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RBF-Chebychev direct method for solving variational problems

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Abstract. This paper establishes a direct method for solving variational problems via a set of Radial basis functions (RBFs) with Gauss-Chebyshev collocation centers. The method consist of reducing a variational problem into a mathematical programming problem. The authors use some optimization techniques to solve the reduced problem. Accuracy and stability of the multiquadric, Gaussian and inverse multiquadric RBF is examined and compared by some numerical experiments.

Keywords: Radial basis functions- direct method - variational problems - Gauss-Chebyshev centers.

2000 Mathematics subject classification: 49M37, Secondary 49M05.

1. Introduction

The calculus of variations investigates methods that permit maximal or minimal values of functionals[6]. Functional minimization problems naturally occur in engineering, mechanics, economics and so forth where minimization of functionals such as Lagrangian, potential and total energy, etc. give the laws governing the system behavior[1].

A direct method converts the variational problem into a mathematical programming problem. The idea of direct methods for solving variational problems consists in replacing the problem of searching for the extremum (usually, for the point of stationary) of a functional in the function space by a problem of searching for a

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Table 1. Some popular RBFs.

Name	$\phi(r,c)$
Gaussian (GA)	$\phi(r) = exp(-cr^2)$
Multiquadric (MQ)	$\phi(r) = \sqrt{c^2 + r^2}$
Inverse multiquadric (IMQ)	$\phi(r) = (c^2 + r^2)^{-1/2}$

solution in a finite set of parameters. The Ritz method [8] usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods for solving variational problems.

In the last decades , Radial Basis Functios (RBFs) have been widely applied in different fields such as multivariate function interpolation and approximation, neural networks and solution of differential and integral equations. The essence of RBF interpolation is to use linear combination of $\phi(\|x - x_j\|)$, which is radially symmetric about its center x_j to approximate the unknown function $y(x)$. Common choices for RBFs are listed in Table 1 where $r = \|x - x_j\|$ and $\|.\|$ denotes the Euclidean norm, and x_js are the centers of RBFs and c is a positive shape parameter which controls the flatness (width) of the basis function.

Franke and Schaback [7] used RBFs for solving PDEs , Golbabai and Seifollahi [10] used RBFs for solving the second kind integral equations (IEs). Moreover, it has been observed that certain class of RBFs such as multiqudric (MQ) and Gaussian(GA) RBFs exhibit superior error convergence properties [19] , [20]. Some advantages of the RBF-Based methods are their ease of implementation , accuracy and efficiency , which are the reasons that this technique is getting popular.

In this paper we use GA-RBFs, MQ-RBFs and Inverse multiquadric(IMQ)-RBFs for solving variational problems . For this purpose, we introduce the RBFs opertaional matrices of differentiation and the product of two RBF vectors. The RBFs operational matrices can be used to solve problems in fields analysis ,calculus of variations and optimal control. The method consists of reducing the variational problem into a linear system of algebraic equations by first expanding the candidate functions as a linear combination of RBFs with unknown coefficients. Gauss-Legendre quadrature is used to approximate integration. Finally we evaluate the coefficients of RBF in such a way that the necessary conditions for extermination of a multivariate function are imposed.

2. Main Results

Approximation of a function $y(x)$ may be expanded as

$$y(x) \cong y^N(x) = \sum_{i=0}^{N} a_i \varphi(\|x - x_i\|), \tag{2.1}$$

where N is the number of RBFs and $a_i s$ are the unknown coefficients and x_i are the Gauss-Chebyshev centers.

proposition 1. Let $y^N(x) = a^T \phi(x)$ where $a^T = [a_1, a_2, ...a_N]$ and $\phi(x) = [\varphi_1(x), ..., \varphi_N(x)]^T$, $\phi_i(x) = (c^2 + \|x - x_i\|^2)^{\frac{-1}{2}}$ for $i = 1, ..., N$. Then there exist symmetric matrices A_{IMQ}, L_{IMQ} and vector $\overline{\phi}(x)$ such that:

$$a)\ (y^N(x))^2 = a^T A_{IMQ} a, \tag{2.2}$$

where $A_{IMQij} = \varphi_i(x)\varphi_j(x)$.

$$b)\ \frac{d}{dx} y^N(x) = a^T \overline{\phi(x)}, \tag{2.3}$$

where $\overline{\phi}(x) = [\overline{\varphi_1}(x), ..., \overline{\varphi_N}(x)]^T$ and $\overline{\varphi_i}(x) = -(x - x_i)(c^2 + \|x - x_i\|^2)^{\frac{-3}{2}}$.

$$c)\ (\frac{d}{dx} y^N(x))^2 = a^T L_{IMQ} a, \tag{2.4}$$

where $L_{IMQij} = (x - x_i)(x - x_j)(c^2 + \|x - x_i\|^2)^{\frac{-3}{2}}(c^2 + \|x - x_j\|^2)^{\frac{-3}{2}}$.

Proof.

The properties can be verified by the fact that if $a,\ b \in R^N$ then one can see that $(a^T b)^2 = a^T b(a^T b) = a^T b b^T a = a^T A$, where $A_{ij} = b_i b_j$ is a symmetric matrix.

Similar properties can be verified for MQ and GA RBFs.

2.1. **RBF-Chebyshev direct method.**

Consider the problem of finding the extremum of the functional

$$min \int_{x_a}^{x_b} L(x, y, \dot{y}) dx, \tag{2.5}$$

where $y : [x_a\ , x_b] \to R^N$ is a suffiecitly smooth function, $y(x_a)\ = A,\ y(x_b) =\ B$ which two points A and B in R^N are given.

Tonelli's theorem [4] says under assumptions

(1) Coercivity of rank $r > 1$ for certain constants $\alpha > 0$ and β we have

$$L(t, x, v) \geq \alpha \|v\|^r + \beta.\ for\ every\ (t, x, v) \epsilon [a\ , b] \times R^N \times R^N$$

(2) Convexity: L_{vv} is everywhere positive semidefinite ($L_{vv} \geq 0$),

there exists a solution of the basic problem 2.5 relative to the class of absolutly continuous (A.C.) functions.

Euler-Lagrange equation leading to a necessary condition for y(x) on extremizing $J(y)$ in the form of the well known Euler-Lagrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}(\frac{\partial L}{\partial \dot{y}}) = 0, \tag{2.6}$$

which can be easily solved only in a few classes. Thus it is more practical to use numerical and direct methods such as the Ritz method.

Using a Gauss-Legendre quadrature with m nodes for integrating yields:

$$\int_{x_a}^{x_b} L(x, y, \dot{y})dx \cong \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, a^T \phi(\|x_k - x_i\|), a^T \frac{d\phi}{dx}(\|x_k - x_i\|)) \tag{2.7}$$

where $x_k = \frac{x_b+x_a}{2} + \frac{x_b-x_a}{2} z_k$, $z_k s$ and $w_k s$ for $k = 1, ..., m$ are nodes and coefficients of Gauss-Legendre quadrature and RBF centers $x_i,\ i = 1, ...N$ are Chebyshev nodes.

Finally the variational problem (2.5) will be reduced to a constrained optimization problem of finding

$$\min J(a) = \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, a^T \phi(\|x_k - x_i)\|, a^T \frac{d\phi}{dx}(\|x_k - x_i\|)), \tag{2.8}$$

subject to

$$a^T \phi(\|x_a - x_i\|) = A,\ a^T \phi(\|x_b - x_i\|) = B. \tag{2.9}$$

To solve the optimization problem (2.8)-(2.9) our choice is to use Lagrange multipliers method , so we establish Lagrangian

$$J^*(a) = J(a) + \lambda_1(a^T \phi(\|x_a - x_i\|) - A) + \lambda_2(a^T \phi(\|x_b - x_i\|) - B), \tag{2.10}$$

then we should solve the following algebraic system in order to calculate coefficients $a^T = [a_0, a_1, ..., a_N]$

$$\frac{\partial J^*}{\partial a} = 0, \tag{2.11}$$

$$\frac{\partial J^*}{\partial \lambda_i} = 0\ i = 1, 2. \tag{2.12}$$

Generally speaking, it is a complicated problem to solve the system of equations 2.11 and 2.12. This problem is simplified if the function $J(y)$ is quadratic in the unknown function and its derivatives. For this case the equations 2.10 are linear in $a_i s$. It is easy to see that if Lagrangian in problem 2.5 is convex then objective function 2.8 will be convex function of unknown parameter $a = [a_1, ... , a_N]^T$.

TABLE 2. results of example 1.

	$m = 32$ [9] $\gamma = 0.00$	$m = 64$ $\gamma = 0.25$	RH [18]	Bern[5]	GA	MQ	IMQ-RBF
(% Δ)	0.042	0.005	1.5812	0.0316	0.0043	0.0096	0.9704
RMS	–	–	0.00580	0.000438	0.000312	0.0000355	0.000258

2.2. **Solving by BiCGSTAB.** For solving a general linear system two different classes are available: first, direct methods, including gaussian elimination and its variants such as LU, Cholskey. Second, iterative methods such as bicg, bicgstab, gmres, lsqr.

Sometimes solving the linear systems by a direct routin such as *linsolve* in MATLAB returns a warning:"Matrix is close to singular or badly scaled, Results may be inaccurate", so there is warning that the results may be unreliable. To avoid this warning we suggest to use an iterative routin such as bicgstab to solve the system.

the biconjugate gradient stabilized method (BiCGSTAB) is an iterative method. It is a variant of the BiCG (biconjugate gradient method) and has faster and smoother convergence than the original BiCG as well as other variants such as the CGS (conjugate gradient square method)

2.3. **Chebyshev nodes.** The Chebychev nodes are important in approximation theory because they form a particularly good set of nodes for polynomial interpolation. For a given natural n, chebyshev nodes in the interval $[-1,\ 1]$ are $x_k = cos(\frac{2k-1}{2n}\pi),\ k = 1, ..., n$. These are the roots of the chebychev polynomial of the first kind of degree n. For nodes over an arbitrary interval $[a,\ b]$ an affine transform can be used

$$x_k = \frac{x_b + x_a}{2} + \frac{x_b - x_a}{2} cos(\frac{2k-1}{2n}\pi),\ k = 1, \ldots, n. \tag{2.13}$$

3. EXAMPLES

Example 3.1. [18], [9], [5] Find the extremal of the following functional

$$J(y) = \int_0^1 y'^2(x) + xy'(x) + y(x)^2 dx. \tag{3.1}$$

The boundary conditions are

$$y(0) = 0\ ,\ y(1) = \frac{1}{4}. \tag{3.2}$$

The exact solution of the problem is $y^*(x) = \frac{1}{2} + c_1 e^x + c_2 e^{-x}$, $c_1 = \frac{2-e}{4(e^2-1)}$, $c_2 = \frac{e-2e^2}{4(e^2-1)}$.

We used RBF direct method to solve the problem (3.1)-(3.2) with both equally spaced and Chebychev node centers. Using the described procedure in sec. 2.1 the

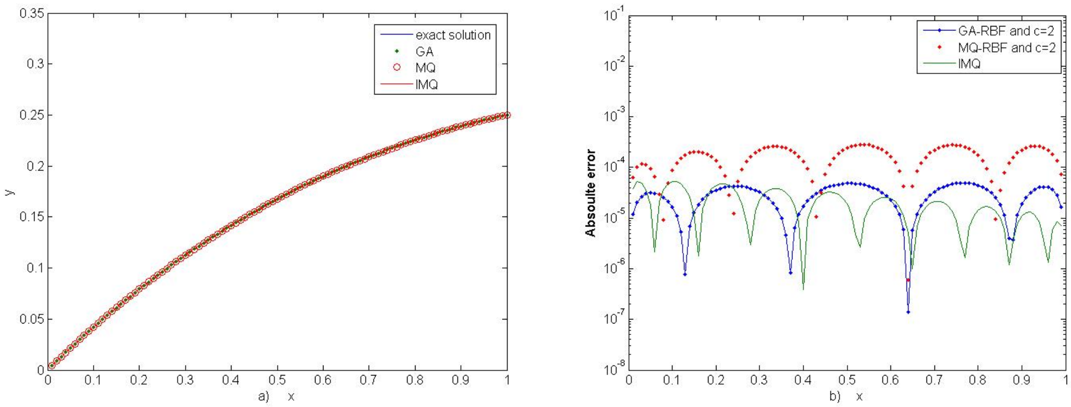

Figure 1. a) exact solution b) Absolute error vs. variable x for some RBFs.

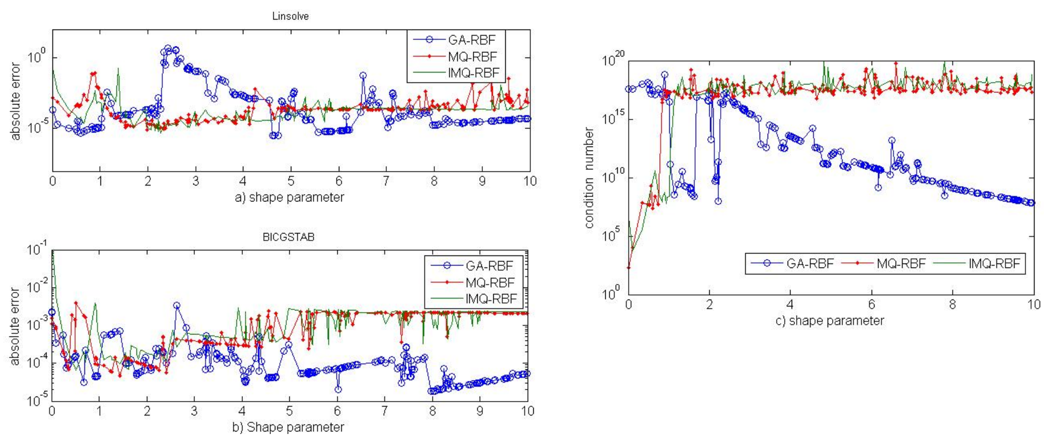

Figure 2. a) Absolute error vs. shape parameter. b)Absolute error when Using Bicgstab method. c) Condition number vs. shape parameter.

problem (3.1)-(3.2) will be reduced to an $(N+2)\times(N+2)$ algebraic linear system can be solved via MATLAB software.

The numerical solution for $c = 1$ and $N = 36$ (number of centers) is compared with [5] , [9] and [18] (Table 3) by two criteria: relative error and RMS:

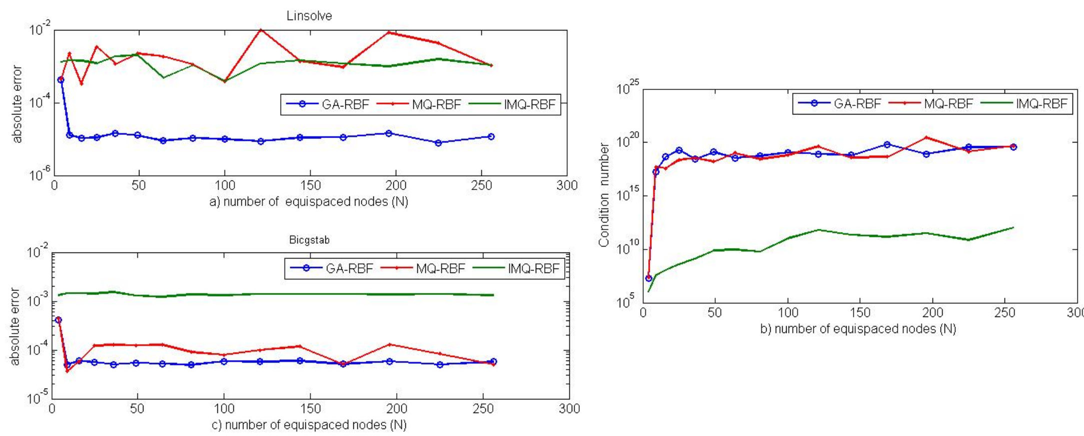

FIGURE 3. a) Absolute error vs. Number of equispaced nodes(N) . b) Condition number vs. number of equispaced nodes(N). c) Absolute error when solving by BiCGSTAB

Relative error $\triangle(\%)$ for approximate solutions obtained here defined as:

$$\Delta(\%) = \sum_{j=0}^{m-1} \|y_j^e - y_j^a\|^2 \sum_{j=0}^{m-1} \|y_j^e\|^2 \times 100\% \tag{3.3}$$

where y_j^e and y_j^a denote values of state y at the jth center point of the analysed interval for respectively the exact solution and the approximate solution for different approximation degrees m and different values of parameter γ and different RBF bass (GA, MQ and IMQ) with $N = 36$ chebyshev center points.

The formula for the root mean square error RMS_{error} is given by

$$RMS_{error} = \sqrt{\frac{1}{M} \sum_{j=1}^{M} \|y_j^e - y_j^a\|^2} \tag{3.4}$$

where M is the number of selected test nodes and y_j^e, y_j^a are as previous.

Figure 2 (a ,b) compares absolute error of GA, MQ and IMQ RBFs for various random shape parameteres between 0 and 10. It is obvious from the figure that solving the system by command linsolve makes the results more fluctuating than when solved by BiCGSTAB.

Figure 2 (c) compares condition number of the algebraic system in GA, MQ and IMQ cases. Obviously increasing in shape paramter yields decreasing in condition number of GA-RBF case. It is obvious from figure 3 that while solution of problem in IMQ-RBF case has desirable condition number and high absolute error , the solution in the GA-RBF case behaves more accurate. While using BiCGSTAB helps to MQ-RBF make better approximations.

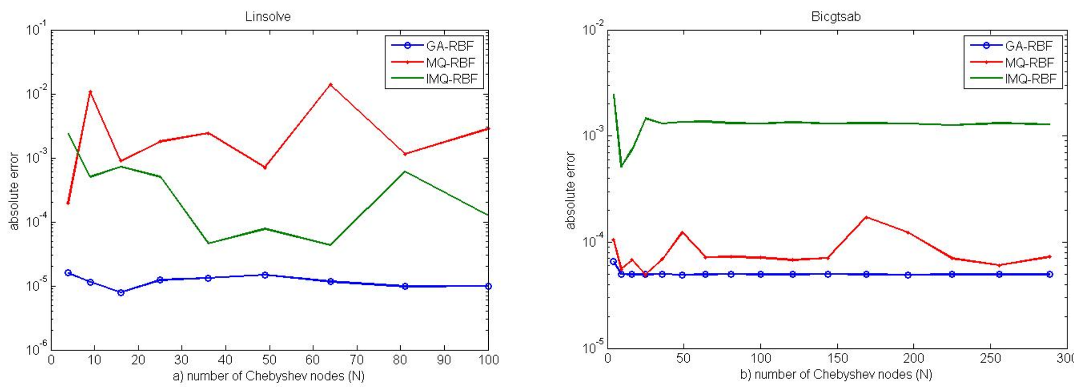

Figure 4. a) Absolute error vs. Number of chebyshev nodes(N) .
b) Absolute error when solving by BiCGSTAB.

Roundoff error is a fundamental part of any numerical computation. Numerically stable methods attempt to control roundoff error and stop it from accumulating too quickly. Figures 3 , 4 demonstrates that increasing number of equally/Chebyshev spaced nodes does not yield uncontrolled increase in error, so it can be inferred that the proposed method is numerically stable. There is not significant difference in behaviour of solution in GA-RBF and MQ-RBF cases.

4. Conclusion

In this paper some properties of RBFs including IMQ-RBF were considered. Then a method that reduces the variational problem to a constrained optimization problem was described.

Numerically with the RMS and relative error criterias, it was proved that GA-RBF and MQ-RBF method accuracy is better than RH and Bernstein method.

The reduced linear algebraic system was proposed to solve by some kind of preconditioner including BICGSTAB that enhances the reliability of the results so there is no more WARNING from MATLAB to solve the corresponding linear system. To improve the efficiency of the method, Chebyshev nodes were used as center points of RBFs. Finally numerical stability of the method was examined and verified by some numerical experiments.

References

[1] Om P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*. J. Math. Anal. Appl. 272 (2002) pp. 368-379.

[2] E. Babolian, R. Mokhtari, M. Salmani, *Using direct method for solving variational problems via triangular orthogonal functions*, Applied mathematics and computation, 191 (2007) pp. 206-217.

[3] C. F. Chen , C. H. Hsiao ,*A walsh series direct method for solving variational problems*, Journal of the Franklin Institue, 300 (1975) pp. 265-1373.

[4] Clarke, F., Functional Analysis, Calculus of Variations and Optimal Control, Springer, (2013).

[5] S. Dixit, V. K. Singh, A. K. Singh, O. P. Singh, *Bernstein Direct Method for Solving Variational Problems* , International Mathematical Forum, 5(2010), 2351 - 2370.

[6] L. Elsgolts ,*Differentiarl equations and Calculus of Variations*, Mir, Moscow. 1977 ,translated from the Russian by G. Yankovsky.

[7] C. Franke , R. Shaback, *Solving Partial Differential Equations by collocation using Radial Basis Dunctions*, Applied Mathematics and Computation, Volume 93, Issue 1 (1998) , pp. 73-82.

[8] I. M. Gelfand, S.V. Fomin, *Calculus of Variations, Prentice Hall*, Englewood cliffs, NJ, 1963.

[9] W. Glabisz, *Direct walsh-wavelet packet method for variational problems* , Applied Mathematics and Computation , 159 (2004), pp. 769-781.

[10] A. Golbabai , S. Seifollahi, *Numerical Solution of the Second kind Integral Equations using Radial Basis Functions networks*, Applied Mathematics and Computation, 174 (2006), pp. 877-883.

[11] I.R. Horng, J. H. Chou,*Shifted Chebyshev direct method for solving variational problems*, International Journal of systems science , 16 (1985) , pp. 855-861.

[12] C. H. Hsio, *Haar wavelet direct method for solving variational problems*, Mathematics and Computers in Simulation, 64 (2004), pp. 569-585.

[13] C. Hwang, Y.P. Shih , *Optimal control of delay systems via block-pulse functions*, Journal of optimization theory and applications, (1985) pp. 101-112.

[14] C. Hwang , Y. P. Shih , *Laguerre series direct method for variational problems*, Journal of optimization theory and applications, (1983) pp. 143-149.

[15] M. Maleki and M. Mashali-Firouzi, *A numerical solution of problems in calculus of variation using direct method and nonclassical parametrization*,J. of Computational and Applied MAthematics., 234 (2010), pp. 1364-1373.

[16] M. Razzaghi and S. Yousefi, *Legendre wavelets direct method for variational problems*, Mathematics and Computers in Simulation, 53 (2000), pp. 185-192.

[17] M. Razzaghi , S. Yousefi , *Sine-Cosine wavelets operational matrix of integration and its applications in the calculus of variations*, International Journal of systems sciences, 2002 , Volume 2, number 10, pp. 805-810.

[18] M. Razzaghi, Y. Ordokhani , *An application of rationalized Haar functions for variational problems*, Applied Mathematics and Computation, 122 (2001) , pp. 353-364.

[19] Sarra, Scott A. "A numerical study of the accuracy and stability of symmetric and asymmetric RBF collocation methods for hyperbolic PDEs." Numerical Methods for Partial Differential Equations 24.2 (2008): 670-686.

[20] Sarra, Scott A. "Radial basis function approximation methods with extended precision floating point arithmetic." Engineering Analysis with Boundary Elements 35.1 (2011): 68-76.