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Centers, Commutators and Abelianization of Crossed Squares

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Abstract. In this work we will introduce center of cat^2-group and related structures in the sense of [7]. By using the equivalence between crossed squares and cat^2-groups we will introduce the center of crossed square.

Keywords: Crossed square, cat^2- group, center, commutator.

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1. Introduction

In [6], J.-L. Loday introduced the notion of crossed squares and it was generalized in [2] to crossed n-cubes, which can be thought as higher dimensional crossed modules [8], and are algebraic models for homotopy $(n+1)$ types. Crossed 1-cubes are equivalent to crossed modules.

In the sense of [6], *a* crossed square of groups, which is equivalent to crossed 2-cubes, is a commutative diagram of groups;

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

together with actions of P on L, M and N (and hence actions of M on L and N via μ) and of N on L and M via υ) and a function $h : M \times N \longrightarrow L$ such that the following axioms are satisfied:

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(i) The maps λ, λ' preserve the actions of P. The maps μ, ν and $\mu\lambda = \lambda'\nu$ are crossed modules with the given actions..
(ii) $h(mm', n) = h({}^{m}m', {}^{m}n)h(m, n)$,
(iii) $h(m, nn') = h(m, n)h({}^{n}m, {}^{n}n')$,
(iv) $\lambda h(m, n) = m\,{}^{n}m^{-1}$,
(v) $\lambda' h(m, n) = {}^{m}n n^{-1}$,
(vi) $h(\lambda(l), n) = l\,{}^{n}l^{-1}$,
(vii) $h(m, \lambda'(l)) = {}^{m}l l^{-1}$,
(viii) $h({}^{p}m, {}^{p}n) = {}^{p}h(m, n)$,
for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

A morphism of crossed square is a quadtuple $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ of group homomorphisms

$$\phi_1 : L_1 \longrightarrow L_2 \qquad \phi_3 : N_1 \longrightarrow N_2$$

$$\phi_2 : M_1 \longrightarrow M_2 \qquad \phi_4 : P_1 \longrightarrow P_2$$

which makes the diagram

$$\begin{tikzcd}
& L_2 \arrow[rr, "\lambda_2"] \arrow[dd, "\lambda'_2"] & & M_2 \arrow[dd, "\mu_2"] \\
L_1 \arrow[ur, "\phi_1"] \arrow[rr, "\lambda_1"] \arrow[dd, "\lambda'_1"] & & M_1 \arrow[ur, "\phi_2"] \arrow[dd, "\mu_1"] & \\
& N_2 \arrow[rr, "\nu_2"] & & P_2 \\
N_1 \arrow[ur, "\phi_3"] \arrow[rr, "\nu_1"] & & P_1 \arrow[ur, "\phi_4"] &
\end{tikzcd}$$

commutative such that $\phi_1(h(m, n)) = h(\phi_2(m_1), \phi_3(n_1))$, for all $m_1 \in M_1$, $n_1 \in N_1$ and each of the homomorphisms ϕ_1, ϕ_2, ϕ_3 is ϕ_4-equivariant.

Consequently, we have the category of crossed squares which we denote by $\mathbf{Crs^2}$.

Example 1.1. Let P be a group. Let M, N be normal subgroups of P, λ, and λ' are inclusions and $L = M \cap N$, with h being the conjugation map is a crossed square.

$$\begin{tikzcd}
M \cap N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"'] & P
\end{tikzcd}$$

Recall from [6] that a *cat^1-group* is a triple (G, s, t) consisting of a group G and endomorphisms s, the source map, and t, the target map of G, satisfying $st = t$, $ts = s$ and $[kers, kert] = 1$.

There exists a natural equivalence between crossed modules and cat^1-groups. The notion was also generalized to cat^n-groups in the same work. For $n = 2$ we get the cat^2-groups. Namely, a cat^2-group is a system (G, s_1, t_1, s_2, t_2) consists of a group G and endomorphisms s_1, t_1, s_2, t_2 such that (G, s_1, t_1) and (G, s_2, t_2) are cat^2-groups and $s_i s_j = s_j s_i,\ t_i t_j = t_j t_i,\ s_i t_j = t_j s_i$, for $i, j = 1, 2,\ i \neq j$. A morphism between cat^2-groups (G, s_1, t_1, s_2, t_2) and $(G', s'_1, t'_1, s'_2, t'_2)$ is a homomorphism $\alpha : G \to G'$ such that $\alpha s_i = s'_i \alpha$ and $\alpha t_i = t'_i \alpha$, for all $i = 1, 2$. Consequently, we have the category of cat^2-groups which will be denoted here by $\mathbf{Cat^2}$. In [6], it was proved that the categories $\mathbf{Crs^2}$ and $\mathbf{Cat^2}$ are naturally equivalent.

Proposition 1.2. [6] *The category $\mathbf{Crs^2}$ of crossed squares is naturally equivalent to the category $\mathbf{Cat^2}$ of cat^2-groups.*

Proof. Here, we only give a sketch of the proof. Details can be found in [6] . Given a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

Since λ' and μ are crossed modules, we have the corresponding semi-direct products $L \rtimes N$ and $M \rtimes P$ which give rise to the crossed module $\partial : L \rtimes N \to M \rtimes P$ with the action of $M \rtimes P$ on $L \rtimes N$ defined by

$$^{(m,p)}(l, n) = (^{m}(^{p}l)h(m, ^{p}n), ^{p}n),$$

for all $(m, n) \in M \rtimes N$ and $(l, n) \in L \rtimes N$. Using this action, we thus form its associated cat^1-group with big group $(L \rtimes N) \rtimes (M \rtimes P)$ and induced endomorphisms s_1, t_1, s_2, t_2. That is, if (L, M, N, P) is a crossed square, then the corresponding cat^2 -group is $((L \rtimes N) \rtimes (M \rtimes P), s_1, t_1, s_2, t_2)$.

Conversely, let (G, s_1, t_1, s_2, t_2) be a cat^2-group. The cat^1-group (G, s_1, t_1) and (G, s_2, t_2) give rise to the crossed square

$$\begin{tikzcd} kers_1 \cap kers_2 \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & Ims_1 \cap kers_2 \arrow[d, "\mu"] \\ kers_1 \cap Ims_2 \arrow[r, "\nu"'] & Ims_1 \cap Ims_2 \end{tikzcd}$$

where $\lambda, \upsilon = t_1|$; $\lambda', \mu = t_2|$ and each morphism is a crossed module with the action defined by conjugation in G and the h-map is the commutator in G. $\square$

Finally, we recall from [7] the notion of a category of interest in the sense of [3]. Let $\mathbb{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group laws and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:
(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(b) the group operations (written additively : $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \setminus \{+\}$, $\Omega_1' = \Omega_1 \setminus \{-\}$ and assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$. Assume further that $\Omega_0 = \{0\}$;
(c) for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;
(d) for each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x * y)$.

If C is an object of $\mathbb{C}$ and $x_1, x_2, x_3 \in C$:

Axiom 1 $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$.

Axiom 2 For each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that

$$(x_1 * x_2)\overline{*}x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1,$$
$$(x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in Ω_2'.

A category of groups with operations satisfying Axiom 1 and Axiom 2 is called a *category of interest*.

Example 1.3. The category of cat^n-groups is a category of interest, for each positive integer n.

2. Centers, Commutators and Abelianizations in $\mathbf{Cat^2}$

In this section, first we recall from [7] and [3], the notions of centers, commutators and abelianizations in categories of interest and consider the particular case, namely cat^2-groups as a preparation for the next section to define the center and consequently the commutator and the abelianization of a crossed square.

Definition 2.1. Let $\mathbf{C}$ be a category of interest and $A \in ob(\mathbf{C})$. Then the center of A is the ideal $Z(A) = \{z \in A : \text{for all} \quad a \in A, \text{a unary operation of } A \text{ and} \quad * \in \Omega_2', \quad a + z = z + a, a + \omega(z) = \omega(z) + a, a * z = 0\}$

Example 2.2. Given a group G and consider the cat^2-group (G, s_1, t_1, s_2, t_2) where $s_i = t_i = id_G$, for $i = 1, 2$. Then the center of (G, s_1, t_1, s_2, t_2) is the ideal $(Z(G), s_1|, t_1|, s_2|, t_2|)$ where $Z(G)$ is the center of G in the category of groups. Let G be a group and $a, b \in G$ such that $ab = ba$. Consider the maps $s : G \to G$, $t : G \to G$ defined by $s(x) = axa^{-1}$ and $t(x) = bxb^{-1}$.

Example 2.3. A direct calculation shows that (G, s_1, t_1, s_2, t_2) is a cat^2 -group where $s_1, s_2 = s$ and $t_1, t_2 = t$. Then the center of (G, s_1, t_1, s_2, t_2) is the normal subobject

$$\begin{aligned} Z(G) &= \{z \in G : z+g=g+z,\ s_i(z)+g=g+s_i(z),\\ &\quad t_i(z)+g=g+t_i(z),\ i=1,2,\ for\ all\ g\in G\}\\ &= \{z\in G : z+g=g+z,\ a+z-a+g=g+a+z-a,\\ &\quad b+z-b+g=g+b+z-b\}\\ &= \{z\in G : g+z=z+g, [-a+g+a,z]=0=[-b+g+b,z]\}. \end{aligned}$$

Definition 2.4. A cat^2-group (G, s_1, t_1, s_2, t_2) is abelian if it coincides with its center.

Consequently, a cat^2-group is abelian if and only if G is an abelian object in the category of groups.

3. Application

In this section, using the natural equivalence between $\mathbf{Cat^2}$ and $\mathbf{Crs^2}$, we will carry the center of corresponding cat^2-group of a crossed square to $\mathbf{Crs^2}$ to get its center. This gives to obtain abelian objects in $\mathbf{Crs^2}$.

Definition 3.1. Let (G_1, s_1, t_1) and (G_0, s_0, t_0) be cat^1-groups. Then we say that (G_0, s_0, t_0) has an action on (G_1, s_1, t_1) iff there exist a split extension

$$(G_1, s_1, t_1) \rightarrowtail (H, s_H t_H) \overset{\curvearrowleft}{\twoheadrightarrow} (G_0, s_0, t_0)$$

of cat^1-groups.

As indicated in [4], we may represent an action of (G_0, s_0, t_0) on (G_1, s_1, t_1) by a homomorphism from (G_0, s_0, t_0) to the actor of (G_1, s_1, t_1).

Definition 3.2. A crossed module of cat^1-groups is a cat^1-group homomorphism

$$\partial : (G_1, s_1, t_1) \to (G_0, s_0, t_0)$$

with an action of (G_0, s_0, t_0) on (G_1, s_1, t_1) such that
1) $\partial(c_0.c_1) = c_0 + \partial(c_1) - c_0$,
2) $\partial(c_1).c_1' = c_1 + c_1' - c_1$,
for all $c_0 \in C_0$ and $c_1 \in C_1$.

The equivalence between the category of crossed module of cat^1-groups and the category of cat^2-groups, leads us to define the center of crossed module of cat^1-groups which is a particular case of the general definition given in [1].

The center of $\partial : (G_1, s_1, t_1) \to (G_0, s_0, t_0)$ is the crossed ideal $\partial| : Z_1 \to Z_0$ where $Z_1 = \{z_1 \in G_1 : g_0.z_1 = z_1, g_0.s_1(z_1) = s_1(z_1), g_0.t_1(z_1) = t_1(z_1),\ \text{for all } g_0 \in G_0\}$
and
$Z_0 = \{z_0 \in G_0 : z_0.g_1 = g_1, s_0(z_0).g_1 = g_1, t_0(z_0).g_1 = g_1 \text{for all} g_1 \in G_1\} \cap Z(G_0, s_0, t_0)$ with inducedsource and target morphisms.

Now, consider the crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

and the corresponding crossed module

$$\begin{array}{rcl} \partial : (L \rtimes N, s_1, t_1) & \to & (M \rtimes P, s_0, t_0) \\ (l,n) & \mapsto & (\lambda(l), \nu(n)). \end{array}$$

The center of $\partial : (L \rtimes N, s_1, t_1) \to (M \rtimes P, s_0, t_0)$ is $\partial| : Z_1 \to Z_0$ where $Z_1 = \{(z_l, z_n) \in L \rtimes N : (m,p).(z_l, z_n) = (z_l, z_n), (m,p).s_1(z_l, z_n) = s_1(z_l, z_n), (m,p).t_1(z_l, z_n) = t_1(z_l, z_n)\}$ and $Z_0 = \{(z_m, z_p) \in M \rtimes P : (z_m, z_p).(l,n) = (l,n), w(z_m, z_p).(l,n) = s_1(l,n), w = s_0, t_0, \text{for all}(l,n) \in L \rtimes N\} \cap Z(M \rtimes P, s_0, t_0)$.

By a direct calculation we have;

$$\begin{array}{rcl} Z_1 & = & \{(z_l, z_n) \in L \rtimes N : m.(p, z_l) + h(m, p.z_n) = z_l, \\ & & p.z_n = z_n, h(m, p.z_n) = 0, h(m, p.(\partial(z_l)z_n)) = 0\} \end{array}$$

and

$$\begin{array}{rcl} Z_0 & = & \{(z_m, z_p) \in M \rtimes P : z_m.(z_p.l) + h(z_m, z_p.n) = l, z_p.n = n, z_p.l = l, \partial(z_m).l = l, \\ & & (\partial(z_m)z_p).n = n\} \cap \{(z_m, z_p) : m = z_p.m, p + z_p = z_p + p, m + z_m = z_m + m, p.z_m = z_m\}. \end{array}$$

So, we get the crossed square

$$\begin{tikzcd} Z_L \arrow[r, "\lambda|"] \arrow[d, "\lambda'|"'] & Z_M \arrow[d, "\mu|"] \\ Z_N \arrow[r, "\nu|"'] & Z_P \end{tikzcd}$$

where $Z_L \cong Ker(s_1) \cap Z_1$, $Z_N \cong Im(s_1) \cap Z_1$, $Z_M \cong Ker(s_0) \cap Z_0$ and $Z_P \cong Im(s_0) \cap Z_0$.

Proposition 3.3.

$$\begin{tikzcd} Z_L \arrow[r, "\lambda|"] \arrow[d, "\lambda'|"'] & Z_M \arrow[d, "\mu|"] \\ Z_N \arrow[r, "\nu|"'] & Z_P \end{tikzcd}$$

is the center of

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

in the sense of [5].

Proof. It is clear from the natural equivalences and the definition of center of a crossed module in a modified category of interest given in [1]. □

Definition 3.4. A crossed square is called abelian if it coincides with its center.

A direct checking shows that, an abelian crossed square is an abelian object in the category of crossed squares with a categorical view point.

4. Conclusion

The work is a preparation for the different kinds of extensions and (co)homology of crossed squares. In further wworks the related concepts can be introduced and many gadgets in group theory can be appliedto crossed squares as their generalizations.

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