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Anti-synchronization and synchronization of T-system

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Abstract. In this paper, we discuss the synchronization and anti-synchronization of two identical chaotic T-systems. The adaptive and nonlinear control schemes are used for the synchronization and anti-synchronization. The stability of these schemes is derived by Lyapunov Stability Theorem. Firstly, the synchronization and anti-synchronization are applied to systems with known parameters, then to systems in which the drive and response systems have one unknown parameter. Numerical simulations show the effectiveness and feasibility of the proposed methods.

Keywords: Lyapunov stability, Chaos, Control, Anti-Synchronization, Synchronization.

2000 Mathematics subject classification: 34H10 34C28 37N35.

1. Introduction

Chaos as an interesting phenomenon in nonlinear dynamical systems, has been studied extensively over the last four decades [16, 18, 19, 11, 23, 2]. Chaotic and hyperchaotic systems as nonlinear deterministic systems display complex and unpredictable behaviors. Moreover, these systems are sensitive to initial conditions. The chaotic and hyperchaotic systems have many important applications in nonlinear sciences, such as laser physics, secure communications, nonlinear circuits, control, neural networks, and active wave propagation [11, 20, 13, 4, 3, 7, 12, 17].

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The synchronization of chaotic systems has been investigated since their introduction in a paper by Pecora and Carrol in 1990 [19] and has been widely studied in many fields, such as physics, chemistry, ecological sciences, and secure communications [10, 6, 23]. Accordingly, various techniques have been proposed to achieve chaos synchronization and anti-synchronization such as adaptive control, active control, sliding mode, and nonlinear control [1, 14, 22, 8, 9, 15]. Fortunately, some existing synchronization methods can be generalized to the anti-synchronization of chaotic systems. The objective of the anti-synchronization is to make the output of the slave system follows the symmetry of output of the slave system.

In this paper, we use the adaptive and nonlinear control schemes for the synchronization and anti-synchronization of two identical chaotic T-systems. The synchronization and anti-synchronization are applied for system with known and unknown parameters. In unknown parameter case, each of the drive and response systems has one unknown parameter.

The rest of this paper is organized as follows: Section 2 briefly introduces the chaotic T-system. In Section 3, the chaos synchronization and anti-synchronization of chaotic T-system with known parameter via nonlinear controller are discussed. In Section 4, the adaptive control is used for synchronization and anti-synchronization of T-systems such that each of the drive and response systems have one unknown parameter. Numerical simulations are given section 5. Finally, the concluding remarks are presented in Section 6.

2. T-system

In 2005, Tigen [21] introduced a new real chaotic nonlinear system called T-system, presented as follows:

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) \\ \dot{x_2} = (c-a)x_1 - ax_1x_3 \\ \dot{x_3} = x_1x_2 - bx_3, \end{cases} \tag{2.1}$$

where x_1, x_2, and x_3 are the state variables and a, b, and c are real positive parameters.

By choosing $a = 2.1$, $b = 0.6$, and some value of $0 < c < 40$, the Lyapunov exponents in FIGURE 1 show that the system (2.1) is a chaotic system because a Lyapunov exponent of system is positive[14, 5]. Also this system can be regarded as a dissipative system, since the sum of its Lyapunov exponents is negative. The attractors of chaotic systems are bounded but not to a fixed point or limit cycle, characterizing a property of chaotic systems [5]. FIGURE 2 displays the attractor of the T-system. In addition, the chaotic systems are sensitive to their initial conditions. FIGURE 3 show the sensitivity of T-system for $a = 2.1$,

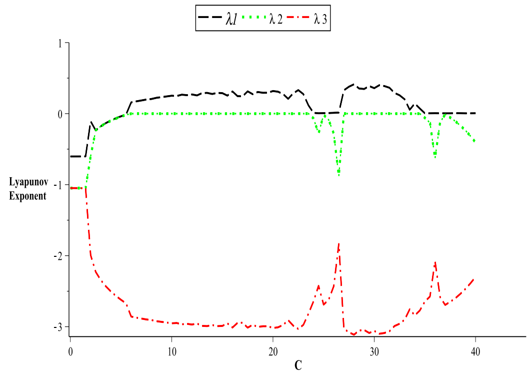

FIGURE 1. Lyapunov exponents of system (2.1), for $a = 2.1$, $b = 0.6$ and $0 < c < 40$.

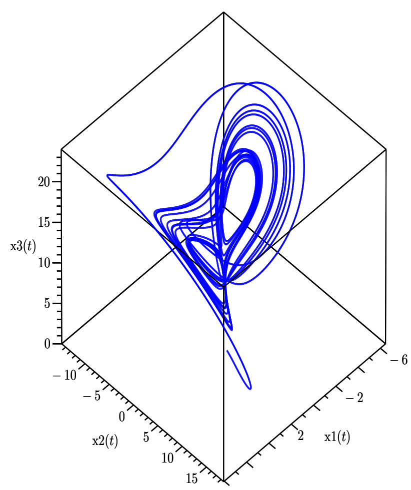

FIGURE 2. An attractor of T-system for $a = 2.1$, $b = 0.6$ and $c = 28$ with initial conditions $(x_1(0), x_2(0), x_3(0)) = (1, 3, 0)$.

$b = 0.6$, and $c = 30$ with close initial conditions $(2, 1, 2)$ and $(2.01, 1, 2)$. Synchronization and anti-synchronization of this system can be used for cryptography and decryption of data in secure communication:

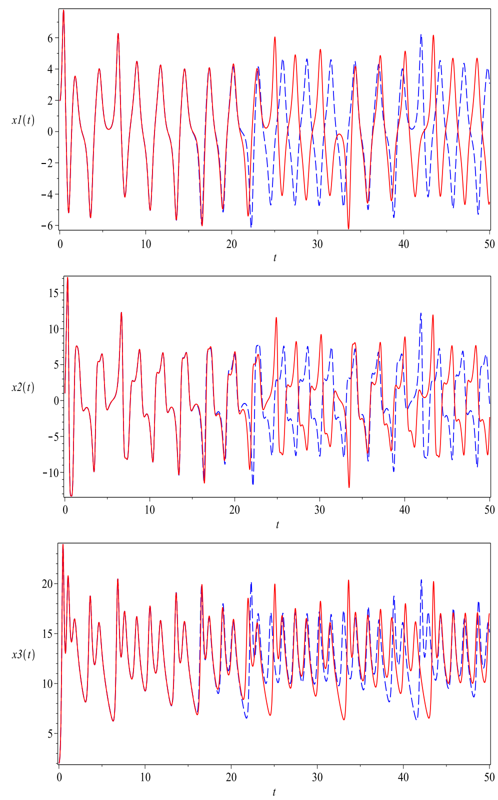

FIGURE 3. Sensitivity of T-system for close initial conditions: Dash line for initial condition $(2, 1, 2)$, and solid line for initial condition $(2.01, 1, 2)$.

3. Synchronization and anti-synchronization with known parameter via nonlinear controller

Let the drive and response systems are defined as:

$$\dot{x} = f(x), \tag{3.1}$$

$$\dot{y} = f(y) + u, \tag{3.2}$$

where $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n$are the state vectors of the systems (3.1) and (3.2) respectively; $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defines a vector field in n-dimensional space and $u = (u_1, u_2, \cdots, u_n)^T$ is an n-dimensional control signal.

Let $e = x - y$ and $\bar{e} = x + y$ are the synchronization and anti-synchronization error vectors, respectively. The goal is to design an appropriate controller u such that for any initial conditions y_0 and x_0, we have:

$$\lim_{t\longrightarrow\infty}\|e\| = \lim_{t\longrightarrow\infty}\|x(t,x_0)-y(t,y_0)\| = 0,$$
$$\lim_{t\longrightarrow\infty}\|\bar{e}\| = \lim_{t\longrightarrow\infty}\|x(t,x_0)+y(t,y_0)\| = 0,$$

where $\|.\|$ is the Euclidean norm.

For synchronization and anti-synchronization with known parameter, we define the drive and response T-systems as follows:

$$\begin{cases} \dot{x_1} = a(y_1 - x_1) \\ \dot{y_1} = (c-a)x_1 - ax_1z_1 \\ \dot{z_1} = x_1y_1 - bz_1, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \dot{x_2} = a(y_2 - x_2) + u_1 \\ \dot{y_2} = (c-a)x_2 - ax_2z_2 + u_2 \\ \dot{z_2} = x_2y_2 - bz_2 + u_3, \end{cases} \tag{3.4}$$

where x_1, x_2, y_1 y_2, z_1 and z_2 are the state variables, u_1, u_2 and u_3 are three control functions to be designed and a, b and c are real parameters and positive, ensuring that the system is chaotic.

3.1. **Synchronization of (3.3) and (3.4).** For synchronization, we subtract (3.4) from (3.3), in a way that the error dynamical system is represented as follow:

$$\begin{cases} \dot{e_1} = a(e_2 - e_1) + u_1 \\ \dot{e_2} = (c-a)e_1 - a(x_2z_2 - x_1z_1) + u_2 \\ \dot{e_3} = x_2y_2 - x_1y_1 - be_3 + u_3. \end{cases} \tag{3.5}$$

In the following theorem, the appropriate nonlinear controller is discussed for synchronization of chaotic systems (3.4) and (3.3).

Theorem 3.1. *Systems (3.3) and (3.4) will be globally asymptotically synchronized for any initial condition with the following control law for all $k > 0$:*

$$\begin{cases} u_1 = -ae_2 \\ u_2 = -(c-a)e_1 + a(x_2z_2 - x_1z_1) - ke_2 \\ u_3 = -x_2y_2 + x_1y_1. \end{cases} \tag{3.6}$$

Proof. We define the lyapunov function as follows:

$$V(t) = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{3.7}$$

Then with the above mentioned conditions we have:

$$\dot{V(t)} = \dot{e_1}e_1 + \dot{e_2}e_2 + \dot{e_3}e_3 = -(ae_1^2 + ke_2^2 + be_3^2) < 0.$$

It is clear that V is positive definite and $\dot{V}$ is negative definite. According to the Lyapunov stability Theorem, the error system (3.5) can converge in to the origin asymptotically. Therefore, the drive system (3.3) and the response system (3.4) can be asymptotically and globally synchronized. This completes the proof. □

3.2. **Anti-synchronization of (3.3) and (3.4).** For anti-synchronization, we add (3.3) to (3.4), so the error dynamical system as follow:

$$\begin{cases} \dot{\bar{e_1}} = a(\bar{e_2} - \bar{e_1}) + u_1 \\ \dot{\bar{e_2}} = (c-a)\bar{e_1} - a(x_2 z_2 + x_1 z_1) + u_2 \\ \dot{\bar{e_3}} = x_2 y_2 + x_1 y_1 - b\bar{e_3} + u_3. \end{cases} \tag{3.8}$$

For the two identical chaotic systems without control ($u_i = 0$), the trajectories of the two identical systems will quickly separate and become irrelevant on the condition that initial values $(x_1(0), y_1(0), z_1(0)) \neq (x_2(0), y_2(0), z_2(0))$. However, with appropriate control schemes, the two systems will approach anti-synchronization for any initial values.

Theorem 3.2. *Systems (3.3) and (3.4) will be globally asymptotically anti-synchronized for any initial condition with the following control law for all $k > 0$:*

$$\begin{cases} u_1 = -a\bar{e_2} \\ u_2 = -(c-a)\bar{e_1} + a(x_2 z_2 + x_1 z_1) - k\bar{e_2} \\ u_3 = -x_2 y_2 - x_1 y_1. \end{cases} \tag{3.9}$$

Proof. We define the Lyapunov function as follows:

$$V(t) = \frac{1}{2}(\bar{e_1}^2 + \bar{e_2}^2 + \bar{e_3}^2). \tag{3.10}$$

Then with the above mentioned conditions, we have:

$$\dot{V(t)} = \dot{\bar{e_1}}\bar{e_1} + \dot{\bar{e_2}}\bar{e_2} + \dot{\bar{e_3}}\bar{e_3} = -(a\bar{e_1}^2 + k\bar{e_2}^2 + b\bar{e_3}^2) < 0,$$

It is clear that V is positive definite and $\dot{V}$ is negative definite. According to the Lyapunov Stability Theorem, the error system (3.8) can converge in to the origin asymptotically. Therefore, the drive system (3.3) and the response system (3.4) can be asymptotically and globally anti-synchronized. This completes the proof. □

4. Anti-synchronization and synchronization with unknown parameter via adaptive control

Let each of the drive and response systems has one unknown parameter. Also let the drive and response T-system, are defined as follow:

$$\begin{cases} \dot{x_1} = a(y_1 - x_1) \\ \dot{y_1} = (\hat{c} - a)x_1 - a x_1 z_1 \\ \dot{z_1} = x_1 y_1 - b z_1, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dot{x_2} = a(y_2 - x_2) + u_1 \\ \dot{y_2} = (c-a)x_2 - ax_2z_2 + u_2 \\ \dot{z_2} = x_2y_2 - \hat{b}z_2 + u_3, \end{cases} \tag{4.2}$$

where $\hat{c}$ and $\hat{b}$ are unknown parameters and estimates of c and b, respectively.

4.1. **Synchronization of (4.1) and (4.2) via adaptive control.** For synchronization, we subtract (4.2) from (4.1), the error dynamical system as follow:

$$\begin{cases} \dot{e_1} = a(e_2 - e_1) + u_1 \\ \dot{e_2} = cx_2 - \hat{c}x_1 - a(x_2z_2 + x_1z_1 + x_2 - x_1) + u_2 \\ \dot{e_3} = x_2y_2 - x_1y_1 + bz_1 - \hat{b}z_2 + u_3. \end{cases} \tag{4.3}$$

The following theorem shows that system (4.1) and system (4.2) can be effectively synchronized, and estimate the unknown parameters.

Theorem 4.1. *By the following controller:*

$$\begin{cases} u_1 = -ae_2 + (a-1)e_1 \\ u_2 = -(\hat{c} - a)e_1 + a(x_2z_2 - x_1z_1) - e_2 \\ u_3 = -x_2y_2 + x_1y_1 - e_3 - \hat{b}e_3, \end{cases} \tag{4.4}$$

and the parameter update law

$$\begin{cases} \dot{\tilde{b}} = \dot{\hat{b}} = e_3z_1 - \tilde{b} \\ \dot{\tilde{c}} = \dot{\hat{c}} = e_2x_2 - \tilde{c}, \end{cases} \tag{4.5}$$

the drive system (4.1) and the response system (4.2) will be asymptotically synchronized. Here, $\hat{b}$ and $\hat{c}$ are the estimates values of b and c respectively; and $\tilde{b} = \hat{b} - b$ and $\tilde{c} = \hat{c} - c$.

Proof. We define the lyapunov function as follows:

$$V(t) = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \tilde{b}^2 + \tilde{c}^2). \tag{4.6}$$

Then with the above mentioned conditions we have:

$$\dot{V(t)} = \dot{e_1}e_1 + \dot{e_2}e_2 + \dot{e_3}e_3 + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} = -(e_1^2 + e_2^2 + e_3^2 + \tilde{b}^2 + \tilde{c}^2) < 0.$$

It is clear that V is positive definite and $\dot{V}$ is negative definite. According to the Lyapunov stability Theorem, the error system (4.7) can be converged in to the origin asymptotically. Therefore, the drive system (4.1) and the response system (4.2) can be asymptotically and globally synchronized. This completes the proof. □

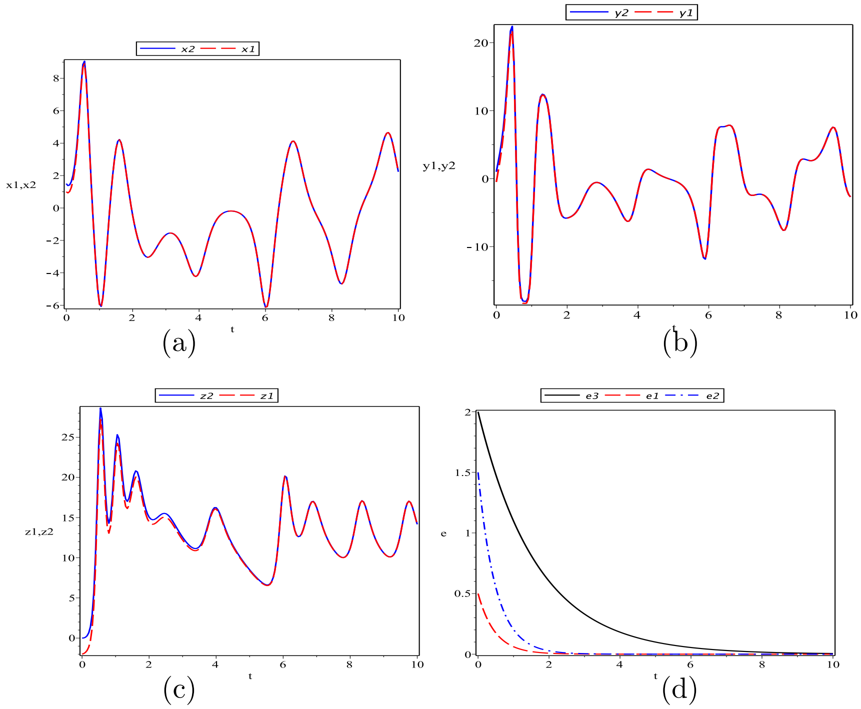

FIGURE 4. Time series trajectory for synchronization via nonlinear control; **a**: x_1, x_2, **b**: y_1, y_2, **c**: z_1, z_2, **d**: e_1, e_2, e_3.

4.2. **Anti-synchronization of (4.1) and (4.2) via adaptive control.** For anti-synchronization, we add (4.1) to (4.2), the error dynamical system is represented as follows:

$$\begin{cases} \dot{\bar{e}}_1 = a(\bar{e}_2 - \bar{e}_1) + u_1 \\ \dot{\bar{e}}_2 = cx_2 + \hat{c}x_1 - a(x_2z_2 + x_1z_1 + x_2 + x_1) + u_2 \\ \dot{\bar{e}}_3 = x_2y_2 + x_1y_1 - bz_1 + \hat{b}z_2 + u_3. \end{cases} \tag{4.7}$$

The following theorem shows that system (4.1) and system (4.2) can be effectively anti-synchronized, and estimate the unknown parameters.

Theorem 4.2. *By the following controller:*

$$\begin{cases} u_1 = -a\bar{e}_2 + (a-1)\bar{e}_1 \\ u_2 = -(\hat{c} - a)\bar{e}_1 + a(x_2z_2 + x_1z_1) - e_2 \\ u_3 = -x_2y_2 - x_1y_1 - e_3 - \hat{b}\bar{e}_3, \end{cases} \tag{4.8}$$

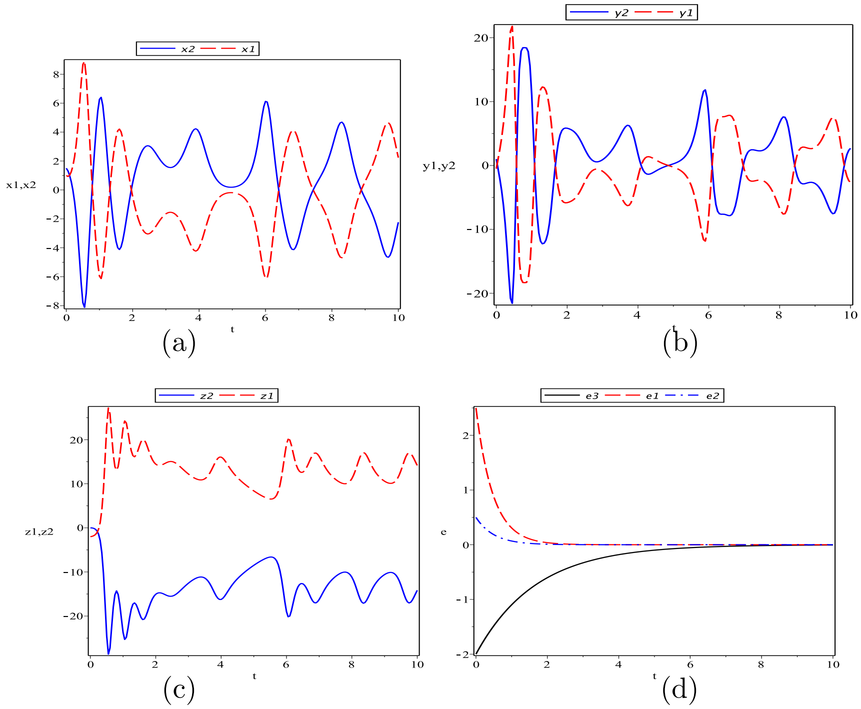

FIGURE 5. Time series trajectory for anti-synchronization via nonlinear control; **a**: x_1, x_2, **b**: y_1, y_2, **c**: z_1, z_2, **d**: $\bar{e_1}, \bar{e_2}, \bar{e_3}$.

and the parameter update law

$$\begin{cases} \dot{\tilde{b}} = \dot{\hat{b}} = -\bar{e_3} z_1 - \tilde{b} \\ \dot{\tilde{c}} = \dot{\hat{c}} = -\bar{e_2} x_2 - \tilde{c}. \end{cases} \tag{4.9}$$

The drive system (4.1) and the response system (4.2) will be globally asymptotically anti-synchronized. Here, $\hat{b}$ and $\hat{c}$ are the estimates values of b and c respectively; and $\tilde{b} = \hat{b} - b$ and $\tilde{c} = \hat{c} - c$.

Proof. We define the Lyapunov function as follows:

$$V(t) = \frac{1}{2}(\bar{e_1}^2 + \bar{e_2}^2 + \bar{e_3}^2 + \tilde{b}^2 + \tilde{c}^2). \tag{4.10}$$

Then with the above mentioned conditions we have:

$$\dot{V(t)} = \dot{\bar{e_1}}\bar{e_1} + \dot{\bar{e_2}}\bar{e_2} + \dot{\bar{e_3}}\bar{e_3} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} = -(\bar{e_1}^2 + \bar{e_2}^2 + \bar{e_3}^2 + \tilde{b}^2 + \tilde{c}^2) < 0.$$

It is clear that V is positive definite and $\dot{V}$ is negative definite. According to the Lyapunov Stability Theorem, the error system (4.7) can be converged in to the origin asymptotically. Therefore, the drive system

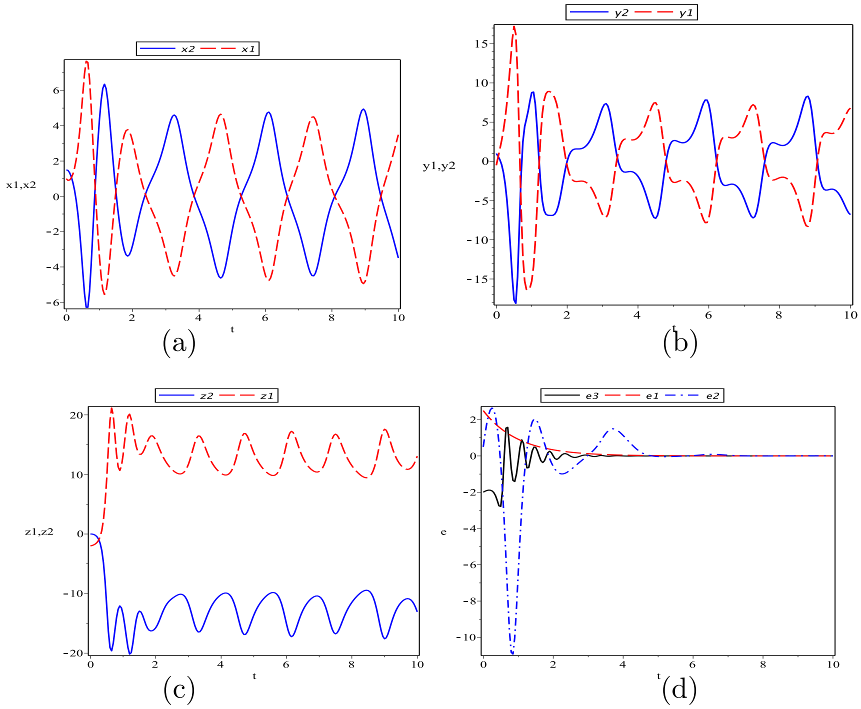

FIGURE 6. Time series trajectory for anti-synchronization via adaptive control; **a**: x_1, x_2, **b**: y_1, y_2, **c**: z_1, z_2, **d**: $\bar{e_1}, \bar{e_2}, \bar{e_3}$.

(4.1) and the response system (4.2) can be asymptotically and globally anti-synchronized. This completes the proof. □

5. Numerical simulations

To demonstrate the validity of the proposed scheme, we present and discuss the numerical results for synchronization and anti-synchronization of chaotic T-system. Fourth-order Runge-Kutta method is used to solve the systems. Numerical simulation are discussed for $a = 2.1, b = 0.6$, and $c = 30$ and the initial values conditions $(x_1(0), y_1(0), z_1(0)) = (4, 0, -2)$, $(x_2(0), y_2(0), z_2(0)) = (1.5, 1, 0)$, and $(\hat{b}(0), \hat{c}(0)) = (-1, 20)$. Figures 5 and 4 show simulations for synchronization and anti-synchronization via nonlinear control, respectively. Numerical results for synchronization and anti-synchronization via adaptive control are given in FIGURES 6 and 7. FIGURES 8 shows the numerical results of parameter estimations via adaptive control.

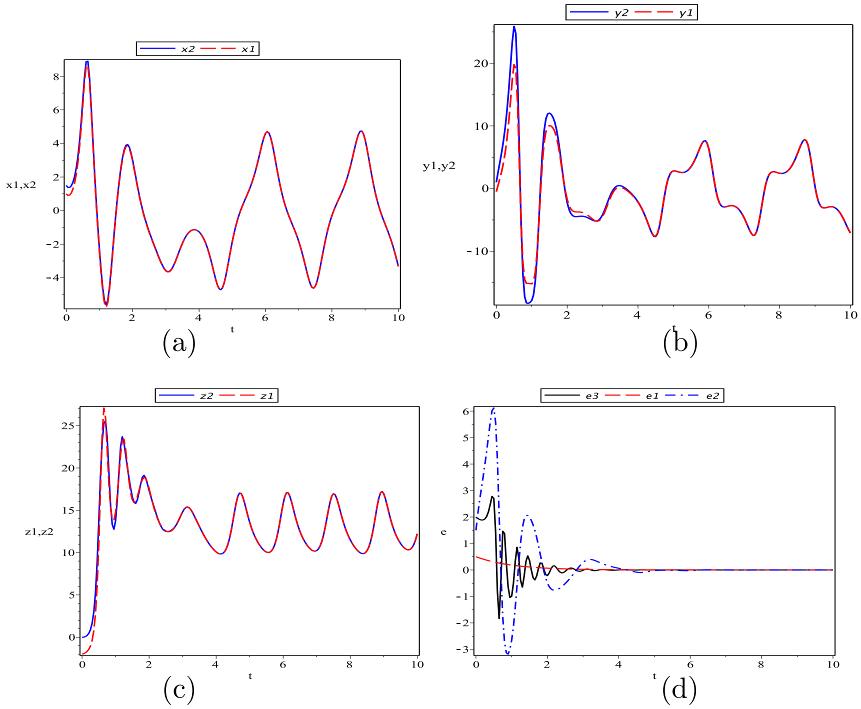

FIGURE 7. Time series trajectory for synchronization via adaptive control **a**: x_1, x_2, **b**: y_1, y_2, **c**: z_1, z_2, **d**: e_1, e_2, e_3.

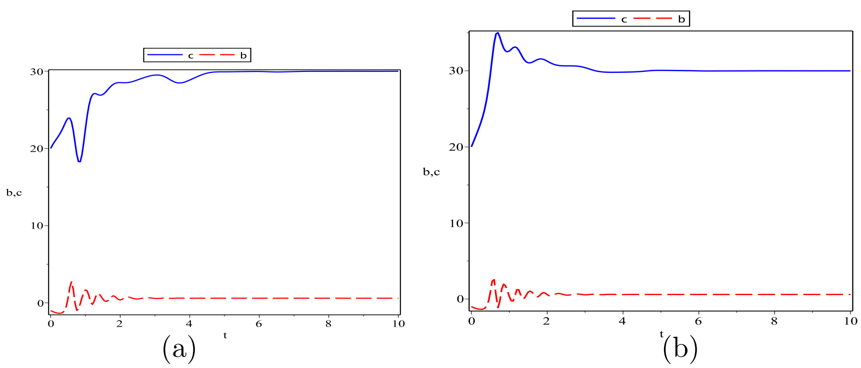

FIGURE 8. Estimation of parameters via adaptive control; **a**: anti-synchronization **b**: synchronization

6. Conclusions

In this paper, we used the nonlinear and adaptive control methods for chaos synchronization and anti-synchronization of chaotic T-system. We also discussed the anti-synchronization and synchronization in two cases, first they were applied for systems with known parameters, then we applied them for systems in which the drive and response systems with one unknown parameter. Sufficient conditions for the synchronization and anti-synchronization were obtained analytically, by Lyapunov stability Theorem. Finally, numerical simulations were given to demonstrate the effectiveness of the proposed scheme.

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