

Eigenfunction expansion in the singular case for q -Sturm-Liouville operators

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ABSTRACT. In this work, we prove the existence of a spectral function for singular q -Sturm-Liouville operator. Further, we establish a Parseval equality and expansion formula in eigenfunctions by terms of the spectral function.

Keywords: q -Sturm-Liouville operator, singular point, Parseval equality, spectral function.

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1. INTRODUCTION

q -calculus, roughly speaking, is calculus without ordinary limits. It allows to deal with sets of not differentiable functions if the classical derivative is replaced with the q -derivative operator. It has a lot of applications in different mathematical areas, such as number theory, orthogonal polynomials, fractal geometry, combinatorics, the calculus of variations, mechanics, orthogonal polynomials, statistic physics, theory of relativity and quantum theory. For a general introduction to the

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q -calculus we refer the reader to the references [1-2, 5,7-9,12-13,17-18, 21-22, 25, 27].

In mathematics, eigenfunction expansions theorems are important for solving various problems. Specially, whenever we seek a solution of a partial differential equation by the Fourier method, we lead to the problem of expanding an arbitrary function as a series of eigenfunctions. There are a lot of studies about eigenfunction expanding problems (see [2-9, 14-15, 19-20, 23, 28]).

It is clear that q -difference equations arise q -analogues of differential equations. In this paper, we consider q -analogue of the Sturm-Liouville equations. Recently, in [11], the Titchmarsh-Weyl theory of Sturm-Liouville equations was extended to the case of q -Sturm-Liouville equation

$$-\frac{1}{q}D_{q^{-1}}D_q y(x) + u(x)y(x) = \lambda y(x),$$

on an interval $[0, a)$, where $0 \leq x \leq a \leq \infty$. In [11], Annaby et al. define q -limit point and q -limit circle singularities. Using Titchmarsh's technique, they give sufficient conditions that the singular points are in a limit point case and derive the eigenfunction expansions.

In this paper, we prove the existence of a spectral function for singular q -Sturm-Liouville operators on semi-unbounded interval. A Parseval equality and an expansion formula in eigenfunctions are established.

2. PRELIMINARIES

Now, we recall some necessary fundamental concepts of quantum analysis. Following the standard notations in [22],[10], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$ and $a \in A$. A q -difference equation is an equation that contains q -derivatives of a function defined on A . Let y be a complex-valued function on A . The q -difference operator D_q , the Jackson q -derivative is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{qx - x} \text{ for all } x \in A \setminus \{0\}.$$

Note that there is a connection between q -deformed Heisenberg uncertainty relation and the Jackson derivative on q -basic numbers (see [26]). In the q -derivative, as $q \rightarrow 1$, the q -derivative is reduced to the classical derivative. The q -derivative at zero is defined by

$$D_q y(0) = \lim_{n \rightarrow \infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x . A *right-inverse* to D_q , the *Jackson q -integration* is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t \quad (a, b \in A).$$

The q -integration for a function over $[0, \infty)$ defined in [16] by the formula

$$\int_0^{\infty} f(t) d_q t = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

A function f which is defined on A , $0 \in A$, is said to be q -regular at zero if

$$\lim_{n \rightarrow \infty} f(xq^n) = f(0),$$

for every $x \in A$. Through the remainder of the paper, we deal only with functions q -regular at zero.

If f and g are q -regular at zero, then we have

$$\int_0^a g(t) D_q f(t) d_q t + \int_0^a f(qt) D_q g(t) d_q t = f(a)g(a) - f(0)g(0).$$

Let $L_q^2(0, \infty)$ be the space of all complex-valued functions defined on $(0, \infty)$ such that

$$\|f\| := \left(\int_0^{\infty} |f(x)|^2 d_q x \right)^{1/2} < \infty.$$

The space $L_q^2(0, \infty)$ is a separable Hilbert space with the inner product

$$(f, g) := \int_0^{\infty} f(x) \overline{g(x)} d_q x, \quad f, g \in L_q^2(0, \infty)$$

(see [11]).

The q -Wronskian of $y(x)$, $z(x)$ is defined to be

$$W_q(y, z)(x) := y(x) D_q z(x) - z(x) D_q y(x), \quad x \in [0, a]. \quad (2.1)$$

3. MAIN RESULTS

Let us consider the q -Sturm-Liouville equations

$$-\frac{1}{q}D_{q^{-1}}D_q y(x) + u(x)y(x) = \lambda y(x), \quad (3.1)$$

with the boundary condition

$$y(0, \lambda) \cos \beta + D_{q^{-1}}y(0, \lambda) \sin \beta = 0, \quad \beta \in \mathbb{R}, \quad (3.2)$$

where λ is a complex eigenvalue parameter, u is a real-valued functions defined on $[0, \infty)$ and continuous at zero and $u \in L_{q,loc}^1(0, \infty)$.

Denote by $\phi(x, \lambda)$, the solution of the system (3.1) subject to the initial conditions

$$\phi(0, \lambda) = \sin \beta, \quad D_{q^{-1}}\phi(0, \lambda) = -\cos \beta. \quad (3.3)$$

Further, we adjoin to problem (3.1)-(3.2) the boundary condition

$$D_{q^{-1}}y(q^{-n}, \lambda) \sin \alpha + y(q^{-n}, \lambda) \cos \alpha = 0, \quad \alpha \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (3.4)$$

It is clear that the problem (3.1), (3.2), (3.4) is a regular problem for a q -Sturm-Liouville equation.

In [10], the authors prove that the boundary value problem (3.1) with the boundary conditions (3.2), (3.4) has a compact resolvent, thus it has a purely discrete spectrum.

Let $\lambda_{m,q^{-n}}$ denote the eigenvalues of this problem and by $y_{m,q^{-n}}(x) = y_{m,q^{-n}}(x, \lambda_{m,q^{-n}})$ the corresponding eigenfunctions which satisfy the conditions (3.3). If $f(\cdot)$ is areal valued function, $f(\cdot) \in L_q^2(0, q^{-n})$ and

$$\alpha_{m,q^{-n}}^2 = \int_0^{q^{-n}} y_{m,q^{-n}}^2(x) d_q x,$$

then we have

$$\int_0^{q^{-n}} f^2(x) d_q x = \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,q^{-n}}^2} \left\{ \int_0^{q^{-n}} f(x) y_{m,q^{-n}}(x) d_q x \right\}^2. \quad (3.5)$$

which is called the Parseval equality (see [10]).

Now, let us define the nondecreasing step function $\varrho_{q^{-n}}$ on $[0, \infty)$ by

$$\varrho_{q^{-n}}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_{m,q^{-n}} < 0} \frac{1}{\alpha_{m,q^{-n}}^2}, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_{m,q^{-n}} < \lambda} \frac{1}{\alpha_{m,q^{-n}}^2} & \text{for } \lambda \geq 0. \end{cases}$$

Then equality (3.5) can be written as

$$\int_0^{q^{-n}} f^2(x) d_q x = \int_{-\infty}^{\infty} F^2(\lambda) d\varrho_{q^{-n}}(\lambda), \quad (3.6)$$

where

$$F(\lambda) = \int_0^{q^{-n}} f(x) y(x, \lambda) d_q x.$$

We will show that the Parseval equality for the problem (3.1), (3.2) can be obtained from (3.6) by letting $n \rightarrow \infty$. For this purpose, we shall prove a lemma.

Lemma 3.1. *For any positive κ , there is a positive constant $\Upsilon = \Upsilon(\kappa)$ not depending on q^{-n} such that*

$$\int_{-\kappa}^{\kappa} \{\varrho_{q^{-n}}(\lambda)\} = \sum_{-\kappa \leq \lambda_{m, q^{-n}} < \kappa} \frac{1}{\alpha_{m, q^{-n}}^2} = \varrho_{q^{-n}}(\kappa) - \varrho_{q^{-n}}(-\kappa) < \Upsilon. \quad (3.7)$$

Proof. Let $\sin \beta \neq 0$. Since $\phi(x, \lambda)$ is continuous at zero, by condition $\phi(0, \lambda) = \sin \beta$, there is a positive number k and near by 0 such that

$$\frac{1}{k} \left(\int_0^k \phi(x, \lambda) d_q x \right)^2 > \frac{1}{2} \sin^2 \beta. \quad (3.8)$$

Let us define $f_k(x)$ by

$$f_k(x) = \begin{cases} \frac{1}{k}, & 0 \leq x < k \\ 0, & x > k. \end{cases}$$

From (3.6) and (3.8), we get

$$\begin{aligned} \int_0^k f_k^2(x) d_q x &= \frac{1}{k} = \int_{-\infty}^{\infty} \left(\frac{1}{k} \int_0^k \phi(x, \lambda) d_q x \right)^2 d\varrho_{q^{-n}}(\lambda) \\ &\geq \int_{-\kappa}^{\kappa} \left(\frac{1}{k} \int_0^k \phi(x, \lambda) d_q x \right)^2 d\varrho_{q^{-n}}(\lambda) \\ &> \frac{1}{2} \sin^2 \beta \{ \varrho_{q^{-n}}(\kappa) - \varrho_{q^{-n}}(-\kappa) \}, \end{aligned}$$

which proves the inequality (3.7).

If $\sin \beta = 0$, then we define the function $f_k(x)$ by the formula

$$f_k(x) = \begin{cases} \frac{1}{k^2}, & 0 \leq x < k \\ 0, & x > k. \end{cases}$$

So, we obtain the inequality (3.7) by applying the Parseval equality. \square

Now, we recall that the following well-known theorems of Helly's.

Theorem 3.2 ([24]). *Let $(w_n)_{n \in \mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing functions on a finite interval $a \leq \lambda \leq b$. Then there exists a subsequence $(w_{n_k})_{k \in \mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k \rightarrow \infty} w_{n_k}(\lambda) = w(\lambda), \quad a \leq \lambda \leq b.$$

Theorem 3.3 ([24]). *Assume $(w_n)_{n \in \mathbb{N}}$ is a real, uniformly bounded, sequence of nondecreasing functions on a finite interval $a \leq \lambda \leq b$, and suppose*

$$\lim_{n \rightarrow \infty} w_n(\lambda) = w(\lambda), \quad a \leq \lambda \leq b.$$

If f is any continuous function on $a \leq \lambda \leq b$, then

$$\lim_{n \rightarrow \infty} \int_a^b f(\lambda) dw_n(\lambda) = \int_a^b f(\lambda) dw(\lambda).$$

Let ϱ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L^2_\varrho(-\infty, \infty)$ the Hilbert space of all functions $f : (-\infty, \infty) \rightarrow \mathbb{R}$ which are measurable with respect to the Lebesgue-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda) d\varrho(\lambda) < \infty,$$

with the inner product

$$(f, g)_\varrho := \int_{-\infty}^{\infty} f(\lambda) g(\lambda) d\varrho(\lambda).$$

The main result of this paper is the following theorem.

Theorem 3.4. *For the q -Sturm-Liouville problem (3.1)-(3.2), there exists a nondecreasing function $\varrho(\lambda)$ on $-\infty < \lambda < \infty$ with the following properties:*

(i) *If $f(\cdot)$ is a real valued function and $f(\cdot) \in L^2_q(0, q^{-n})$, then there exists a function $F \in L^2_\varrho(0, \infty)$ such that*

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left\{ F(\lambda) - \int_0^{q^{-n}} f(x) y(x, \lambda) d_q x \right\} d\varrho(\lambda) = 0, \quad (3.9)$$

and the Parseval equality

$$\int_0^{\infty} f^2(x) d_q x = \int_{-\infty}^{\infty} F^2(\lambda) d\varrho(\lambda) \quad (3.10)$$

holds.

(ii) *The integral*

$$\int_{-\infty}^{\infty} F(\lambda) y(x, \lambda) d\varrho(\lambda),$$

converges to f in $L^2_q(0, \infty)$. That is,

$$\lim_{n \rightarrow \infty} \int_0^{q^{-n}} \left\{ f(x) - \int_{-n}^n F(\lambda) y(x, \lambda) d\varrho(\lambda) \right\}^2 d_q x = 0.$$

We note that the function ϱ is called a spectral function for the boundary value problem (3.1)-(3.2).

Proof. Assume that the function $f_\xi(x)$ satisfies the following conditions.

- 1) $f_\xi(x)$ vanishes outside the interval $[0, q^{-\xi}]$, $q^{-\xi} < q^{-n}$.
- 2) The functions $f_\xi(x)$ and $D_q f_\xi(x)$ are q -regular at zero
- 3) $f_\xi(x)$ satisfies the boundary condition (3.3).

If we apply to $f_\xi(x)$ the Parseval equality (3.6), we obtain

$$\int_0^{q^{-\xi}} f_\xi^2(x) d_q x = \int_{-\infty}^{\infty} F^2(\lambda) d\rho(\lambda) \quad (3.11)$$

where

$$F(\lambda) = \int_0^{q^{-\xi}} f_\xi(x) y(x, \lambda) d_q x. \quad (3.12)$$

Since $y(x, \lambda)$ satisfies the system (3.1), we see that

$$y(x, \lambda) = \frac{1}{\lambda} \left[-\frac{1}{q} D_{q^{-1}} D_q y(x, \lambda) + u(x) y(x, \lambda) \right].$$

By (3.12), we get

$$F_n(\lambda) = \frac{1}{\lambda} \int_0^{q^{-n}} f_\xi(x) \left[-\frac{1}{q} D_{q^{-1}} D_q y(x, \lambda) + u(x) y(x, \lambda) \right] d_q x.$$

Since $f_\xi(x)$ vanishes in a neighborhood of the point q^{-n} and $f_\xi(x)$ and $y(x, \lambda)$ satisfy the boundary condition (3.3), we obtain

$$F_n(\lambda) = \frac{1}{\lambda} \int_0^{q^{-n}} y(x, \lambda) \left[-\frac{1}{q} D_{q^{-1}} D_q f_\xi(x) + u(x) f_\xi(x) \right] d_q x,$$

by q -integration by parts.

For any finite $\kappa > 0$, using (3.6), we have

$$\begin{aligned} & \int_{|\lambda| > \kappa} F_n^2(\lambda) d\rho_{q^{-n}}(\lambda) \\ & \leq \frac{1}{\kappa^2} \int_{|\lambda| > \kappa} \left\{ \int_0^{q^{-n}} \left[y(x, \lambda) \left[-\frac{1}{q} D_{q^{-1}} D_q f_\xi(x) + u(x) f_\xi(x) \right] \right] d_q x \right\}^2 d\rho_{q^{-n}}(\lambda) \\ & \leq \frac{1}{\kappa^2} \int_{-\infty}^{\infty} \left\{ \int_0^{q^{-n}} \left[y(x, \lambda) \left[-\frac{1}{q} D_{q^{-1}} D_q f_\xi(x) + u(x) f_\xi(x) \right] \right] d_q x \right\}^2 d\rho_{q^{-n}}(\lambda) \\ & = \frac{1}{\kappa^2} \int_0^{q^{-\xi}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_\xi(x) + u(x) f_\xi(x) \right]^2 d_q x. \end{aligned}$$

From (3.11), we see that

$$\begin{aligned} & \left| \int_0^{q^{-\xi}} f_{\xi}^2(x) d_q x - \int_{-\kappa}^{\kappa} F_n^2(\lambda) d\rho_{q^{-n}}(\lambda) \right| \\ & < \frac{1}{\kappa^2} \int_0^{q^{-\xi}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_{\xi}(x) + u(x) f_{\xi}(x) \right]^2 d_q x. \quad (3.13) \end{aligned}$$

By Lemma 3.1, the set $\{\rho_{q^{-n}}(\lambda)\}$ is bounded. Using Theorems 3.2 and 3.3, we can find a sequence $\{q^{-n_k}\}$ such that the function $\rho_{q^{-n_k}}(\lambda)$ converges to a monotone function $\rho(\lambda)$. Passing to the limit with respect to $\{q^{-n_k}\}$ in (3.13), we get

$$\begin{aligned} & \left| \int_0^{q^{-\xi}} f_{\xi}^2(x) d_q x - \int_{-\kappa}^{\kappa} F_n^2(\lambda) d\rho(\lambda) \right| \\ & < \frac{1}{\kappa^2} \int_0^{q^{-\xi}} \left[-\frac{1}{q} D_{q^{-1}} f_{\xi}(x) + y(x) f_{\xi}(x) \right]^2 d_q x. \end{aligned}$$

Hence, letting $\kappa \rightarrow \infty$, we obtain

$$\int_0^{q^{-\xi}} f_{\xi}^2(x) d_q x = \int_{-\infty}^{\infty} F_n^2(\lambda) d\rho(\lambda).$$

Now, let f be an arbitrary real valued function on $L_q^2(0, \infty)$. It is known that there exists a sequence of function $\{f_{\xi}(x)\}$ satisfying the condition 1-3 and such that

$$\lim_{\xi \rightarrow \infty} \int_0^{\infty} (f(x) - f_{\xi}(x))^2 d_q x = 0.$$

Let

$$F_{\xi}(\lambda) = \int_0^{\infty} f_{\xi}(x) y(x, \lambda) d_q x.$$

Then, we have

$$\int_0^{\infty} f_{\xi}^2(x) d_q x = \int_{-\infty}^{\infty} F_{\xi}^2(\lambda) d\rho(\lambda).$$

Since

$$\int_0^{\infty} (f_{\xi_1}(x) - f_{\xi_2}(x))^2 d_q x \rightarrow 0 \text{ as } \xi_1, \xi_2 \rightarrow \infty,$$

we have

$$\int_{-\infty}^{\infty} (F_{\xi_1}(\lambda) - F_{\xi_2}(\lambda))^2 d\rho(\lambda) = \int_0^{\infty} (f_{\xi_1}(x) - f_{\xi_2}(x))^2 d_q x \rightarrow 0$$

as $\xi_1, \xi_2 \rightarrow \infty$. Consequently, there is a limit function F which satisfies

$$\int_0^{\infty} f^2(x) d_q x = \int_{-\infty}^{\infty} F^2(\lambda) d\rho(\lambda),$$

by the completeness of the space $L^2_\varrho(-\infty, \infty)$.

Our next goal is to show that the function

$$K_\xi(\lambda) = \int_0^{q^{-\xi}} f(x) y(x, \lambda) d_q x$$

converges as $\xi \rightarrow \infty$ to F in the metric of space $L^2_\varrho(-\infty, \infty)$. Let g be another function in $L^2_q(0, \infty)$. By a similar arguments, $G(\lambda)$ be defined by g . It is clear that

$$\int_0^\infty (f(x) - g(x))^2 d_q x = \int_{-\infty}^\infty \{F(\lambda) - G(\lambda)\}^2 d\varrho(\lambda).$$

Set

$$g(x) = \begin{cases} f(x), & x \in [0, q^{-\xi}] \\ 0, & x \in (q^{-\xi}, \infty) \end{cases}.$$

Then we have

$$\int_{-\infty}^\infty \{F(\lambda) - K_\xi(\lambda)\}^2 d\varrho(\lambda) = \int_{q^{-\xi}}^\infty f^2(x) d_q x \rightarrow 0 \quad (\xi \rightarrow \infty),$$

which proves that K_ξ converges to F in $L^2_\varrho(-\infty, \infty)$ as $\xi \rightarrow \infty$. This proves (i).

Now, we will prove (ii). Suppose that the functions $f(\cdot)$, $g(\cdot) \in L^2_q(0, q^{-n})$, and $F(\lambda)$ and $G(\lambda)$ are their Fourier transforms. Then $F \mp G$ are transforms of $f \mp g$. Consequently, by (3.10), we have

$$\begin{aligned} \int_0^\infty [f(x) + g(x)]^2 d_q x &= \int_{-\infty}^\infty (F(\lambda) + G(\lambda))^2 d\varrho(\lambda), \\ \int_0^\infty [f(x) - g(x)]^2 d_q x &= \int_{-\infty}^\infty (F(\lambda) - G(\lambda))^2 d\varrho(\lambda). \end{aligned}$$

Subtracting the second relation from the first, we get

$$\int_0^\infty f(x) g(x) d_q x = \int_{-\infty}^\infty F(\lambda) G(\lambda) d\varrho(\lambda) \quad (3.14)$$

which is called the generalized Parseval equality.

Set

$$f_\tau(x) = \int_{-\tau}^\tau F(\lambda) y(x, \lambda) d\varrho(\lambda),$$

where F is the function defined in (3.9). Let $g(\cdot)$ be a vector-function which equals zero outside the finite interval $[0, q^{-\mu}]$. Thus, we obtain

$$\begin{aligned} \int_0^{q^{-\mu}} f_\tau(x) g(x) d_q x &= \int_0^{q^{-\mu}} \left\{ \int_{-\tau}^\tau F(\lambda) y(x, \lambda) d_\varrho(\lambda) \right\} g(x) d_q x \\ &= \int_{-\tau}^\tau F(\lambda) \left\{ \int_0^{q^{-\mu}} y(x, \lambda) g(x) d_q x \right\} d_\varrho(\lambda) \\ &= \int_{-\tau}^\tau F(\lambda) G(\lambda) d_\varrho(\lambda). \end{aligned} \quad (3.15)$$

From (3.14), we get

$$\int_0^\infty f(x) g(x) d_q x = \int_{-\infty}^\infty F(\lambda) G(\lambda) d_\varrho(\lambda). \quad (3.16)$$

Subtracting (3.15) and (3.16), we have

$$\int_0^\infty (f(x) - f_\tau(x)) g(x) d_q x = \int_{|\lambda| > \tau} F(\lambda) G(\lambda) d_\varrho(\lambda).$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left| \int_0^\infty (f(x) - f_\tau(x)) g(x) d_q x \right|^2 &\leq \int_{|\lambda| > \tau} F^2(\lambda) d_\varrho(\lambda) \int_{|\lambda| > \tau} G^2(\lambda) d_\varrho(\lambda) \\ &\leq \int_{|\lambda| > \tau} F^2(\lambda) d_\varrho(\lambda) \int_{-\infty}^\infty G^2(\lambda) d_\varrho(\lambda). \end{aligned}$$

Apply this inequality to the function

$$g(x) = \begin{cases} f_\tau(x) - f(x), & x \in [0, q^{-\mu}] \\ 0, & x \in (q^{-\mu}, \infty) \end{cases},$$

we get

$$\int_0^{q^{-\mu}} (f(x) - f_\tau(x))^2 d_q x \leq \int_{|\lambda| > \tau} F^2(\lambda) d_\varrho(\lambda).$$

Letting $\tau \rightarrow \infty$ yields the desired result since the right side does not depend on μ . \square

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