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A local approach to the entropy of countable fuzzy partitions

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Abstract. This paper defines and investigates the ergodic properties of the entropy of a countable partition of a fuzzy dynamical system at different points of the state space. It ultimately introduces the local fuzzy entropy of a fuzzy dynamical system and proves it to be an isomorphism invariant.

Keywords: Local entropy, countable partition, local diameter.

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1. Introduction

The concept of entropy was first proposed as a mathematical tool in physics by Clausius and plays a significant role in physical phenomena. In physics, this concept is understood as a measure of disorder. In 1948, Shannon introduced the concept of entropy to Information Theory. Examining a random phenomenon as a member of a σ-algebra, Kolmogorov [6] introduced the concept of entropy to ergodic theory in 1958. One of the main issues in ergodic theory is the classification of dynamical systems based on isomorphism. Objects remaining invariant under isomorphism are thus very significant. In the classical setting entropy fits in this role [3, 10, 12]. The entropy of fuzzy dynamical systems is obtained by replacing partitions with fuzzy partitions and was

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first introduced by Dumetriscu [1]. In 2006, Ebrahimi [2] defined the entropy of a dynamical system through examining countable partitions and proved the concept to be an isomorphism invariant. This paper is an attempt at defining the local entropy of fuzzy dynamical systems; that is, the entropy of a system can be calculated at any point in the state space. The entropy of a countable partition of a fuzzy dynamical system is first defined at any point in the system state space and its properties are investigated. The entropy of a fuzzy dynamical system is then defined at any point in relation to an arbitrary countable partition and is proved to be an isomorphism invariant.

2. Preliminary facts

We recall that a fuzzy set in a non-empty set X is an element of the family I^X of all functiuns from X to closed unit interval $I = [0,1]$. A fuzzy σ- algebra M on a non-empty set X is a subset of I^X which satisfies the following conditions:

(i) $1_X \in M$,
(ii) If $f, g \in M$ then $f.g \in M$ and $(f-g)^+ \in M$ where $(f-g)^+(x) := max\{(f-g)(x), 0\}$,
(iii) If $\{f_n\}_{n\geq 1} \subseteq M$ then $\oplus_{n=1}^{\infty} f_n \in M$ where $\oplus_{n=1}^{\infty} f_n = \min\left\{ \sum_{n=1}^{\infty} f_n, 1\right\}$.

A function $m : M \to [0,\infty)$ is called a fuzzy set measure, If

(i) $m(0_X) = 0$,
(ii) $m(\oplus_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} m(f_n)$, whenever $f_n \in M$, $\sum_{n=1}^{\infty} m(f_n) \leq 1$.

A fuzzy set measure is said to be preserved by a transformation $T : X \to X$, if for any $f \in M$ the following implication holds:

$$m(foT) = m(f).$$

The set of all fuzzy set measures $m : M \to [0,\infty]$, satisfying $m(1_X) = 1$ is denoted by $F^*(X)$. The set of all fuzzy set measures in $F^*(X)$, preserved by T, is defined as follows:

$$F^*(X,T) = \{m \in F^*(X) : m(foT) = m(f)\ \forall f \in M\}.$$

By a fuzzy countable partition we mean a countable collection $\alpha = \{f_i\}_{i\in\mathbb{N}}$ of fuzzy subsets such that $\sum_{i=1}^{\infty} f_i = 1$ on X.

Definition 2.1. *A fuzzy countable partition $\alpha = \{f_i\}_{i\in\mathbb{N}}$ is a refinement of a fuzzy countable partition $\beta = \{g_j\}_{j\in\mathbb{N}}$, if there are disjoint sets $I(1), I(2), ..., I(m) \subseteq \{1, 2, ..., n\}$ Such that $g_j = \sum_{j\in I(i)} f_j$, for every $i \in \mathbb{N}$. If α is a refinement of β, we write $\beta < \alpha$.*

Definition 2.2. *A common refinement of two countable partitions* $\alpha = \{f_i\}_{i\in N}$ *and* $\beta = \{g_j\}_{j\in\mathbb{N}}$ *is defined as*

$$\alpha \odot \beta = \Big\{ f_i . g_j ;\ i,j \in \mathbb{N} \Big\}.$$

Clearly , $\alpha \odot \beta > \alpha$ *and* $\alpha \odot \beta > \beta$.

3. Local entropy of countable partitions

In this section we define a fuzzy entropy for countable dynamical systems through a local approach.
It is assumed that X is a metric space and $T : X \to X$ is continuous. Note that (X, T, M) is called a (discrete) fuzzy dynamical system, and if $x \in X$ then the set $O_T(x) = \Big\{ T^k(x), k \in \mathbb{N} \cup \{0\} \Big\}$ is called the orbit of x .

Definition 3.1. *Let* (X, T, M) *be a fuzzy dynamical system and* $m \in F^*(X, T)$. *For* $x \in X$ *and* $f \in M$ *we define*

$$m_x(f) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f o T^k(x).$$

Note that if we consider the crisp set $f = \chi_A$ *in above, we will have:*

$$\begin{aligned} m_x(f) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A \Big(T^k(x) \Big) \\ &= \limsup_{n\to\infty} \frac{1}{n} card \Big\{ 0 \le k \le n-1, T^k(x) \in A \Big\} \end{aligned}$$

Definition 3.2. *Let* $\alpha = \{f_i\}_{i\in N}$ *be a fuzzy countable partition and* $x \in X$ *we define:*

$$P_T(x, \alpha) = -\log \sup_{i\in\mathbb{N}} m_x(f_i),$$

and for $g \in M$,

$$P_T(x, \alpha|g) = -\log \sup_{i\in\mathbb{N}}\ m_x(f_i|g).$$

Where

$$m_x(f_i|g) = \frac{m_x(f_i\ .g)}{m_x(g)}\ (m_x(g) \neq 0).$$

Corollary 3.3. *The following properties are easily verified:*

(i) *If* $\alpha = \{1_X\}$ *then* $P_T(x, \alpha) = 0$ *for all* $x \in X$.
(ii) $P_T(x, \alpha) \ge 0$.
(iii) *For* $x \in X$, *the map* m_x *is a finitely sub-additive probability fuzzy set measure, i.e.,*

(1) $m_x(0_X) = 0$.

(2) *If* $\{f_i\}_{i=1}^{\infty}$ *is a family of fuzzy sets with* $\sum_{i=1}^{n}(f_i) \leq 1$ *then*

$$m_x(\oplus_{i=1}^{n} f_i) \leq \sum_{i=1}^{n} m_x(f_i).$$

(3) $m_x(1_X) = 1$.

(iv) $P_T(x,\alpha) = P_T(x, T^{-1}\alpha)$.

Convention: $P_T(x,\alpha|0_X) = 0$.

Theorem 3.4. *Let* $\alpha = \{f_i\}_{i\in\mathbb{N}}$ *and* $\beta = \{g_j\}_{j\in\mathbb{N}}$ *be fuzzy countable partitions and* $x \in X$. *Then:*

(i) $\alpha < \beta$ *if and only if* $P_T(x,\alpha|h) \leq P_T(x,\beta|h)$ *for each* $h \in M$.

(ii) $\alpha < \beta$ *if and only if* $P_T(x,\alpha) \leq P_T(x,\beta)$.

Proof :

(i) Suppose $\alpha < \beta$, and then for each $g_j \in \beta$ there exisits $f_i \in \alpha$ such that $g_j \leq f_i$. This implies that $g_j\ .h \leq f_i\ .h$ for each $h \in M$. Then

$$\begin{aligned} m_x(g_j.h) \leq m_x(f_i.h) \quad &\Rightarrow \quad \sup_{j\in\mathbb{N}} \frac{m_x(g_j.h)}{m_x(h)} \leq \sup_{i\in\mathbb{N}} \frac{m_x(f_i.h)}{m_x(h)} \\ &\Rightarrow \quad \log \sup_{j\in\mathbb{N}} \frac{m_x(g_j.h)}{m_x(h)} \leq \log \sup_{i\in\mathbb{N}} \frac{m_x(f_i.h)}{m_x(h)} \\ &\Rightarrow \quad P_T(x,\beta|h) \geq P_T(x,\alpha|h). \end{aligned}$$

Conversely, if $P_T(x,\beta|h) \geq P_T(x,\alpha|h)$, we have

$$\sup_{j\in\mathbb{N}} m_x(g_j.h) \leq \sup_{i\in\mathbb{N}}\ m_x(f_i.h)$$

and hence for each $j \in \mathbb{N}, g_j \leq f_i$ for some $i \in \mathbb{N}, i.e, \alpha < \beta$.

(ii) It is clear.

□

Corollary 3.5. *Let* $\alpha = \{f_i\}_{i\in N}$ *and* $\beta = \{g_j\}_{j\in N}$ *be fuzzy countable partitions and* $x \in X$. *Then*

(i) $P_T(x,\alpha\odot\beta) \geq P_T(x,\alpha), P_T(x,\alpha\odot\beta) \geq P_T(x,\beta)$.

(ii) $P_T(x,\alpha\odot h) \geq P_T(x,\alpha)$, *for all* $h \in M$.

(iii) $P_T(x,\alpha\odot h) \geq P_T(x,\alpha|h)$, *for all* $h \in M$.

(iv) $\alpha < \beta$ *if and only if* $P_T(x,\beta\odot h) \geq P_T(x,\alpha\odot h)$ *for all* $h \in M$.

Definition 3.6. *If* α *is a fuzzy countable partition of* (X,T,M) *, the local diameter of* α *at* $x \in X$ *is defined as follows:*

$$d_x(\alpha) = \sup_{A_i\in\alpha}\ m_x(A_i).$$

Definition 3.7. *Let $\alpha=\{f_i\}_{i\in\mathbb{N}}$ and $\beta=\{g_j\}_{j\in\mathbb{N}}$ fuzzy countable partitions of (X,T,M). We define:*

$$P_T(x,\alpha|\beta)=-\log\sup_{i\in\mathbb{N}}\frac{d_x(f_i\odot\beta)}{d_x(\beta)}=-\log\sup_{j\in\mathbb{N}}\frac{d_x(\alpha\odot g_j)}{d_x(\beta)}.$$

Theorem 3.8. *If α,β and γ are fuzzy countable partitions of (X,T,M) and $x\in X$, then*

(i) $P_T(x,\alpha|\beta)\geq 0$.
(ii) *If $\alpha<\gamma$ then $P_T(x,\alpha|\beta)\leq P_T(x,\alpha\odot\gamma)$ especially, $P_T(x,\alpha|\beta)\leq P_T(x,\alpha\odot\beta)$.*
(iii) *If $\alpha<\beta$, them $P_T(x,\alpha|\gamma)\leq P_T(x,\beta)$.*

Proof :

(iii) Since $\alpha<\beta$, for each $g_j\in\beta$ there exists $f_i\in\alpha$ such that $g_j\leq f_i$. Then $g_j.h_k\leq f_i.h_k$ for all $h_k\in\gamma$. Therefore

$$\begin{aligned}m_x(g_j.h_k)\leq m_x(f_i.h_k)\quad&\Leftrightarrow\quad\frac{\sup_{j\in N}\ m_x(g_j\ .h_k)}{\sup_{k\in N}m_x(h_k)}\leq\frac{\sup_{j\in\mathbb{N}}\ m_x(f_i\ .h_k)}{\sup_{k\in\mathbb{N}}\ m_x(h_k)}\\&\Leftrightarrow\quad P_T(x,\alpha|\gamma)\leq P_T(x,\beta|\gamma).\end{aligned}$$

□

Definition 3.9. *Let α be a countable fuzzy partition. The local fuzzy entropy of T with respect to α is defined as follows:*

$$h(x,T,\alpha)=\limsup_{n\to\infty}\ \frac{1}{n}P_T(x,\odot_{i=0}^{n-1}T^{-i}\alpha)$$

Note that

$$\odot_{i=0}^{n-1}T^{-i}\alpha=\Big\{\prod_{k=0}^{n-1}f_{i_k}oT^k,f_{i_k}\in\alpha,i_k\in\mathbb{N}\Big\},$$

is the partition induced by n successive performance of T on the fuzzy partition α.

Example 3.10. *Let $X=(0,1]$ and $T:X\to X$ be the doubling map $T(x)=2x(mod1)$. We know that T preserves Lebesgue measure m and is ergodic. Hence by Birkhoff ergodic Theorem [12] for each $x\in X$ and $A\subset X$ we have $m_x(A)=m(A)$. Let $\alpha=\{f_i\}_{i\in\mathbb{N}}$ such that:*

$$f_1=\chi_{(\frac{1}{2},1]},$$

$$f_2=\chi_{(\frac{1}{4},\frac{1}{2}]},$$

and for $i\geq 3$,

$$f_i=\chi_{(\frac{1}{2^i},\frac{1}{2^{i-1}}]}.$$

Then α is a fuzzy countable partition of X. For each $x \in X$ we can now calculate

$$P_T(x, \alpha) = -\log \sup_{i \in \mathbb{N}} m_x(f_i) = -\log \frac{1}{2} = \log 2.$$

So we can easily obtain that $\frac{1}{n} P_T(x, T, \odot_{i=1}^{n-1} T^{-i}\alpha) = \log 2$ *and thus letting* $n \to \infty$ *gives that* $h(x, T, \alpha) = \log 2$.

Theorem 3.11. *Suppose that $T : X \to X$ is a continuous map on the metric space X. If α, β are countable fuzzy partitions and $x \in X$ then*

(i) *If $\alpha < \beta$ then $h(x, T, \alpha) \leq h(x, T, \beta)$.*
(ii) $h(x, T, T^{-1}\alpha) = h(x, T, \alpha)$.
(iii) *If $k \geq 1$ then $h(x, T, \odot_{i=0}^{k-1} T^{-i}\alpha) = h(x, T, \alpha)$.*
(iv) *If T is invertible and $k \geq 1$. Then $h(x, T, \alpha) = h(x, T, \odot_{i=-k}^{k} T^{-i}\alpha)$.*

Proof :

(i) If $\alpha < \beta$ then $\odot_{i=0}^{n-1} T^{-i}\alpha < \odot_{i=0}^{n-1} T^{-i}\beta$ for all $n \geq 1$. This easily leads to the result.
(ii) Since

$$P_T(x, \odot_{i=1}^{n} T^{-i}\alpha) = P_T(x, \odot_{i=0}^{n-1} T^{-i}\alpha)$$

Then we will easily have

$$h(x, T, T^{-1}\alpha) = h(x, T, \alpha).$$

(ii) We have

$$\begin{aligned} h(x, T, \odot_{i=1}^{k} T^{-i}\alpha) &= \limsup_{n\to\infty} \frac{1}{n} P_T\Big(x, \odot_{j=0}^{n-1} T^{-j}(\odot_{i=0}^{k} T^{-i}\alpha)\Big) \\ &= \limsup_{n\to\infty} \frac{1}{n} P_T(x, \odot_{i=0}^{n+k-1} T^{-i}\alpha) \\ &= \limsup_{n\to\infty} (\frac{k+n}{n})(\frac{1}{k+n}) P_T(x, \odot_{i=0}^{n+k-1} T^{-i}\alpha) \\ &= h(x, T, \alpha). \end{aligned}$$

(iv) $h(x, T, \odot_{i=-k}^{k} T^{-i}\alpha) = h(x, T, \odot_{i=0}^{2k} T^{-i}\alpha) = h(x, T, \alpha)$.

□

Theorem 3.12. *Suppose that $T : X \to X$ is a continuous map on the metric space X. If α is a fuzzy countable partition, $x \in X$ and $k \in \mathbb{N}$, then*

$$h(x, T^k, \alpha) = kh(x, T, \alpha).$$

Proof :

$$\begin{aligned} h(x,T^k,\alpha) &= \limsup_{n\to\infty} \frac{1}{n} P_T\Big(x, \odot_{j=0}^{n-1} T^{-kj}(\odot_{i=0}^{k-1} T^{-i}\alpha)\Big) \\ &= \limsup_{n\to\infty} \frac{k}{nk} P_T(x, \odot_{i=0}^{nk-1} T^{-i}\alpha) \\ &= kh(x,T,\alpha). \end{aligned}$$

□

Theorem 3.13. *Suppose that $T : X \to X$ is a continuous map on the metric space X with $T^k = id$ for some $k \in \mathbb{N}$, then for each $x \in X$ we have,*

$$h(x,T,\alpha) = 0.$$

Proof : Since $T^k = id$ so $h(x,T^k,\alpha) = 0$. Therefore,

$$h(x,T,\alpha) = \frac{1}{k} h(x,T^k,\alpha) = 0.$$

□

Definition 3.14. *We say that two dynamical systems (X_1,T_1,M_1) and (X_2,T_2,M_2) are isomorphic if there exists a homeomorphism $\varphi : X_1 \to X_2$ such that $\varphi o T_1 = T_2 o \varphi$.*

Theorem 3.15. *If (X_1,T_1,M_1) and (X_2,T_2,M_2) are isomorphic dynamical systems via the homeomorphism $\varphi : X_1 \to X_2$, then for any fuzzy countable partition α of (X_2,T_2,M_2) and $x \in X_1$, we have,*

$$h(x,T_1,\varphi^{-1}(\alpha)) = h(\varphi(x),T_2,\alpha).$$

Proof : Let α be any countable fuzzy partition of (X_2,T_2,M_2). For $x \in X_1$ and $f \in M_1$, we have ${m_x}^{T_1}(f) = {m_{\varphi(x)}}^{T_2}(\varphi(f))$. Therefore, $P_{T_1}(x,\alpha) = P_{T_2}(\varphi(x),\varphi(\alpha))$. So,

$$\begin{aligned} h(\varphi(x),T_2,\alpha) &= \limsup_{n\to\infty} \frac{1}{n} P_{T_2}\Big(\varphi(x), \odot_{i=0}^{n-1} T^{-i}\alpha\Big) \\ &= \limsup_{n\to\infty} \frac{1}{n} P_{T_1}(x, \varphi^{-1}(\odot_{i=0}^{n-1} T^{-i}\alpha) \\ &= \limsup_{n\to\infty} \frac{1}{n} P_{T_1}(x, \odot_{i=0}^{n-1} T^{-i}\varphi^{-1}\alpha)) \\ &= h(x,T_1,\varphi^{-1}\alpha). \end{aligned}$$

□

4. Conclusions

This paper generalizes the concept of entropy of fuzzy dynamical systems in relation to countable partitions and examines its properties. The local entropy of a continuous dynamical system is finally proved to be an isomorphism invariant. This study can facilitate the study of entropy for fuzzy dynamical systems with countable partitions.

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