Caspian Journal of Mathematical Sciences(CJMS)
University of Mazandaran, Iran
http://www.cjms.umz.ac.ir
ISSN: 1735-0611

CJMS. **1**(1)(2012), 43-47

AN APPLICATION OF FIBONACCI NUMBERS INTO INFINITE TOEPLITZ MATRICES

E.E. KARA[1] AND M. BASARIR[2,*]

ABSTRACT. The main purpose of this paper is to define a new regular matrix by using Fibonacci numbers and to investigate its matrix domain in the classical sequence spaces ℓ_p, ℓ_∞, c and c_0, where $1 \leq p < \infty$.

Keywords: Regular matrix transformations, Sequence spaces.
Classification 2000 MSC: 11B39, 46B45

1. INTRODUCTION

The Fibonacci numbers has been discussed in so many articles and books (see [1-3]). The Fibonacci numbers are the sequence of numbers $\{f_n\}_{n=1}^{\infty}$ defined by the linear recurrence equations

$$f_0 = 0 \text{ and } f_1 = 1,\ f_n = f_{n-1} + f_{n-2};\ n \geq 2\ .$$

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converges to the golden section which is important in sciences and arts. Also, some basic properties of Fibonacci numbers are given as follows (see [2,3]):

$$\sum_{k=1}^{n} f_n = f_{n+2} - 1;\ \ n \geq 1,$$

$$\sum_{k=1}^{n} f_n^2 = f_n f_{n+1};\ \ n \geq 1,$$

$$\sum_{k=1}^{\infty} 1/f_k \text{ converges.}$$

Received: 16 Dec 2011
AAccepted: 17 Jan 2012
* *Corresponding author:basarir@sakarya.edu.tr.*

In this paper, we define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$ by using Fibonacci numbers f_n and introduce some new sequence spaces related to matrix domain of F in the sequence spaces ℓ_p, ℓ_∞, c and c_0, where $1 \leq p < \infty$.

2. The Fibonacci Matrix F and Some New Sequence Spaces

Let ω be the space of real sequences. Any vector subspace of ω is called as a sequence space. By ℓ_∞, c, c and ℓ_p $(1 \leq p < \infty)$, we denote the sets of all bounded, convergent, null sequences and p-absolutely convergent series, respectively.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}^0 = \{1, 2, 3, ...\}$. We write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=1}^{\infty}$. Then, we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{A_n(x)\}_{n=1}^{\infty}$, the $A-$transform of x, is in Y; where

$$A_n(x) = \sum_{n=1}^{\infty} a_{nk} x_k \quad \left(n \in \mathbb{N}^0\right). \tag{2.1}$$

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right side of (2.1) converges for each $n \in \mathbb{N}^0$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$. The matrix domain X_A of an infinite matrix A in sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{2.2}$$

which is a sequence space. The approach is constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors, see for instance [4-6]. A sequence space X is called FK *space* if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{R}$ $(n \in \mathbb{N}^0)$, where $\mathbb{R}$ denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}^0$. A BK *space* is a normed FK *space*, that is, a BK *space* is a Banach space with continuous coordinates. The space ℓ_p $(1 \leq p < \infty)$ is BK space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$ and c_0, c and ℓ_∞ are BK spaces with $\|x\|_\infty = \sup_k |x_k|$.

The following lemma (known as the Toeplitz Theorem) contains necessary and sufficient condition for regularity of a matrix.

Lemma 2.1 ([7, Lemma 2.2])**.** *Matrix* $A = (a_{nk})_{n,k=1}^{\infty}$ *is regular if and only if the following three conditions hold:*

(1) There exists $M > 0$ *such that for every* $n = 1, 2, ...$ *the following inequality holds:*

$$\sum_{k=1}^{\infty} |a_{nk}| \leq M;$$

(2) $\lim_{n\to\infty} a_{nk} = 0$ *for every* $k = 1, 2, ...;$

(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$

Let (q_k) be a sequence of positive numbers and

$$Q_n = \sum_{k=1}^{n} q_k; \quad (n \geq 1).$$

Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n} & (1 \leq k \leq n) \\ 0 & (k > n), \end{cases}$$

It is known that the Riesz matrix R^q is a Toeplitz matrix if and only if $Q_n \to \infty$, as $n \to \infty$ [8].

Now, we define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$ by

$$f_{nk} = \begin{cases} \frac{f_k^2}{f_n f_{n+1}} & (1 \leq k \leq n) \\ 0 & (k > n). \end{cases}$$

that is,

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 & 0 & \dots \\ \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & \frac{9}{15} & 0 & 0 & \dots \\ \frac{1}{40} & \frac{1}{40} & \frac{4}{40} & \frac{9}{40} & \frac{25}{40} & 0 & .. \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is obvious that the matrix F is a triangle, that is $f_{nn} \neq 0$ and $f_{nk} = 0$ for $k > n$ ($n = 1, 2, 3, ...$). Also, it follows by Lemma 2.1 that the method F is regular.

Throughout, let X denotes any of the classical sequence spaces ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$. Then the Fibonacci sequence space $X(F)$ is defined by

$$X(F) = \{x = (x_k) \in w : y = (y_k) \in X\},$$

where the sequence $y = (y_k)$ is the F-transform of a sequence $x = (x_k)$, i.e,

$$y_k = F_k(x) = \frac{1}{f_k f_{k+1}} \sum_{j=1}^{k} f_j^2 x_j \text{ for all } k \in \mathbb{N}^0. \tag{2.3}$$

With the notation (2.2), we can redefine the space $X(F)$ as the matrix domain of the triangle F in the space X, that is

$$X(F) = X_F. \tag{2.4}$$

Theorem 2.2. *The space $X(F)$ is a BK space with the norm given by*

$$\|x\|_{X(F)} = \|F(x)\|_X = \|y\|_X = \begin{cases} \sup_k |y_k| & \text{if } X \in \{\ell_\infty, c, c_0\} \\ \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p} & \text{if } X = \ell_p;\ 1 \leq p < \infty. \end{cases} \tag{2.5}$$

Proof. Since the matrix F is a triangle, we have the result by (2.5) and Theorem 4.3.12 of Wilansky [9, p.63].

Theorem 2.3. *The sequence space $X(F)$ is isometrically isomorphic to the space X, that is, $X(F) \cong X$.*

Proof. To prove this, we should show the existence of an isometric isomorphism between the spaces $X(F)$ and X. Consider the transformation T defined, with the notation of (2.3), from $X(F)$ to X by $x \to y = Tx$. Then, we have $Tx = y = F(x) \in X$ for every $x \in X(F)$. Also, the linearity of T is trivial. Further, it is easy to see that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Furthermore, let $y = (y_k) \in X$ be given and then define the sequence $x = (x_k)$ by

$$x_k = \frac{f_{k+1}}{f_k} y_k - \frac{f_{k-1}}{f_k} y_{k-1}; \quad (k \in \mathbb{N}^0). \tag{2.6}$$

Then, by using (2.3) and (2.6), we have for every $k \in \mathbb{N}^0$ that

$$\begin{aligned} F_k(x) &= \frac{1}{f_k f_{k+1}} \sum_{j=1}^{k} f_j^2 x_j \\ &= \frac{1}{f_k f_{k+1}} \sum_{j=1}^{k} f_j \left(f_{j+1} y_j - f_{j-1} y_{j-1} \right) \\ &= y_{k.} \end{aligned}$$

This shows that $F(x) = y$ and, since $y \in X$ we obtain that $F(x) \in X$. Thus, we deduce that $x \in X(F)$ and $Tx = y$. Hence, T is surjective.

Moreover, for any $x \in X(F)$, we have by (2.5) of Theorem 2.2 that

$$\|T(x)\|_X = \|y\|_X = \|F(x)\|_X = \|x\|_{X(F)}$$

which shows that T is norm preserving. Hence, T is isometry. Consequently, the spaces $X(F)$ and X are isometrically isomorphic. This concludes the proof.

Lemma 2.4. *Let $\{f_n\}_{n=1}^{\infty}$ be Fibonacci numbers sequences. Then we have*

$$\sup_k \left(f_k^2 \sum_{n=k}^{\infty} \frac{1}{f_n f_{n+1}} \right) < \infty.$$

Proof. This is a consequence of [5, Lemma 4.11] since the sequence $\left(\frac{1}{f_n f_{n+1}} \right)$ is in ℓ_1.

Theorem 2.5. *The inclusion $X \subset X(F)$ holds.*

It is clear that the inclusions $c_0 \subset c_0(F)$ and $c \subset c(F)$ since the matrix F is a regular matrix.

Now, let $x = (x_k) \in \ell_\infty$. Then, there is a constant $K > 0$ such that $|x_k| \le K$ for all $k \in \mathbb{N}^0$. Thus, we have for every $n \in \mathbb{N}^0$ that

$$\begin{aligned} |F_n(x)| &\le \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2 \, |x_k| \\ &\le \frac{K}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2 = K \end{aligned}$$

which shows that $F(x) \in \ell_\infty$. Therefore, we deduce that $x = (x_k) \in \ell_\infty$ implies $x = (x_k) \in \ell_\infty(F)$.

Finally, let $1 < p < \infty$ and take any $x = (x_k) \in \ell_p$. Then, for every $n \in \mathbb{N}^0$, by applying the Hölder's inequality

$$\begin{aligned} |F_n(x)|^p &\leq \left[\sum_{k=1}^{n} \frac{f_k^2}{f_n f_{n+1}} |x_k|\right]^p \qquad (2.1) \\ &\leq \left[\sum_{k=1}^{n} \frac{f_k^2}{f_n f_{n+1}} |x_k|\right]^p \left[\sum_{k=1}^{n} \frac{f_k^2}{f_n f_{n+1}}\right]^{p-1} \\ &= \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2 |x_k|^p . \end{aligned}$$

Thus, by the fact in (2.7) that

$$\begin{aligned} \sum_{n=1}^{\infty} |F_n(x)|^p &\leq \sum_{n=1}^{\infty} \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2 |x_k|^p \\ &= \sum_{n=1}^{\infty} |x_k|^p f_k^2 \sum_{n=k}^{\infty} \frac{1}{f_k f_{k+1}} \end{aligned}$$

and hence

$$\|x\|_{\ell_p(F)}^p \leq K \sum_{n=1}^{\infty} |x_k|^p = K \|x\|_{\ell_p}^p , \tag{2.8}$$

where $K = \sup_k [f_k^2 \sum_{n=k}^{\infty} 1/f_n f_{n+1}] < \infty$ by Lemma 2.4. This shows that $x \in \ell_p(F)$. Hence, we deduce that the inclusion $\ell_p \subset \ell_p(F)$ holds for $1 < p < \infty$. By the similar discussions, it may be easily proved that the inequality (2.8) also holds in the case $p = 1$ and so we omit the detail. This completes the proof.

Now, we show that the converse of Theorem 2.5 is also true. In [4], Mursaleen and Noman have defined the matrix $\Lambda = (\lambda_{nk})$ by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} & (1 \leq k \leq n) \\ 0 & (k > n) \end{cases} ,$$

where $\lambda = (\lambda_k)$ is strictly increasing sequence of positive reals which tends to infinity, that is

$$0 < \lambda_1 < \lambda_2 < ... \quad \text{and} \quad \lambda_k \to \infty \text{ as } k \to \infty.$$

If we take $\lambda_n = f_n f_{n+1}$ $(n \in \mathbb{N}^0)$ in the matrix Λ, then $F = \Lambda$ $(\lambda_k - \lambda_{k-1} = f_k^2$ for every $k \in \mathbb{N}^0)$. Thus, we conclude the following result:

Theorem 2.6. *The inclusion $X(F) \subset X$ holds.*

Proof. Let $\lambda_n = f_n f_{n+1}$ $(n \in N^0)$. Then we have that

$$\begin{aligned} \lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} &= \lim_{n\to\infty} \frac{f_{n+2}}{f_n} \\ &= 1 + \lim_{n\to\infty} \frac{f_{n+1}}{f_n} \\ &= 1 + \varphi > 1, \end{aligned}$$

where φ is the golden section, that is, $\varphi = (1+\sqrt{5})/2$. Hence, from [4, Corollary 4.7] and [5, Corollary 4.19], $X(F) \subset X$ for $X \in \{\ell_p, c, c_0\}$, where $1 \leq p \leq \infty$. This completes the proof.

Corollary 2.7. *The equality $X(F) = X$ holds.*

Proof. This is an immediate consequence of Theorems 2.5 and 2.6.

Remark 2.8. We may note that if we put $q_k = f_k^2$ for all k, then the matrix F is the special case of the matrix R^q($Q_n = \sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$).

3. Conclusion

Consequently, the Fibonacci matrix F is a regular matrix which is a special case of Riesz matrix. Also, $X_F = X$ for $X \in \{\ell_p, c, c_0\}$, where $1 \leq p \leq \infty$. One can ask if X is an arbitrary normed or paranormed space, does the equality $X_F = X$ holds? Morever, One can study the spectrum of the matrix F in the classical sequence spaces.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, 2001.

[2] S. Vajda, *Fibonacci and Lucas Numbers, and Golden Section: Theory and Applications*, Chichester: Ellis Horword, 1989.

[3] D. Kalman, R. Mena, The Fibonacci numbers-Exposed, Mathematics Magazine, 76(3), June 2003.

[4] M. Mursaleen, A.K. Noman, On the space of λ-convergent and bounded sequences, Thai J. Math., 8(2) (2010) 311-329.

[5] M. Mursaleen, A.K. Noman, On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I, FILOMAT, 25(2) (2011) 33-51.

[6] B. Altay, F. Başar, M. Mursaleen, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I, Infomations Science, 176(2006) 1450-1462.

[7] E. Kováč, On φ-convergence and φ-density, Math. Slovaca, 55(3) (2005) 329-351.

[8] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, Istanbul, 2011.

[9] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies 85, Elsevier Science Publishers, Amsterdam: New York: Oxford, 1984.

[1]*Department of Mathematics, Bilecik University,11210, Bilecik, Turkey.*
E-mail address: `emrah.kara@bilecik.edu.tr`

[2]*Department of Mathematics, Sakarya University, 54187, Sakarya, Turkey.*
E-mail address: `basarir@sakarya.edu.tr`