On the determination of eigenvalues for differential pencils with the turning point

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ABSTRACT. In this paper, we investigate the boundary value problem for differential pencils on the half-line with a turning point. Using a fundamental system of solutions, we give a asymptotic distribution of eigenvalues.

Keywords: Eigenvalues, Differential pencil, Turning point.

2010 Mathematics subject classification: 34L15, 34K10, 34E20.

1. INTRODUCTION

We consider the differential equation

$$y''(x) + (\rho^2 r(x) + i\rho q_1(x) + q_0(x))y(x) = 0 , \qquad x \geq 0, \qquad (1.1)$$

on the half-line with a nonlinear dependence on the spectral parameter ρ. Let $0 < a < 1$, and

$$r(x) = \begin{cases} -(x-1)^2, & 0 \leq x < a, \\ \\ 1, & x \geq a, \end{cases} \qquad (1.2)$$

i.e., the sign of the weight function changes in an interior point $x = a$, which is called the turning point. The functions $q_j(x)$, $j = 0, 1$, are

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1

complex-valued, $q_1(x)$ is absolutely continuous and $(1+x)q_j^{(l)} \in L(0, \infty)$ for $0 \le l \le j \le 1$.

Differential equations with turning points are widely used to describe many important phenomena and dynamic processes in physics, geophysics, mechanics (see [5,7] for details). The classical Sturm-Liouville operators with turning points in the finite interval have been studied fairly completely in [2]. Indefinite differential pencils with turning points produce significant qualitative modification in the investigation of the inverse problem. Some aspects of the inverse problem theory for differential pencils without turning points have been studied in [3,8]. Here we investigate a boundary value problem for differential pencils with a turning point. Similar problems for Sturm-Liouville operators have been studied in [9]. In this work, the weight function is a polynomial of degree two before the turning point.

In this paper, we will study the solution for Eq. (1.1). In Section 2, we determine the asymptotic form of the characteristic function and then give the eigenvalues.

2. Main result

We consider the boundary value problem L for Eq. (1.1) with the spectral boundary condition

$$U(y) := y'(0) + (\beta_1 \rho + \beta_0)y(0) = 0, \qquad (2.1)$$

where the coefficients β_1 and β_0 are complex numbers and $\beta_1 \ne \pm 1$.

Denote $\Pi_\pm := \{\rho : \pm Im\rho > 0\}$ and $\Pi_0 := \{\rho : Im\rho = 0\}$. By the well-known method (see [4,7]), we obtain a solution for Eq. (1.1) which is called the Jost-type solution.

Theorem 1. The Eq. (1.1) has a unique solution $y = e(x, \rho)$, $\rho \in \Pi_\pm$, $x \ge a$, with the following properties :

1. For each fixed $x \ge a$, the functions $e^{(\nu)}(x, \rho)$, $\nu = 0, 1$, are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$ (i.e., they are piecewise holomorphic).

2. The functions $e^{(\nu)}(x, \rho)$, $\nu = 0, 1$, are continuous for $x \ge a$, $\rho \in \overline{\Pi_+}$ and $\rho \in \overline{\Pi_-}$. In other words, for real ρ, there exist the finite limits

$$e_\pm^{(\nu)}(x, \rho) = \lim_{z \longrightarrow \rho, \ z \in \Pi_\pm} e^{(\nu)}(x, z).$$

Moreover, the functions $e^{(\nu)}(x, \rho), \nu = 0, 1$, are continuously differentiable with respect to $\rho \in \overline{\Pi_+} \setminus \{0\}$ and $\rho \in \overline{\Pi_-} \setminus \{0\}$.

3. For $x \longrightarrow \infty$, $\rho \in \overline{\Pi_\pm} \setminus \{0\}$, $\nu = 0, 1$,

$$e^{(v)}(x, \rho) = (\pm i\rho)^v \exp(\pm(i\rho x - Q(x)))(1 + o(1)), \qquad (2.2)$$

where $\quad Q(x) = \frac{1}{2}\int_0^x q_1(t)dt.$

4. For $|\rho| \longrightarrow \infty, \rho \in \overline{\Pi_\pm}$, $\nu = 0, 1$, uniformly in $x \geq a$,

$$e^{(\nu)}(x, \rho) = (\pm i\rho)^\nu \exp(\pm(i\rho x - Q(x)))[1], \qquad (2.3)$$

where $\quad [1] := 1 + O\left(\rho^{-1}\right).$

We extend $e(x, \rho)$ to the segment $[0, a]$ as a solution of Eq. (1.1) which is smooth for $x \geq 0$, i.e.,

$$e^{(\nu)}(a - 0, \rho) = e^{(\nu)}(a + 0, \rho), \quad \nu = 0, 1. \qquad (2.4)$$

Then the properties $1 - 2$ remain true for $x \geq 0$.

Let the function $\varphi(x, \rho)$ be the solution of Eq. (1.1) under the initial conditions $\varphi(0, \rho) = 1$ and $U(\varphi) = 0$. For each fixed $x \geq 0$, the functions $\varphi^{(\nu)}(x, \rho)$, $\nu = 0, 1$ are entire in ρ.

Denote

$$\Delta(\rho) := U(e(x, \rho)). \qquad (2.5)$$

The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ, there exist the finite limits

$$\Delta_\pm(\rho) = \lim_{z \longrightarrow \rho,\ z \in \Pi_\pm} \Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho \in \overline{\Pi_\pm} \setminus \{0\}$.

Definition 1. The values of the parameter ρ, for which the Eq. (1.1) has nontrivial solutions satisfying the conditions $U(y) = 0$, $y(\infty) = 0$ (i.e., $\lim_{x \longrightarrow \infty} y(x) = 0$) are called eigenvalues of L, and the corresponding solutions are called eigenfunctions.

Theorem 2. For $|\rho| \longrightarrow \infty, \rho \in \overline{\Pi_\pm}$, the following asymptotical formula holds:

$$\Delta(\rho) = \frac{\rho}{2(a - 1)} \exp(\pm(i\rho a - Q(a)))$$

$$\times \left((a \mp i)(1 + \beta_1) \exp\left(\frac{\rho}{2}(a^2 - 2a) - \frac{iQ(a)}{a - 1}\right)[1] \right.$$

$$\left. -(a \pm i)(1 - \beta_1) \exp\left(-\frac{\rho}{2}(a^2 - 2a) + \frac{iQ(a)}{a - 1}\right)[1] \right).$$

Proof. Denote $\Pi_\pm^1 := \{\rho : \pm Re\rho > 0\}$. It is known (see [4,7]) that for $x \in [0, a]$, $\nu = 0, 1$, $\rho \in \overline{\Pi_\pm^1}$, $|\rho| \longrightarrow \infty$, there exists the Birkhoff-type fundamental system of solutions $\{y_k(x, \rho)\}_{k=1,2}$ of Eq. (1.1) of the form

$$y_k^{(\nu)}(x, \rho) = ((-1)^k \rho(x - 1))^\nu \exp\left((-1)^k \left(\frac{\rho(x^2 - 2x)}{2} - \frac{iQ(x)}{x - 1}\right)\right)[1].$$

$$(2.6)$$

Using these solutions, one has

$$e^{(\nu)}(x,\rho) = h_1(\rho)y_1^{(\nu)}(x,\rho) + h_2(\rho)y_2^{(\nu)}(x,\rho), \quad x \in [0,a]. \quad (2.7)$$

Taking Cramers rule, we calculate

$$h_1(\rho) = \frac{a \pm i}{2(a-1)} \exp(\pm(i\rho a - Q(a))) \exp\left(\frac{-\rho}{2}(a^2 - 2a) + \frac{iQ(a)}{a-1}\right)[1],$$

$$h_2(\rho) = \frac{a \mp i}{2(a-1)} \exp(\pm(i\rho a - Q(a))) \exp\left(\frac{\rho}{2}(a^2 - 2a) - \frac{iQ(a)}{a-1}\right)[1].$$

Now, substituting (2.6) and coefficients $h_j(\rho)$, $j = 1, 2$ in (2.7), we have

$$\begin{aligned} e^{(\nu)}(x,\rho) &= \frac{(\rho(x-1))^{\nu}}{2(a-1)} \exp(\pm(i\rho a - Q(a))) \\ &\quad \times((-1)^{\nu}(a \mp i)\exp(k(x,\rho))[1] + (a \pm i)\exp(-k(x,\rho))[1]), \end{aligned}$$

where

$$k(x,\rho) = \frac{-\rho}{2}\left((x^2 - 2x) - (a^2 - 2a)\right) + i\left(\frac{Q(x)}{x-1} - \frac{Q(a)}{a-1}\right).$$

Together with (2.1) and (2.5), this yields the characteristic function. Theorem 2 is proved. $\diamondsuit$

Now we obtain the eigenvalues for the boundary value problem L.

Theorem 3. 1) For sufficiently large k, the function $\Delta(\rho)$ has simple zeros of the form

$$\rho_k = \frac{1}{a^2 - 2a}\left(2k\pi i + 2\frac{iQ(a)}{a-1} + \kappa_1 \pm \kappa_2\right) + O\left(k^{-1}\right), \quad (2.8)$$

where

$$\kappa_1 = ln\frac{1-\beta_1}{1+\beta_1}, \quad \kappa_2 = ln\frac{a+i}{a-i}.$$

2) For real $\rho \neq 0$, L has no eigenvalues.

3) Let $\Lambda' = \Lambda'_+ \cup \Lambda'_-$, where $\Lambda'_\pm = \{\rho \in \Pi_\pm; \Delta(\rho) = 0\}$. The set Λ' coincides with the set of all non-zero eigenvalues of L. For $\rho_k \in \Lambda'$, the functions $e(x, \rho_k)$ and $\varphi(x, \rho_k)$ are eigenfunctions and

$$e(x, \rho_k) = \gamma_k\varphi(x, \rho_k), \quad \gamma_k \neq 0. \quad (2.9)$$

Proof. Using characteristic function and Rouche's theorem [1], we obtain a countable set of the zeros for the function $\triangle(\rho)$ of the form (2.8). Let $\rho_0 \neq 0$ be real. Then the function $y(x, \rho_0) = c_1e_+(x, \rho_0) + c_2e_-(x, \rho_0)$ vanishes at infinity only if $c_1 = c_2 = 0$. Thus for real $\rho \neq 0$, BVP(L) has no eigenvalues. For prove part 3, let $\rho_k \in \Lambda'$. Therefore $\Delta(\rho_k) = U(e(x, \rho_k)) = 0$. Also $\lim_{x \to \infty} e(x, \rho_k) = 0$. Thus ρ_k is an eigenvalue. Since the Wronskian of the functions $\varphi(x, \rho)$ and $e(x, \rho)$, i.e., $< \varphi(x, \rho), e(x, \rho) >= \triangle(\rho)$ (see [6]), we arrive at (2.9). Conversely,

let ρ_k (complex value) be an eigenvalue and $y(x, \rho_k)$ be a corresponding eigenfunction. Since $U(y(x, \rho_k)) = 0$ and $\lim_{x \longrightarrow \infty} y(x, \rho_k) = 0$, one gets $y(x, \rho_k) = c_{k0}\varphi(x, \rho_k)$ and $y(x, \rho_k) = c_{k1}e(x, \rho_k)$ for $c_{ks} \neq 0$, $s = 0, 1$, respectively. These yield (2.9) and $\Delta(\rho_k) = U(e(x, \rho_k)) = 0$. The proof of Theorem 3 is completed. $\diamondsuit$

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