{ds^*} = \left[1 - \lambda \varepsilon_t \varepsilon_N \kappa(s)\right] T(s) + \left[\lambda \varepsilon_n \tau(s) - \mu \varepsilon_{b_1} \left(\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)(s)\right] B_1(s)$$

for all $s \in I$. Thus, by the relation (ii), we have

$$\frac{d\alpha^*(s^*)}{ds^*} = \left[\lambda \varepsilon_n \tau(s) - \mu \varepsilon_{b_1} \left(\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)(s)\right] \left[\gamma T(s) + B_1(s)\right],$$

for all $s \in I$. Also we get

$$\left\| \frac{d\alpha^*(s^*)}{ds^*} \right\| = \eta \left[\lambda \varepsilon_n \tau(s) - \mu \varepsilon_{b_1} \left(\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)(s)\right] \sqrt{\gamma^2 + 1}$$

where $\eta = \mp 1$. Then we can write

$$s^* = \varphi(s) = \int_0^s \left\| \frac{d\alpha^*(t)}{dt} \right\| dt\ ,\ (\forall s \in I)$$

where $\varphi : I \longrightarrow I^*$ is a regular C^∞-function, and we obtain

$$\varphi'(s) = \eta \left[\lambda \varepsilon_n \tau(s) - \mu \varepsilon_{b_1} \left(\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)(s)\right] \sqrt{\gamma^2 + 1}$$

for all $s \in I$. Thus α^* can be writen as

$$\alpha^*(s^*) = \alpha^*(\varphi(s)) = \alpha(s) + \lambda(s)N(s) + \mu(s)B_2(s) \tag{3.21}$$

for all $s \in I$. Differentiating in equation (3.21) with respect to s and using the Frenet equations, we obtain

$$\varphi'(s)\frac{d\alpha^*(s^*)}{ds^*} = \left[\lambda \varepsilon_n \tau(s) - \mu \varepsilon_{b_1} \left(\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)(s)\right] \left[\gamma T(s) + B_1(s)\right]. \tag{3.22}$$

or

$$T^*(\varphi(s)) = \eta(\gamma^2 + 1)^{-\frac{1}{2}}(\gamma T(s) + B_1(s)) \tag{3.23}$$

for all $s \in I$. Differentiating in equation (3.23) with repect to s and using the Frenet frame equations, we obtain

$$\begin{gathered}\varphi'(s)\frac{T^*(\varphi(s))}{ds^*}\\ = \eta(\gamma^2+1)^{-\frac{1}{2}}\left[(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))N(s)+\varepsilon_n\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)B_2(s)\right]\end{gathered} \tag{3.24}$$

and

$$\begin{gathered}\varepsilon_N\kappa^*(\varphi(s)) = \left\|\frac{T^*(\varphi(s))}{ds^*}\right\|\\ = \frac{\sqrt{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))^2+(\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))^2}}{\varphi'(s)\sqrt{\gamma^2+1}}\end{gathered} \tag{3.25}$$

By the fact that $\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s) \neq 0$ for all $s \in I$, we obtain

$$\varepsilon_N\kappa^*(\varphi(s)) = \left\|\frac{T^*(\varphi(s))}{ds^*}\right\| > 0.$$

From Frenet equations for the curve $\alpha^*(s^*)$, we have

$$\frac{T^*(\varphi(s))}{ds^*} = \varepsilon_N\kappa^*(\varphi(s))N^*(\varphi(s))$$

Then we can write

$$\begin{gathered}N^*(\varphi(s))\\ = \frac{1}{\varepsilon_N\kappa^*(\varphi(s))}\frac{T^*(\varphi(s))}{ds^*}\\ = \frac{\eta(\gamma^2+1)^{-\frac{1}{2}}(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))N(s)+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)B_2(s)}{\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+(\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))^2}}\end{gathered}$$

for all $s \in I$. If we denote

$$\begin{aligned} m(s) &= \frac{\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s)}{\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+\left(\varepsilon_n\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right)^2}}\\ n(s) &= \frac{\varepsilon_n\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)}{\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+\left(\varepsilon_n\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right)^2}}\end{aligned}$$

we obtain

$$N^*(\varphi(s)) = m(s)N(s)+n(s)B_2(s). \tag{3.26}$$

and we can easily show that $m(s)$ and $n(s)$ are constant functions. So differentiating in equation (3.26) with respect to s and by using the Frenet equations, we have

$$\varphi'(s)\frac{N^*(\varphi(s))}{ds^*} = mN'(s)+nB_2'(s)$$

or

$$\begin{gathered}\frac{N^*(\varphi(s))}{ds^*} =\\ -\frac{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))\varepsilon_t\varepsilon_N\kappa(s)}{\eta\varphi'(s)\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+(\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))^2}}T(s)\\ +\frac{\varepsilon_n\tau(s)(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))-\varepsilon_n\varepsilon_{b_1}\left[(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)\right]^2}{\eta\varphi'(s)\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+(\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))^2}}B_1.\end{gathered}$$

for all $s \in I$. Also by using equations (3.23) and (3.25) we can write

$$\varepsilon_N \kappa^*(\varphi(s))T^*(\varphi(s)) = \frac{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+(\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))^2}{\eta\varphi'(s)\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}}(\gamma T(s)+B_1(s)). \tag{3.27}$$

and, then from the Frenet Equations for the curve $\alpha^*$and in equation (3.27), respectively we have

$$\frac{N^*(\varphi(s))}{ds^*} + \varepsilon_N\kappa^*(\varphi(s))T^*(\varphi(s)) = \varepsilon_n\tau^*(\varphi(s))B_1(\varphi(s))$$

$$\frac{N^*(\varphi(s))}{ds^*} + \varepsilon_t\varepsilon_N\kappa^*(\varphi(s))T^*(\varphi(s)) = \frac{P(s)}{R(s)}T(s) + \frac{Q(s)}{R(s)}B_1(s))$$

where we can easily show

$$P(s) = -\left[\left[\gamma^2-1\right]\varepsilon_t\varepsilon_N\kappa(s)\tau(s) + \gamma\left[\begin{array}{c}(\varepsilon_N\kappa(s))^2-(\varepsilon_t\tau(s))^2\\ -\left(\varepsilon_n\left[\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]\right)^2\end{array}\right]\right]$$

$$Q(s) = \gamma\left[\left[\gamma^2-1\right]\varepsilon_t\varepsilon_N\kappa(s)\tau(s) + \gamma\left[\begin{array}{c}(\varepsilon_N\kappa(s))^2-(\varepsilon_t\tau(s))^2\\ -\left(\varepsilon_n\left[\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)\right]\right)^2\end{array}\right]\right]$$

$$R(s) = \eta\varphi'(s)\left(\gamma^2+1\right)\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t k(s))^2+\varepsilon_n\left(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)(s)^2} \neq 0.$$

Thus we obtain

$$\begin{aligned}\left\|\frac{N^*(\varphi(s))}{ds^*} + \varepsilon_N\kappa^*(\varphi(s))T^*(\varphi(s))\right\| &= \|\varepsilon_n\tau^*(\varphi(s))B_1(\varphi(s))\| \\ &= \frac{1}{R(s)}\sqrt{P^2(s)+Q^2(s)}.\end{aligned}$$

Then

$$\begin{array}{c}\tau^*(\varphi(s)) \\ = \frac{1}{\varepsilon_n R(s)}\sqrt{P^2(s)+Q^2(s)} \\ = \frac{\left[\gamma^2-1\right]\varepsilon_t\varepsilon_N\kappa(s)\tau(s)+\gamma\left[\varepsilon_N^2\kappa(s)^2-\varepsilon_t^2\tau^2-(\varepsilon_n[(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)])^2\right]}{\eta\varepsilon_n\varphi'(s)(\gamma^2+1)\sqrt{(\gamma\varepsilon_n\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}}\end{array}$$

for all $s \in I$. Thus we can define a unit vector fieds $B_1^*(s^*)$ along α^* by

$$\begin{array}{c}B_1^*(s^*) = B_1^*(\varphi(s)) = \\ \frac{1}{\varepsilon_n\tau^*(\varphi(s))}\frac{1}{\eta}\frac{1}{\sqrt{\gamma^2+1}}\frac{N^*(\varphi(s))}{ds^*}(-T(s)+\gamma B_1(s)) \\ +\frac{1}{\varepsilon_n\tau^*(\varphi(s))}\frac{1}{\eta}\frac{1}{\sqrt{\gamma^2+1}}\varepsilon_t\varepsilon_N\kappa^*(\varphi(s))T^*(\varphi(s))(-T(s)+\gamma B_1(s))\end{array}$$

for all $s \in I$. Also we can define a unit vector fieds $B_2^*(s^*)$ along α^* by

$$\begin{array}{c}B_2^*(s^*) = B_2^*(\varphi(s)) = \\ -\frac{\{(\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))\}N(s)}{\eta\sqrt{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}} \\ -\frac{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))B_2(s)}{\eta\sqrt{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}}\end{array}$$

that is

$$B_2^*(\varphi(s)) = -n(s)N(s) + m(s)B_1(s)$$

for all $s \in I$. Now we obtain by (24), (27), (29) and (30),

$$\det(T^*(\varphi(s)),\ N^*(\varphi(s)), B_1^*(\varphi(s)), B_2^*(\varphi(s))) = 1,$$

and $\{T^*(\varphi(s)),\ N^*(\varphi(s)), B_1^*(\varphi(s)), B_2^*(\varphi(s))\}$ is orthonormal for all $s \in I$. Thus $\{T^*(\varphi(s)),\ N^*(\varphi(s)), B_1^*(\varphi(s)), B_2^*(\varphi(s))\}$ the Frenet frame along α^* in E_2^4 is of orthonormal . And And we have

$$Span\{N,\ B_2\} = Span\{N^*,\ B_2^*\}$$

where $(N,\ B_2)$ normal plane of α and $\{N^*,\ B_2^*\}$ normal plane of α^*. Consequently, α is a quaternionic $\{N,\ B_2\}$ Bertrand curve in E_2^4.

Theorem 3.5. *Let α be a quaternionic $\{N,\ B_2\}$ Bertrand curve , α^* be a quaternionic $\{N,\ B_2\}$ Bertrand mate of α in E_2^4. And $\varphi : I \longrightarrow I^*, s^* = \varphi(s)$ is a regular C^∞-function such that each points $\alpha(s)$ of α correspond to the points $\alpha^*(s^*) = \alpha^*(\varphi(s))$ of α^* for all $s \in I$. Then the distance between the points $\alpha(s)$ and $\alpha^*(s^*)$ is constant for all $s \in I$.*

□

Proof. Let α be a quaternionic $\{N,\ B_2\}$ Bertrand curve in E_2^4.and α^* be a quaternionic $\{N,\ B_2\}$ Bertrand mate of α. We assume that α^* is distinct from α. Let the pair of $\alpha(s)$,and $\alpha^*(s^*) = \alpha^*(\varphi(s))$,Then we can write,

$$\alpha^*(s^*) = \alpha^*(\varphi(s)) = \alpha(s) + \lambda(s)N(s) + \mu(s)B_2(s)$$

where λ and μ are non-zero constants. Thus, we can rewrite

$$\alpha^*(s^*) - \alpha(s) = \lambda(s)N(s) + \mu(s)B_2(s)$$

and

$$\|\alpha^*(s^*) - \alpha(s)\| = \sqrt{\lambda^2 + \mu^2}.$$

Since, $d(\alpha^*(s^*) - \alpha(s)) = \text{constant}$. □

Corollary 3.6. *Let α be a quaternionic $\{N,\ B_2\}$ -Bertrand curve in E_2^4 with curvature functions $\kappa(s),\ \tau(s),\ (\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)$ and α^* be a quaternionic $\{N,\ B_2\}$ Bertrand mate of α with curvature functions $\kappa^*(s),\ \tau^*(s),\ (\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)^*(s)$. Then the relations between these curvature functions are*

$$\kappa^*(\varphi(s)) = \frac{\sqrt{(\gamma\varepsilon_N\kappa(s) - \varepsilon_t\tau(s))^2 + (\varepsilon_n(\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s))^2}}{\varepsilon_N\varphi'(s)\sqrt{\gamma^2+1}}$$

$$\tau^*(\varphi(s)) = \frac{\left[\gamma^2-1\right]\varepsilon_t\varepsilon_N\kappa(s)\tau(s)}{\varepsilon_n\varphi'(s)(\gamma^2+1)\sqrt{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}} + \frac{\gamma\left[(\varepsilon_N\kappa(s))^2-(\varepsilon_t k(s))^2-(\varepsilon_n[(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)])^2\right]}{\varepsilon_n\varphi'(s)(\gamma^2+1)\sqrt{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}}$$

$$(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)^*(\varphi(s)) = \frac{\varepsilon_n\left[(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)\right]\varepsilon_N\sqrt{\gamma^2+1}}{\sqrt{(\gamma\varepsilon_N\kappa(s)-\varepsilon_t\tau(s))^2+\varepsilon_n(\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)(s)^2}}.$$

Proof. It is obvious the proof of theorem 2. □

Example 3.7. Consider a quaternionic curve in E_2^4 defined by

$$\alpha(s): I\subset R\longrightarrow E_2^4,$$
$$\alpha(s)=\tfrac{1}{\sqrt{3}}(\sinh 2s,\ cosh s, \cos h 2s,\ \sinh s)$$

for all $s\in I$. α is regular curve and s is the arc-length parameter of α and its curvature functions are given as

$$\kappa(s)=\varepsilon_N\ \left\|\alpha''(s)\right\|=\sqrt{5}, \tau=\varepsilon_t\varepsilon_N\frac{2}{\sqrt{5}} \text{ and } \sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa=\varepsilon_N\frac{2}{\sqrt{5}}.$$

where

$$\lambda=\frac{\sqrt{5}}{\varepsilon_N\varepsilon_t},\ \mu=-\frac{\sqrt{5}}{\varepsilon_n},\ \gamma=-\frac{1}{\varepsilon_n} \text{ and } \delta=-\frac{7}{\varepsilon_n\varepsilon_N}$$

constants are for all $s\in I$. The curvature of quaternionic curve α satisfies the relations *(i), (ii), (iii), (iv).* So α is a quaternionic $\{N,\ B_2\}$-Bertrand curve and we obtain its quaternionic $\{N, B_2\}$ Bertrand mate curve of α^* as follows

$$\alpha^*(s^*)=\frac{1}{\sqrt{3}}(4\sinh 2s,\ 2cosh s, 4\cos h 2s,\ 2\sinh s).$$

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