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Impulsive integrodifferential Equations and Measure of noncompactness

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Abstract. This paper is concerned with the existence of mild solutions for impulsive integro-differential equations with nonlocal conditions. We apply the technique measure of noncompactness in the space of piecewise continuous functions and by using Darbo-Sadovskii's fixed point theorem, we prove reasults about impulsive integro-differential equations for convex-power condensing operators.
Keywords: impulsive integrodifferential equations; nonlocal conditions; Hausdorff measure of noncompactness; fixed point theorem; convex-power condensing map.

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1. Introduction

In this paper, we discuss the existence of mild solutions for the following impulsive integrodifferential equation with nonlocal conditions:

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t), \int_0^t k(t,s,u(s))ds) \qquad t \in J = [0,b], \qquad t \neq t_i \\ u(0) &= g(u) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.1) \\ \Delta u(t_i) &= I_i(u(t_i)) \qquad i = 1,2,...,s \end{aligned}$$

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Where $A : D(A) \subseteq X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$ in a Banach space X.
$0 = t_0 < t_1 < t_2 < ... < t_s < t_{s+1} = b$,
$\Delta u(t_i) = u(t_i^+) - u(t_i^-)$,
$u(t_i^+), u(t_i^-)$
denote the right and left limit of u at t_i respectively, f, g, k, I_i are appropriate functions to be specified later.

Impulsive integrodifferential equations are recognized as excellent models to study the evolution processes that are subject to sudden changes in their states; see the monographs of Lakshmikantham*et al.* [21], Benchohra *et al.* [7].
In recent years impulsive differential equations in Banach spaces have been investigated by many authors; see [8,17,18,19,20,22] and references therein. Liu [24] discussed the existence and uniqueness of mild solutions for a semilinear impulsive Cauchy problem with Lipschitz impulsive functions.
Non-Lipschitzian impulsive equations are considered by Nieto *et al.* [26]. Cardinali and Rubbioni [12] proved the existence of mild solutions for the impulsive Cauchy problem controlled by a semilinear evolution differential inclusion.
In [1], Abada *et al.* studied the existence of integral solutions for some nondensely defined impulsive semilinear functional differential inclusions.
On the other hand, the study of abstract nonlocal initial value problems was initiated by Byszewski, and the importance of the problem consists in the fact that it is more general and has better effect than the classical initial conditions $u(0) = u_0$ alone. Therefore, it has been studied extensively under various conditions. Here we mention some results. Byszewski and Lakshmikantham [9], Byszewski [10] obtained the existence and uniqueness of mild solutions and classical solutions in the case that Lipschitz-type conditions are satisfied.
In [16], Fu and Ezzinbi studied neutral functional-differential equations with nonlocal conditions. Aizicovici [3], Xue [30,31] discussed the case when A generates a nonlinear contraction semigroup on X and obtained the existence of integral solutions for nonlinear differential equations. In particular, the measure of noncompactness has been used as an important tool to deal with some similar functional differential and integral equations; see [2,4,11,13,31].

From the viewpoint of theory and practice, it is natural for mathematics to combine impulsive conditions and nonlocal conditions. Recently, the nonlocal impulsive differential problem of type following:

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)) \qquad t \in J = [0, b], \qquad t \neq t_i \\ u(0) &= g(u) \\ \Delta u(t_i) &= I_i(u(t_i)) \qquad i = 1, 2, ..., s \end{aligned} \tag{1.2}$$

has been discussed in the papers of Liang *et al.* [23] and Fan *et al.* [14,15], where a semigroup $T(t)$ is supposed to be compact, and g is Lipschitz continuous, compact, and strongly continuous, respectively. Very recently, Zhu *et al.* [32] obtain the existence results when a nonlocal item g is Lipschitz continuous by using the Hausdorff measure of noncompactness and operator transformation. Compared with the results in [14,15,23], in this paper we do not require the compactness of the semigroup $T(t)$ and Lipschitz continuity of f.
More important, by using the property of the measure of noncompactness in $PC([0, b]; X)$ given by us (see Lemma 1.5[28]), the impulsive conditions and nonlocal conditions can be considered in a unified way under various conditions, including compactness conditions, Lipschitz conditions and mixed-type conditions. Hence, our results generalize and partially improve the results in [15,23,28,32].
In the sequel of this section,we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $(X, \| . \|)$ be a real Banach space. We denote by $C([0, b] : X)$ the space of X- valued continuous functions on $[0, b]$ with the norm $\| x \| = sup\{\|x(t)\|, t \in [0, b]\}$ and by $L^1([0, b] : X)$ the space of X- valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| dt$. The semigroup $T(t)$ is said to be equicontinuous if $\{T(t)x : x \in B\}$ is equicontinuous at $t > 0$ for any bounded subset $B \subset X$ (cf.[6]). Through out this paper, we suppose that

(HA) The semigroup $\{T(t) : t \geq 0\}$ generated by A is equicontinuous. Moreover,
there exists a positive number M such that $M = sup_{0 \leq t \leq b} \| T(t) \|$.
For the sake of simplicity, we put $J = [0, b]; J_0 = [0, t_1]; J_i = (t_i, t_{i+1}], i = 1, ..., s$. In order to define a mild solution of the problem (1.1), we introduce the set
$PC([0, b]; X) = \{u : [0, b] \longrightarrow X : u$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$, and the right limit $u(t_i^+)$ exists, $i = 1, ..., s\}$. It is easy to verify that $PC([0, b]; X)$ is a Banach space with the norm $\|u\|_{PC} = \{sup\|u(t)\|, t \in [0, b]\}$.

Definition 1.1. A function $u \in PC([0,b];X)$ is a mild solution of the problem (1.1) if

$$u(t) = T(t)g(u) + \int_0^t T(t-s)f(s,u(s),\int_0^s k(s,\tau,u(\tau))d\tau)ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))$$

for all $t \in [0,b]$.
Now, we introduce the Hausdorff measure of noncompactness (in short MNC) $\beta(.)$ defined by

$$\beta(B) = inf\{\varepsilon > 0; \text{B has a finite } \varepsilon - \text{net in X}\},$$

for each bounded subset B in a Banach space X. We recall the following properties of the Hausdorff measure of noncompactness β.

Proposition 1.2. *([5]) Let X be a real Banach space and $B, C \subseteq X$ be bounded. Then the following properties are satisfied:*
(1) B is relatively compact iff $\beta(B) = 0$;
(2) $\beta(B) = \beta(\overline{B}) = \beta(convB)$, where $\overline{B}$ and $convB$ mean the closure and convex hull of B , respectively;
(3) $\beta(B) \leq \beta(C)$ when $B \subseteq C$;
(4) $\beta(B + C) \leq \beta(B) + \beta(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;
(5) $\beta(B \bigcup C) \leq max\{\beta(B), \beta(C)\}$;
(6) $\beta(\lambda B) \leq \mid \lambda \mid \beta(B)$ for any $\lambda \in \mathbb{R}$.
(7) If the map $Q : D(Q) \subseteq X \longrightarrow Z$ is Lipschitz continuous with a constant k, then $\beta_z(QB) \leq k\beta(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space.

The map $Q : D \subseteq X \longrightarrow X$ is said to be β - condensing if Q is continuous and bounded , and for any nonprecompact bounded subset $B \subset D$, we have $\beta(QB) < \beta(B)$, where X is a Banach space.

Theorem 1.3. *(see[5], Darbo-Sadovskii) If $D \subset X$ is bounded, closed, and convex, the continuous map $Q : D \longrightarrow D$ is β - condensing, then Q has at least one fixed point in D.*

In order to remove the strong restriction on the coefficient in Darbo-Sadovskii's fixed point theorem, Sun and Zhang [29] generalized the the definition of a β-condensing operator.
At first,we give some notation:

Let $D \subset X$ be closed and convex, the map $Q : D \longrightarrow D$ and $x_0 \in D$ for every $B \subset D$, set

$$Q^{(1,x_0)}(B) = Q(B), \qquad Q^{(n,x_0)}(B) = Q(\overline{conv}Q^{(n-1,x_0)}B, x_0)$$

where $\overline{conv}$ *means the closure of convex hull,* $n = 2, 3, ...$

J.Sun and X.Zhang introduce the operator notion of convex-power condensing in [29].

Definition 1.4. ([29]) Let $D \subset X$ be closed and convex. The map $Q : D \longrightarrow D$ is said to be β - convex - power condensing. if Q is continuous, bounded and there exist $x_0 \in D, n_0 \in \mathbb{N}$ such that for every noncompact bounded subset $B \subset D$, we have

$$\beta(Q^{(n_0,x_0)}(B)) < \beta(B).$$

Obviously, if $n_0 = 1$, then a β - convex-power condensing operator is β-condensing. Thus, the convex-power condensing operator is a generalization of the condensing operator.

Theorem 1.5. ([29]) *If* $D \subset X$ *is bounded, closed, and convex, the continuous map* $Q : D \longrightarrow D$ *is* β *- convex -power condensing, then* Q *has at least one fixed point in* D*.*

Lemma 1.6. ([5]) *If* $W \subseteq C([0, b]; X)$ *is bounded, then* $\beta(W(t)) \leq \beta(W)$ *for all* $t \in [0, b]$*, where* $W(t) = \{u(t) : u \in W\} \subseteq X$*. Furthermore, if* W *is equicontinuous on* $[0, b]$*, then* $\beta(W(t))$ *is continuous on* $[0, b]$ *where* $\beta(W) = sup\{\beta(W(t)), t \in [0, b]\}$*.*

Shaochun and Gang [28] by applying Lemma 1.4, extended the result to the space $PC([0, b], X)$.

Lemma 1.7. ([28]) *If* $W \subseteq PC([0, b]; X)$ *is bounded, then*
$\beta(W(t)) \leq \beta(W)$ *for all* $t \in [0, b]$*, where* $W(t) = \{u(t); u \in W\} \subseteq X$*.*
Furthermore, suppose the following conditions are satisfied:
(1) W *is equicontinuous on* $j_0 = [0, t_1]$ *and each*
$J_i = (t_i, t_{i+1}], i = 1, ..., s$*;*
(2) W *is equicontinuous at* $t = t_i^+, i = 1, ..., s$*.*
Then $sup_{t\in[0,b]}\beta(W(t)) = \beta(W)$*.*

Lemma 1.8. ([5]) *If* $W \subset C([0, b]; X)$ *is bounded and equicontinuous, then* $\beta(W(t))$ *is continuous and*

$$\beta(\int_0^t W(s)ds) \leq \int_0^t \beta(W(s))ds,$$

for all $t \in [0, b]$, *where* $\int_0^t W(s)ds = \{\int_0^t x(s)ds : x \in W\}$.

Lemma 1.9. ([28]) *If the hypothesis (HA) is satisfied, i.e,* $\{T(t) : t \geq 0\}$ *is equicontinuous, and* $\eta \in L^1([0, b]; R^+)$*, then the set* $\{\int_0^t T(t-s)u(s)ds : \| u(s) \| \leq \eta(s) \quad$ *for a.e ;* $s \in [0, b]\}$ *is equicontinuous for* $t \in [0, b]$.

2. Main results

In this section we give the existence results for the problem (1.1) under different conditions on g and I_i when the semigroup is not compact and f is not compact or Lipschitz continuous, by using Lemma1.5 and the generalized β-condensing operator. More precisely, Theorem 2.1 is concerned with the case that compactness conditions are satisfied.
Theorem 2.2 deal with the case Lipschitz conditions are satisfied. And mixed-type conditions are considered in Theorem 2.3 and Theorem 2.4
Let r be a finite positive constant, and set
$B_r = \{x \in X :\| x \| \leq r\}, W_r = \{u \in PC([0, b]; X) : u(t) \in B_r, t \in [0, b]\}$.
We define the solution map $G : PC([0, b]; X) \longrightarrow PC([0, b]; X)$ by

$$(Gu)(t) = T(t)g(u) + \int_0^t T(t-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)) \quad (2.1)$$

$$\begin{aligned}(G_1u)(t) &= T(t)g(u) \\ (G_2u)(t) &= \int_0^t T(t-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds \\ (G_3u)(t) &= \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))\end{aligned}$$

for all $t \in [0, b]$. It is easy to see that u mild solution of the problem (1.1) if and only if u is a fixed point of the map G. In this section, by use the problem (1.1) have shown research about β-convex-power condensing operator [28], then we continue on basis f define as 3-arguments function.
We list the following hypotheses:
(Hf) $f : [0, b] \times X \times X \longrightarrow X$ satisfies the following conditions:
(i) $f(t, ., .) : X \times X \longrightarrow X$ is continuous for a.e $t \in [0, b]$ and
$f(., x, y) : [0, b] \longrightarrow X$ is measurable for all $x, y \in X$.
(ii)there exists a constant$L > 0$ such that
$\| f(t, x, y) \| \leq L(\| x \| + \| y \|); x, y \in X$.
(iii) there exists a constant $L' > 0$ such that for any bounded subsets $D_1, D_2 \subset X, \beta(f(t, D_1, D_2)) \leq L'(\beta(D_1) + \beta(D_2))$ for$a.e$ $t \in [0, b]$.

(Hk1) $k : [0,b] \times [0,b] \times X \longrightarrow X$ satisfies the following conditions:
(i) $k(t,s,.) : X \longrightarrow X$ is continuous for every $t, s \in [0,b]$ and
$k(.,.,x) : [0,b] \times [0,b] \longrightarrow X$ is measurable for all $x \in X$.
(ii)there exists a constant$N_0 > 0$ such that $\| k(t,s,x) \| \leq N_0 \| x \|$ for every $x \in X; t, s \in [0,b]$.
(iii) there exists a constant $L'' > 0$ such that for any bounded subset $V \subset PC([0,b];X), \beta(k(t,s,V)) \leq L''\beta(V)$ for $t, s \in [0,b]$.
(Hg1)$g : PC([0,b];X) \longrightarrow X$ continuous and compact.
(HI1)$I_i : X \longrightarrow X$ is continuous and compact for $i = 1, ..., s$.

Theorem 2.1. *Assume that the hypotheses (HA),(Hf),(Hk1),(Hg1) and (HI1) are satisfied, then the nonlocal impulsive problem (1.1) has at least one mild solution on $[0,b]$ provied that there exists a constant $r > 0$ such that*

$$M[sup_{u \in W_r} \| g(u) \| + rLb(1 + \frac{N_0 b}{2}) + sup_{u \in W_r} \sum_{i=1}^{s} \| I_i(u(t_i)) \|] < r$$

and

$$(L' + L'')Mb \leq \frac{1}{2}. \qquad (2.2)$$

Proof. We can use from the fixed point theorem about the β-convex-power condensing operator to show that the solution map G has a fixed point.

Step 1:

We show that the map G is continuous on $PC([0,b];X)$. Also, let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $PC([0,b];X)$ with $lim_{n \longrightarrow \infty} u_n = u$ in $PC([0,b];X)$. By the continuity of f with respect to the second and third arguments and so by the continuity of k with respect to the third argument. We deduce that for each $s, \tau \in [0,b] f(s, u_n(s), \int_0^s k(s,\tau,u_n(\tau))d\tau)$ converges to $f(s,u(s), \int_0^s k(s,\tau,u(\tau))d\tau)$ in $X \times X$. And we have

$$\| Gu_n - Gu \| \leq M[\| g(u_n) - g(u) \| + \sum_{i=1}^{s} \| I_i(u_n(t_i)) - I_i(u(t_i)) \|]$$

$$+ M \int_0^b \| f(s, u_n(s), \int_0^s k(s,\tau,u_n(\tau))d\tau) - f(s,u(s), \int_0^s k(s,\tau,u(\tau))d\tau) \| ds.$$

By the continuity of f with respect to the second and third arguments and by continuity of g, I_i and using the dominated convergence theorem, we get

$$lim_{n\longrightarrow\infty}Gu_n = Gu \qquad \text{in} \qquad PC([0,b];X)$$

Step2:
We show that $GW_r \subseteq W_r$. In fact, for any $u \in W_r \subseteq PC([0,b];X)$, from (2.1) and (2.2), we have

$$\| (Gu)(t) \| \leq \| T(t)g(u) \| + \| \int_0^t T(t-s)f(s,u(s),\int_0^s k(s,\tau,u(\tau))d\tau)ds \| + \sum_{0<t_i<t} \| T(t-t_i)I_i(u(t_i)) \|$$

$$\leq M[\| g(u) \| + \int_0^t \| f(s,u(s),\int_0^s k(s,\tau,u(\tau))d\tau \| ds + \sum_{0<t_i<t} \| I_i(u(t_i)) \|]$$

$$\leq M[\| g(u) \| + \int_0^t L(\| u(s) \| + \| \int_0^s k(s,\tau,u(\tau))d\tau \|)ds + \sum_{0<t_i<t} \| I_i(u(t_i)) \|]$$

$$\leq M[\| g(u) \| + \int_0^t L(r + \int_0^s \| k(s,\tau,u(\tau)) \| d\tau)ds + \sum_{0<t_i<t} \| I_i(u(t_i)) \|]$$

$$\leq M[\| g(u) \| + \int_0^t L(r + \int_0^s N_0 \| u(\tau) \| d\tau)ds + \sum_{0<t_i<t} \| I_i(u(t_i)) \|]$$

$$\leq M[\| g(u) \| + \int_0^t L(r + N_0 rs)ds + \sum_{0<t_i<t} \| I_i(u(t_i)) \|]$$

$$\leq M[\| g(u) \| + rL(t + N_0\frac{t^2}{2}) + \sum_{0<t_i<t} \| I_i(u(t_i)) \|]$$

$$\leq M[\| g(u) \| + rLb(1 + N_0\frac{b}{2}) + \sum_{0<t_i<t} \| I_i(u(t_i)) \|] \leq r$$

for each $t \in [0,b]$. It implies that $GW_r \subseteq W_r$.

Step 3:
We prove that GW_r is equicontinuous on $J_0 = [0,t_1], J_i = (t_i,t_{i+1}]$ and is also equicontinuous at $t = t_i^+, i = 1,...,s$. In continue, we show that GW_r is equicontinuous on $[t_1,t_2]$, as the cases for other subintervals are the same. For $u \in W_r, t_1 \leq s \leq t \leq t_2$, we have, using the semigroup property,

$$\begin{aligned}\| T(t)g(u) - T(s)g(u) \| &= \| T(t-s)g(u) \| \\ &= \| T(s)(T(t-s) - T(0))g(u) \| \\ &\leq M \| (T(t-s) - T(0))g(u) \|\end{aligned}$$

Thus, G_1W_r is equicontinuous on $[t_1,t_2]$ due to the compactness of g and the strong continuity of $T(.)$. The same idea can be used to prove the equicontinuity of G_3W_r on $[t_1, t_2]$. That is, for $u \in W_r$, $t_1 \leq s < t \leq t_2$, we have

$$\| T(t-t_1)I_1(u(t_1)) - T(s-t_1)I_1(u(t_1)) \| \leq M \| [T(t-s) - T(0)]I_1(u(t_1)) \|$$

which implies the equicontinuity of G_3W_r on $[t_1, t_2]$ due to the compactness of I_1 and the strong continuity of $T(.)$. Moreover, from Lemma 1.7, we have that G_2W_r is equicontinuous on $[0, b]$.Therefore, we have that the functions in $GW_r = (G_1 + G_2 + G_3)W_r$ are equicontinuous on each $[t_i, t_{i+1}], i = 0, 1, ..., s$.

Step 4:
Let $W = \overline{conv}G(W_r)$, where $\overline{conv}$ means the closure of convex hull. Obviously, that G maps W into itself and W is equicontinuous on each $\overline{J_i} = [t_i, t_{i+1}], i = 0, 1, ..., s$. Now, we show that $G : W \longrightarrow W$ is a convex-power condensing operator. Take $x_0 \in W$, we shall prove that there exists a positive integer n_0 such that

$$\beta(G^{(n_0,x_0)}(B)) < \beta(B),$$

for every nonprecompact bounded subset $B \subset W$. From proposition 1.1 and Lemma 1.6 noticing the compactness of g and I_i, we have

$$\begin{aligned}
\beta((G^{(1,x_0)})B(t)) &= \beta((GB)(t)) \\
&\leq \beta(T(t)g(B)) + \beta(\int_0^t T(t-s)f(s, B(s), \int_0^s k(s, \tau, B(\tau))d\tau)ds) \\
&+ \beta(\sum_{0<t_i<t} T(t-t_i)I_i(B(t_i))) \\
&\leq \int_0^t \beta(T(t-s)f(s, B(s), \int_0^s k(s, \tau, B(\tau))d\tau))ds \\
&\leq M \int_0^t \beta(f(s, B(s), \int_0^s k(s, \tau, B(\tau))d\tau))ds \\
&\leq M \int_0^t (L'(\beta(B(s)) + \beta(\int_0^s k(s, \tau, B(\tau))d\tau))ds \\
&\leq M(\int_0^t L'\beta(B(s))ds + L' \int_0^t \int_0^s \beta(k(s, \tau, B(\tau)))d\tau ds) \\
&\leq ML't\beta(B) + ML' \int_0^t \int_0^s L''\beta(B(\tau))d\tau ds \\
&\leq ML't\beta(B) + ML' \int_0^t sL''\beta(B(s))ds \\
&\leq ML't\beta(B) + ML'L''\frac{t^2}{2!}\beta(B) \leq ML'(t + \frac{L''t^2}{2!})\beta(B)
\end{aligned}$$

for $t \in [0, b]$. Further,

$$\beta((G^{(2,x_0)}B)(t)) = \beta((G\overline{conv}\{G^{(1,x_0)}B, x_0\})(t))$$

$$\begin{aligned}
&\leq \beta(T(t)g(\overline{conv}\{G^{(1,x_0)}B, x_0\})) \\
&+ \beta(\int_0^t T(t-s)f(s, \overline{conv}\{G^{(1,x_0)}B(s), x_0(s)\}, \int_0^s k(s,\tau, \overline{conv}\{G^{(1,x_0)}B(\tau), x_0(\tau)\})d\tau)ds) \\
&+ \beta(\sum_{0<t_i<t} T(t-t_i)I_i(\overline{conv}\{G^{(1,x_0)}B(t_i), x_0(t_i)\})) \\
&\leq \beta(\int_0^t T(t-s)f(s, \overline{conv}\{G^{(1,x_0)}B(s), x_0(s)\}, \int_0^s k(s,\tau, \overline{conv}\{G^{(1,x_0)}B(\tau), x_0(\tau)\})d\tau)ds) \\
&\leq M\int_0^t \beta(f(s, \overline{conv}\{G^{(1,x_0)}B(s), x_0(s)\}, \int_0^s k(s,\tau, \overline{conv}\{G^{(1,x_0)}B(\tau)\tau, x_0(\tau)\})d\tau)ds) \\
&\leq M\int_0^t L'(\beta(\overline{conv}\{G^{(1,x_0)}B(s), x_0(s)\}) + \beta(\int_0^s k(s,\tau, \overline{conv}\{G^{(1,x_0)}B(\tau), x_0(\tau)\})d\tau)ds) \\
&\leq ML'\int_0^t \beta((G^{(1,x_0)}B)(s))ds + ML'\int_0^t\int_0^s \beta(k(s,\tau, \overline{conv}\{G^{(1,x_0)}B(\tau), x_0(\tau)\}))d\tau ds \\
&\leq ML'(\int_0^t M(sL' + \frac{L'L''s^2}{2!})\beta(B(s))ds) + ML'\int_0^t\int_0^s L''\beta(\overline{conv}\{G^{(1,x_0)}B(\tau), x_0(\tau)\})d\tau ds \\
&\leq ML'(ML'\frac{t^2}{2!} + \frac{ML'L''t^3}{3!})\beta(B) + ML'\int_0^t\int_0^s L''\beta(G^{(1,x_0)}B(\tau))d\tau ds \\
&\leq M^2(\frac{L'^2t^2}{2!} + \frac{L'^2L''t^3}{3!})\beta(B) + ML'L''\int_0^t\int_0^s M(\tau L' + L'\frac{L''\tau^2}{2!})\beta(B)d\tau ds
\end{aligned}$$

for $t \in [0, b]$

$$M^2(\frac{L'^2t^2}{2!} + 2\frac{L'^2L''t^3}{3!} + \frac{L'^2L''^2t^4}{4!})\beta(B)$$

or

$$M^2(\binom{2}{2}\frac{L'^2t^2}{2!} + \binom{2}{1}\frac{L'^2L''t^3}{3!} + \binom{2}{0}\frac{L'^2L''^2t^4}{4!})\beta(B)$$

We can continue this iterative procedure and get that

$$\beta(G^{(n,x_0)}B)(t) \leq (\sum_{r=0}^{n}\binom{n}{r}\frac{L'^nL''^{n-r}b^n}{(2n-r)!}M^n)\beta(B)$$

for $t \in [0, b]$. As $G^{(n,x_0)}(B)$ is equicontinuous on each $[t_i, t_{i+1}]$, by Lemma 1.5, we that

$$\beta(G^{(n,x_0)}B) = sup_{t\in[0,b]}((G^{(n,x_0)}B)(t)) \leq (\sum_{r=0}^{n}\binom{n}{r}\frac{L'^nL''^{n-r}b^n}{(2n-r)!}M^n)\beta(B)$$

By the Weierstrass M-test and Geometric series can conclude,that $\sum_{r=0}^{n}\binom{n}{r}\dfrac{L'^{n}L''^{n-r}b^{n}}{(2n-r)!}M^{n}$ is convergent as $n \longrightarrow \infty$. Then, by(2.2)

$$\sum_{r=0}^{\infty}\frac{\binom{n}{r}L'L''^{n-r}M^{n}b^{n}}{(2n-r)!} \leq 1,$$

we know that there exists a large enough positive integral n_0 such that

$$\sum_{r=0}^{n_0}\frac{\binom{n}{r}L'^{n}L''^{n-r}M^{n}b^{n}}{(2n-r)!} < 1$$

which impulsive that $G : W \longrightarrow W$ is a convex-power condensing operator. From Theorem 1.3, G has at least one fixed point in W, which is just a mild solution of the nonlocal impulsive problem (1.1). □

Remark 2.2. By using the method of the measure of noncompactness, we require f to satisfy some proper conditions of MNC,but do not require the compactness of a semigroup $T(t)$.
Note that if f is compact or Lipschitz continuous, then the condition(Hf)(ii),(iii) are satisfied. And our work improves many previous results,where they need the compactness of $T(t)$ or f,or the Lipschitz continuity of f.
Lemma 1.5 plays an important role for the impulsive integrodifferential equations, which provides us with the way to calculate the measure of noncompactness in $PC([0, b]; X)$.
The use of noncompact measures in functional differential and integral equations can also be seen in[2,4,11,28,31].

Remark 2.3. When we apply Darbo-Sadovskii's fixed point theorem to get the fixed point of a map,a strong inequality is needed to guarantee its condensing property. By using the β-convex-power condensing operator developed by Sun *et al.* [29] This generalized condensing operator also can be seen in Liu *et al.* [25], where nonlinear Volterra integral equations are discussed.

In the following, by using Lemma 1.5 and Darbo-Sadovskii's fixed point theorem, we give the existence results of the problem (1.1) under Lipschitz conditions and mixed-type conditions, respectively.

We give the following hypotheses:
(Hg2) $g : PC([0, b]; X) \longrightarrow X$ is Lipschitz continuous with the Lipschitz constant k'.
(HI2) $I_i : X \longrightarrow X$ is Lipschitz continuous with the Lipschitz constant k_i; that is,

$$\| I_i(x) - I_i(y) \| \leq k_i \| x - y \|$$

for $x, y \in X, i = 1, ..., s$.

Theorem 2.4. *Assume that the hypotheses (HA),(Hf),(Hk1),(Hg2) and (HI2) are satisfied, then the nonlocal impulsive problem (1.1) has at least one mild solution on* $[0, b]$ *provided that*

$$M(k' + \sum_{i=1}^{s} k_i + L'b + L'L''\frac{b^2}{2}) < 1 \tag{2.3}$$

and (2.2) are satisfied.

Proof. By use the proof of Theorem 2.1, we get the solution operator G is continuous and maps W_r into itself. It remains to show that G is β-condensing in W_r. By the conditions (Hg2) and (HI2), we get that $G_1+G_3 : W_r \longrightarrow PC([0, b]; X)$ is Lipschitz continuous with the Lipschitz constant $M(k' + \sum_{i=1}^{s} k_i)$. In fact, for $u, v \in W_r$, we have

$$\begin{aligned} \| (G_1 + G_3)u - (G_1 + G_3)(v) \|_{PC} &= sup_{t\in[0,b]}[\| T(t)(g(u) - g(v)) \| \\ &\quad + \sum_{0<t_i<t} \| T(t - t_i)(I_i(u(t_i)) - I_i(v(t_i))) \|] \\ &\leq M(\| g(u) - g(v) \| + \sum_{i=1}^{s} \| I_i(u(t_i)) - I_i(v(t_i)) \|) \\ &\leq M(k' + \sum_{i=1}^{s} k_i) \| u - v \|_{PC} \end{aligned}$$

which implies that

$$\| (G_1 + G_3)u - (G_1 + G_3)v \|_{PC} \leq M(k' + \sum_{i=1}^{s} k_i) \| u - v \|_{PC}$$

Thus, from proposition 1.1(7), we obtain that

$$\beta((G_1 + G_3)W_r) \leq M(k' + \sum_{i=1}^{s})k_i)\beta(W_r). \tag{2.4}$$

For the operator $(G_2u)(t) = \int_0^t T(t-s)f(s,u(s),\int_0^s k(s,\tau,u(\tau))d\tau)ds$, from Lemma 1.4, Lemma 1.6 and Lemma 1.7, we have

$$\begin{aligned}
\beta(G_2W_r) &= sup_{t\in[0,b]}\beta((G_2W_r)(t)) \\
&\le sup_{t\in[0,b]}\beta(\int_0^t T(t-s)f(s,W_r(s),\int_0^s k(s,\tau,W_r(\tau))d\tau)ds) \\
&\le sup_{t\in[0,b]}\int_0^t \beta(T(t-s)f(s,W_r(s),\int_0^s k(s,\tau,W_r(\tau))d\tau)ds) \\
&\le M sup_{t\in[0,b]}\int_0^t (L'(\beta(W_r(s)) + \beta(\int_0^s k(s,\tau,W_r(\tau))d\tau)))ds \\
&\le M sup_{t\in[0,b]}\int_0^t (L'(\beta(W_r(s)) + \int_0^s L''\beta(W_r(\tau))d\tau))ds \\
&\le ML'b\beta(W_r) + \int_0^t ML'\int_0^s L''\beta(W_r(\tau))d\tau ds \\
&\le ML'b\beta(W_r) + M\frac{b^2}{2}L'L''\beta(W_r) \\
&\le (ML'b + M\frac{b^2}{2}L'L'')\beta(W_r) \qquad (2.5)
\end{aligned}$$

Combining (2.4) and(2.5)

$$\begin{aligned}
\beta(GW_r) &\le \beta((G_1+G_3)W_r) + \beta(G_2W_r) \\
&\le M(k' + \sum_{i=1}^{s} k_i)\beta(W_r) + ML'(b + \frac{b^2}{2}L'')\beta(W_r) \\
&\le M(k' + \sum_{i=1}^{s} k_i + L'b + L'L''\frac{b^2}{2})\beta(W_r)
\end{aligned}$$

From the condition (2.3), $M(k' + \sum_{i=1}^{s} k_i + L'b + L'L''\frac{b^2}{2}) < 1$, the solution map G is β- condensing in W_r. By Darbo-Sadovskii's fixed point theorem, G has a fixed point in W_r, which is just a mild solution of the nonlocal impulsive problem (1.1). □

Among the previous works on nonlocal impulsive differential equations, few are concerned with the mixed-type conditions.
Here, by using Lemma 1.5, we can also deal with the mixed-type conditions in a similar way.

Theorem 2.5. *Assume that the hypotheses (HA),(Hf),(Hk1),(Hg1)and (HI2) are satisfied, then the nonlocal impulsive problem (1.1) has at least* *one mild solution on* $[0, b]$ *provided that*

$$M(L'b + \frac{b^2}{2}L'L'' + \sum_{i=1}^{s} k_i) < 1 \tag{2.6}$$

and (2.2) are satisfied.

Proof. By Darbo-Sadovskii's fixed point theorem ,we will obtain a fixed point of the solution operator G. By the proof of Theorem 2.1, we have that G is continuous and maps W_r into itself.
Subsequently, we show that G is β - condensing in W_r. From the compactness of g and the strong continuity of $T(.)$, we get that $\{T(.)g(u) : u \in W_r\}$ is equicontinuous on $[0, b]$. Then by Lemma 1.4, we have that

$$\beta(G_1 W_r) = sup_{t\in[0,b]}\beta((G_1 W_r)(t)) = sup_{t\in[0,b]}\beta(T(t)g(W_r)) = 0 \tag{2.7}$$

On the other hand, for $u, v \in W_r$, we have

$$\begin{aligned} \| G_3 u - G_3 v \|_{PC} &= sup_{t\in[0,b]} \| \sum_{0<t_i<t} T(t - t_i)(I_i(u(t_i)) - I_i(v(t_i))) \| \\ &\leq M \sum_{i=1}^{s} \| I_i(u(t_i)) - I_i(v(t_i)) \| \\ &\leq M \sum_{i=1}^{s} k_i \| u - v \|_{PC} \end{aligned}$$

Then by proposition 1.1(7), we obtain that

$$\beta(G_3 W_r) \leq M \sum_{i=1}^{s} k_i \beta(W_r) \tag{2.8}$$

combining (2.7), (2.9) and (2.10), we get that

$$\begin{aligned} \beta(GW_r) &\leq \beta(G_1 W_r) + \beta(G_2 W_r) + \beta(G_3 W_r) \\ &\leq (ML'b + M\frac{b^2}{2}L'L'')\beta(W_r) + M\sum_{i=1}^{s} k_i \beta(W_r) \\ &\leq M(L'b + \frac{b^2}{2}L'L'' + \sum_{i=1}^{s} k_i)\beta(W_r) \end{aligned}$$

From the condition (2.6), the map G is β-condensing in W_r. So, G has a fixed point in W_r due to Darbo-Sadovskii's fixed point theorem, which is just a mild solution of the nonlocal impulsive problem (1.1). □

Theorem 2.6. *Assume that the hypotheses* $(HA), (Hf), (Hk_1), (Hg_2), (HI_1)$ *are satisfied, then the nonlocal impulsive problem (1.1) has at least one* *mild solution on* $[0, b]$ *provided that*

$$M(k' + L'b + \frac{b^2}{2}L'L'') < 1 \tag{2.9}$$

and (2.2) are satisfied.

Proof. By the proof of Theorem 2.1, we have that the solution operator G is continuous and maps W_r into itself. In continue, we shall show that G is β- condensing in W_r. By the Lipschitiz continuity of g, we have that for $u, v \in W_r$,

$$\| G_1 u - G_1 v \|_{PC} = sup_{t\in[0,b]} \| T(t)[g(u) - g(v)] \| \leq Mk' \| u - v \|_{pc}$$

which impulsive that

$$\beta(G_1 W_r) \leq Mk'\beta(W_r). \tag{2.10}$$

Similar to the discussion in Theorem 2.1, from the compactness of I_i and the strong continuity $T(.)$, we get that $G_3 W_r$ is equicontinuous on each $\overline{J_i} = [t_i, t_{i+1}], i = 0, 1, 2, ..., s$. Then by Lemma 1.5, we have that

$$\beta(G_3 W_r) = sup_{t\in[0,b]}((G_3 W_r)(t)) \leq \sum_{i=1}^{s} \beta(T(t - t_i)I_i(W_r(t_i))) = 0 \tag{2.11}$$

combining (2.7),(2.10) and (2.11), we have that

$$\begin{aligned}\beta(GW_r) &\leq \beta(G_1 W_r) + \beta(G_2 W_r) + \beta(G_3 W_r)\\ &\leq Mk'\beta(W_r) + (ML'b + M\frac{b^2}{2}L'L'')\beta(W_r)\\ &\leq M(k' + L'b + \frac{b^2}{2}L'L'')\beta(W_r)\end{aligned}$$

From condition (2.9), the map G is β- condensing in W_r. So, G has a fixed point in W_r due to Darbo-Sadovskii's fixed point theorem, which is just a mild solution of the nonlocal impulsive problem (1.1). □

Remark 2.7. With the assumption of compactness on the associated semigroup, the existence of mild solution to functional differential equations has been discussed in[15,20,23,28]. By using the method of the measure of noncompactness, we deal with the four cases of impulsive integrodifferential equations in a unified way and get the existence results when the semigroup is not compact.

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