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# More connectedness in topological spaces

Shyamapada Modak [1] and Md. Monirul Islam [2]
[1,2] Department of Mathematics
University of Gour Banga
P.O. Mokdumpur; Malda 732103
West Bengal; India
[1] Email: spmodak2000@yahoo.co.in
[2] Email: moni.math007@gmail.com

Abstract. In this paper, we introduce the concept of $S_\beta$-connectedness which lies between semi-connectedness and connectedness. We also characterize this type of connectedness and discuss its relationships with the various types of connectedness from the literature. We further consider the components of this type of connectedness and its properties.

Keywords: Semi-connected, $\beta$-connected, Preconnected, Hyperconnected; $S_\beta$-connected.

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## 1. **Introduction**

The study of connectedness via generalized open sets is not a new idea in topological spaces. The authors Pipitone and Russo in 1975 have introduced and studied semi-connectedness [5] via Levine's semi-open sets [10], Popa in 1987 has studied preconnectedness [6] via preopen sets [11],

[1]Corresponding author: Shyamapada Modak (Email: spmodak2000@yahoo.co.in)
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Aho, Nieminen, Popa, Noiri, and Jafari have studied semipreconnectedness (= $\beta$-connectedness) [2, 4, 7], via semipreopen [3] (= $\beta$-open [1]) sets, and some other connectedness have been introduced by Modak and Noiri in [12, 13, 14, 15] and Noorena and Khan in [16].

In this paper, we are able to place a new type of connected space which is called $S_\beta$-connected between the semi-connected spaces and connected spaces. We also characterize $S_\beta$-connected space and interrelate it with the various types of connected space. For this job we will further study $S_\beta$-open sets [9] and its properties.

## 2. Preliminaries

Let $(X, \tau)$ be a topological space, then we will denote '$Cl(A)$' and '$Int(A)$' the 'closure of $A$' and 'interior of $A$' respectively. A topological space $(X, \tau)$ will be shortly denoted by $X$.

**Definition 2.1.** A subset $A$ of $X$ is said to be semi-open [10] (resp. $\beta$-open [1]) if $A \subseteq Cl(Int(A))$ (resp. $A \subseteq Cl(Int(Cl(A)))$).

The complement of a semi-open (resp., $\beta$-open) set is said to be semi-closed (resp., $\beta$-closed).

**Definition 2.2.** [9] A semi-open subset $A$ of a topological space $X$ is said to be $S_\beta$-open if for each $x \in A$ there exists a $\beta$-closed set $F$ such that $x \in F \subseteq A$. A subset $B$ of a topological space $X$ is $S_\beta$-closed, if $X \setminus B$ is $S_\beta$-open.

The family of all semi-open (resp., $\beta$-open, $S_\beta$-open) subsets of a space (means topological space ) $X$ is denoted by $SO(X)$ (resp., $\beta O(X)$, $S_\beta O(X)$). The family of all semi-closed (resp., $\beta$-closed, $S_\beta$-closed) subsets of a space $X$ is denoted by $SC(X)$ (resp., $\beta C(X)$, $S_\beta C(X)$).

**Definition 2.3.** [9] A point $x \in X$ is said to be an $S_\beta$-interior point of $A$, if there exists an $S_\beta$-open set $U$ containing $x$ such that $x \in U \subseteq A$. The set of all $S_\beta$-interior points of $A$ is said to be $S_\beta$-interior of $A$ and it is denoted by $S_\beta Int(A)$.

**Definition 2.4.** [9] The intersection of all $S_\beta$-closed sets containing $F$ is called the $S_\beta$-closure of $F$ and it is denoted by $S_\beta Cl(F)$.

The examples below illustrate that $S_\beta$-open sets are obtained from open sets but this collection is neither a sub-collection of open sets nor a collection containing the collection of open sets (see following example). Thus, the study of $S_\beta$-open sets is meaningful.

**Example 2.5.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $SO(X) = \beta O(X) = S_\beta O(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. Here $\{a, c, d\} \in S_\beta O(X)$ but $\{a, c, d\} \notin \tau$.

**Example 2.6.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Then $SO(X) = \beta O(X) = \tau$ and $S_\beta O(X) = \{\emptyset, \{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Here $\{a\} \in \tau$ but $\{a\} \notin S_\beta O(X)$.

However the following result holds:

**Lemma 2.7.** *Let $A$ be a subset of a topological space $X$. If $A$ is both open and closed, then $A$ is both $S_\beta$-open and $S_\beta$-closed.*

*Proof.* Let $A$ be a subset of $X$ which is both open and closed in $X$. Then $Int(A) = A = Cl(A)$ and $A \subseteq Cl(Int(Cl(A)))$. Thus $A$ is $\beta$-open. Let $B = X \setminus A$, then $B$ is $\beta$-closed. Since $A$ is both open and closed, $B$ is also open and closed. So $B \subseteq Cl(Int(Cl(B)))$ and thus $B$ is $\beta$-open. Hence $A = X \setminus B$ is $\beta$-closed. Thus $A$ and $B$ are both $\beta$-open and $\beta$-closed in $X$. Again $A$ and $B$ are semi-open as $A$ and $B$ both are open. Therefore for the semi-open set $A$, and for each $x \in A$ there exists a $\beta$-closed set $A$ such that $x \in A \subseteq A$. Thus $A$ is $S_\beta$-open. By similar argument $B = X \setminus A$ is $S_\beta$-open. Hence the result. □

The converse of the above Lemma 2.7 need not hold in general:

**Example 2.8.** From Example 2.5, $\{b\}$ and {a,c,d} are both $S_\beta$-open and $S_\beta$-closed but they are neither both open and closed.

**Definition 2.9.** [11] A topological space $X$ is said to

(1) Locally indiscrete if every open subset of $X$ is closed.
(2) Hyperconnected if every nonempty open subset of $X$ is dense.

**Definition 2.10.** [9] Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is $S_\beta$-continuous at a point $x \in X$, if for each open set $V$ of $Y$ containing $f(x)$, there exists an $S_\beta$-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq V$. If $f$ is $S_\beta$-continuous at every point $x$ of $X$, then it is called $S_\beta$-continuous.

**Proposition 2.11.** [9] *Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is $S_\beta$-continuous if and only if the inverse image of every open set in $Y$ is $S_\beta$-open in $X$.*

## 3. $S_\beta$-connected spaces

**Definition 3.1.** Two nonempty subsets $A$ and $B$ of a topological space $X$ are said to $S_\beta$-separated if $A \cap S_\beta Cl(B) = \emptyset = S_\beta Cl(A) \cap B$.

It is obvious that two $S_\beta$-separated sets are disjoint.

If $A$ and $B$ are two $S_\beta$-separated sets in $X$ with $\emptyset \neq C \subseteq A$ and $\emptyset \neq D \subseteq B$, then $C$ and $D$ are also $S_\beta$-separated sets in $X$.

Next example proves the existence of $S_\beta$-separated sets:

**Example 3.2.** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $SO(X) = \beta O(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. We take $A = \{a, c, d\}$ and $B = \{b\}$. Then $A, B \in S_\beta O(X)$ and also $A, B \in S_\beta C(X)$. Thus $S_\beta Cl(A) = A$ and $S_\beta Cl(B) = B$. Therefore $A \cap S_\beta Cl(B) = S_\beta Cl(A) \cap B = A \cap B = \emptyset$. Thus $A$ and $B$ are two $S_\beta$-separated subsets of $X$.

**Definition 3.3.** A subset $S$ of a topological space $X$ is said to be $S_\beta$-connected if $S$ is not the union of two $S_\beta$-separated sets in $X$.

Below we give an example of $S_\beta$-connected space:

**Example 3.4.** Let $X = \{a, b, c, d\}$ , $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$, then $SO(X) = \beta O(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, X\}$ and $S_\beta O(X) = \{\emptyset, X\}$. We cannot express $X$ as the union of two $S_\beta$-separated sets in $X$ and so $X$ is $S_\beta$-connected.

**Theorem 3.5.** *For a topological space $X$, following hold:*

(1) *If $X$ is semi-connected, then $X$ is $S_\beta$-connected.*
(2) *If $X$ is $S_\beta$-connected, then $X$ is connected.*

*Proof.* 1. Proof is obvious from the fact that $S_\beta O(X) \subseteq SO(X)$.
2. See later. □

The following example shows that the converse of the Theorem 3.5 (2) is not true.

**Example 3.6.** $(i)$ Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $SO(X) = \beta O(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. Only subsets of $X$ which are both open and closed in $X$ are $\emptyset$ and $X$. Next $\{a, c, d\}$ and $\{b\}$ are both $S_\beta$-open and $S_\beta$-closed in $X$. Hence $X$ is not $S_\beta$-connected.

$(ii)$ Suppose that every $S_\beta$-connected space is semi-connected. As every semi-connected space is connected, then every $S_\beta$-connected space is connected. This contradicts the above. Thus every $S_\beta$-connected space is not necessarily a semi-connected space.

For further counterexamples against the Figure 1, see [4].

The following diagram, borrowed from [4] , highlights the relationship between several types of connectedness.

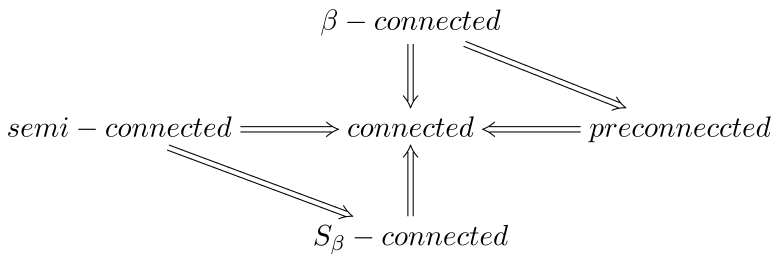


Figure 1

The inverse implications in the above diagram are not true in general.

**Theorem 3.7.** *A topological space $X$ is $S_\beta$-connected if and only if $X$ cannot be expressed as the union of two disjoint nonempty $S_\beta$-open subsets of $X$.*

*Proof.* Let $X$ be $S_\beta$-connected. Let $U$ and $V$ be two disjoint nonempty $S_\beta$-open subsets of $X$ such that $X = U \cup V$. Put $A = X \setminus U$ and $B = X \setminus V$. Then $A$ and $B$ are $S_\beta$-closed in $X$. Thus $A \cap S_\beta Cl(B) = \emptyset = S_\beta Cl(A) \cap B$ and $X = A \cup B$. Thus $X$ is not $S_\beta$-connected. This is a contradiction. Thus $X$ cannot be expressed as the union of two disjoint nonempty $S_\beta$-open subsets of $X$.

Conversely suppose that the condition holds. Suppose $X = A \cup B$, $A \neq \emptyset \neq B$ and $A \cap S_\beta Cl(B) = \emptyset = S_\beta Cl(A) \cap B$. Put $U = X \setminus S_\beta Cl(A)$ and $V = X \setminus S_\beta Cl(B)$. Then $U$ and $V$ are nonempty $S_\beta$-open sets, and $U \cup V = (X \setminus S_\beta Cl(A)) \cup (X \setminus S_\beta Cl(B)) = X \setminus (S_\beta Cl(A) \bigcap S_\beta Cl(B)) \subseteq X$. This implies that $X = U \cup V$. Again $U \cap V = (X \setminus S_\beta Cl(A)) \cap (X \setminus S_\beta Cl(B)) = X \setminus (S_\beta Cl(A) \bigcup S_\beta Cl(B)) = \emptyset$ (since $X = A \cup B$ ). This is a contradiction. Thus $X$ is $S_\beta$-connected. □

**Corollary 3.8.** *Let $\tau_1$ and $\tau_2$ be two topologies on $X$ with $\tau_2 \subseteq \tau_1$. If $(X, \tau_1)$ is $S_\beta$-connected, then $(X, \tau_2)$ is also $S_\beta$-connected.*

The following example shows that the converse of Corollary 3.8 is not true:

**Example 3.9.** Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a, b\}, X\}$. Then $\tau_2 \subseteq \tau_1$. Now in $(X, \tau_1)$, $SO(X) = S_\beta O(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, and in $(X, \tau_2)$, $SO(X) = \tau_2$, $S_\beta O(X) = \{\emptyset, X\}$. So $(X, \tau_2)$ is $S_\beta$-connected, but $(X, \tau_1)$ is not $S_\beta$-connected.

**Theorem 3.10.** *For a topological space $X$, the following statements are equivalent:*

(1) *$X$ is $S_\beta$-connected;*
(2) *The only subsets of $X$ which are both $S_\beta$-open and $S_\beta$-closed are $X$ and the empty set;*

(3) *$X$ cannot be expressed as the union of two disjoint nonempty $S_\beta$-open sets;*
(4) *There is no nonconstant onto $S_\beta$-continuous function from $X$ to a discrete space which contains more than one point.*

*Proof.* (1) $\Rightarrow$ (2):

Let $X$ be $S_\beta$-connected. Let $A \subseteq X$ which is both $S_\beta$-open and $S_\beta$-closed in $X$. Then $B = X \setminus A$ is also $S_\beta$-open and $S_\beta$-closed in $X$. Since $A$, $B$ are $S_\beta$-closed, therefore $S_\beta Cl(A) = A$ and $S_\beta Cl(B) = B$. Therefore $S_\beta Cl(A) \cap B = A \cap B = \emptyset$ and $S_\beta Cl(B) \cap A = B \cap A = \emptyset$. Since $X$ is $S_\beta$-connected, one of the sets $A$ and $B$ must be empty or $X$.

(2) $\Rightarrow$ (3): Obvious from Theorem 3.7.

(3) $\Rightarrow$ (4):

Let $Y$ be a discrete space with more than one point and let $f : X \to Y$ be an onto $S_\beta$-continuous function. Let $Y = U \cup V$, where $U$ and $V$ are two disjoint nonempty $S_\beta$-open sets in $Y$. Since $f : X \to Y$ is onto, $f(X) = Y = U \cup V \Rightarrow X = f^{-1}(Y) = f^{-1}(U) \cup f^{-1}(V)$. Since the topology of $Y$ is discrete, both $U$ and $V$ are open in $Y$. Again since $f$ is $S_\beta$-continuous, the inverse image of every open set in $Y$ is $S_\beta$-open in $X$. Consequently, $f^{-1}(U)$ and $f^{-1}(V)$ both are (nonempty) $S_\beta$-open in $X$, which contradicts (3).

(4) $\Rightarrow$ (1):

By the way of contradiction, suppose that $X$ is not $S_\beta$-connected. We decompose $X$ as $A \cup B$, where $A$ and $B$ are nonempty subsets of $X$ such that $S_\beta Cl(A) \cap B = \emptyset$ or $S_\beta Cl(B) \cap A = \emptyset$. We see that both $A$ and $B$ are $S_\beta$-open sets in $X$. In fact, $B = X \setminus S_\beta Cl(A)$ and $S_\beta Cl(A)$ is the smallest $S_\beta$-closed set containing $A$ and hence $S_\beta$-closed in $X$. So $B$ is $S_\beta$-open in $X$. Let $Y = \{0, 1\}$ with discrete topology. We define a map $f : X \to Y$ by

$$f(x) = \begin{cases} 0 & if\ x \in A \\ 1 & if\ x \in B \end{cases}$$

Then $f^{-1}(\emptyset) = \emptyset$, which is $S_\beta$-open in $X$, $f^{-1}(Y) = f^{-1}(\{0\} \cup \{1\}) = f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = A \cup B = X$, $f^{-1}(\{0\}) = A$ is $S_\beta$-open in $X$ and $f^{-1}(\{1\}) = B$ is $S_\beta$-open in $X$. Again $\emptyset, \{0\}, \{1\}, X$ are open in $Y = \{0, 1\}$ with the discrete topology. Hence we have the inverse image of every open set in $Y$ is $S_\beta$-open in $X$. Thus $f$ is $S_\beta$-continuous and onto which contradicts (4) for $X$. □

Following is the proof of the second part of Theorem 3.5:

*Proof.* Let $X$ be $S_\beta$-connected. Then only subsets of $X$ which are both $S_\beta$-open and $S_\beta$-closed in $X$ are $\emptyset$ and $X$. Suppose $X$ is not connected.

Then there exists a nonempty proper subset $A$ of $X$ which is both open and closed in $X$. By Lemma 2.7, $A$ is also both $S_\beta$-open and $S_\beta$-closed in $X$. Hence $A$ is a nonempty proper subset of $X$, and both $S_\beta$-open and $S_\beta$-closed in $X$, a contradiction. Thus $X$ is connected. □

**Theorem 3.11.** *If a topological space $X$ is hyperconnected, then it is $S_\beta$-connected.*

*Proof.* Proof follows from Proposition 2.16 of [9]. □

**Theorem 3.12.** *Let $X$ be a topological space and $\{x\}$ is closed in $X$ for each $x \in X$. Then the space $X$ is $S_\beta$-connected if and only if $X$ is semi-connected.*

**Corollary 3.13.** *Let $X$ be a Housdroff space. Then the space $X$ is $S_\beta$-connected if and only if $X$ is semi-connected.*

**Theorem 3.14.** *Let $X$ be a locally indiscrete topological space. Then the space $X$ is $S_\beta$-connected if and only if $X$ is semi-connected.*

*Proof.* Obvious from Proposition 2.19 of [9] □

**Definition 3.15.** [8] Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is said to be $S_\beta$-irresolute if the inverse image of every $S_\beta$-open set in $Y$ under $f$ is $S_\beta$-open in $X$.

**Theorem 3.16.** *Let $X$ and $Y$ be two topological spaces. Let $f : X \to Y$ be an onto $S_\beta$-irresolute function. If $X$ is $S_\beta$-connected, then $f(X)$ is $S_\beta$-connected.*

*Proof.* Let $X$ and $Y$ be two topological spaces and $f : X \to Y$ be an onto $S_\beta$-irresolute function. Suppose that $X$ is $S_\beta$-connected. If $A$ is a subset of $Y$ which is both $S_\beta$-open and $S_\beta$-closed, then $f^{-1}(A)$ is both $S_\beta$-open and $S_\beta$-closed in $X$. Since $X$ is $S_\beta$-connected so $f^{-1}(A)$ must be all of $X$ or the empty set (i.e.,$f^{-1}(A) = Y$ or $f^{-1}(A) = \emptyset$). Therefore $A = f(Y) = X$ or $A = \emptyset$ and hence $Y$ is $S_\beta$-connected. □

**Theorem 3.17.** *Let $A$ be a $S_\beta$-connected set of a topological space $X$ and $U$, $V$ are $S_\beta$-separated subsets of $X$ such that $A \subseteq U \cup V$. Then either $A \subseteq U$ or $A \subseteq V$.*

*Proof.* Since $A = (A \cap U) \cup (A \cap V)$, we have $(A \cap U) \cap S_\beta Cl(A \cap V) \subseteq U \cap S_\beta Cl(V) = \emptyset$. Similarly we have $(A \cap V) \cap S_\beta Cl(A \cap U) = \emptyset$. If $A \cap U$ and $A \cap V$ are nonempty, then $A$ is not $S_\beta$-connected, which is a contradiction. Therefore, either $A \cap U = \emptyset$ or $A \cap V = \emptyset$. Then either $A \subseteq U$ or $A \subseteq V$. □

**Theorem 3.18.** *If $A$ is a $S_\beta$-connected set of a topological space $X$ and $A \subseteq N \subseteq S_\beta Cl(A)$, then $N$ is $S_\beta$-connected.*

*Proof.* Assume $N$ is not $S_\beta$-connected. Then there exist $S_\beta$-separated sets $U$ and $V$ such that $N = U \cup V$. Then $U$ and $V$ are nonempty and $U \cap S_\beta Cl(V) = \emptyset = S_\beta Cl(U) \cap V$. Thus we have either $A \subseteq U$ or $B \subseteq V$ (from Theorem 3.17).

(i) Suppose $A \subseteq U$. Then $S_\beta Cl(A) \subseteq S_\beta Cl(U)$ and $V \cap S_\beta Cl(A) = \emptyset$. Next by hypothesis, $V \subseteq N \subseteq S_\beta Cl(A)$ and $V = S_\beta Cl(A) \cap V = \emptyset$. This is a contradiction to the fact that $V$ is nonempty.

(ii) Suppose $A \subseteq V$. Then from (i), $U$ is empty, a contradiction. Thus $N$ is $S_\beta$-connected. □

**Corollary 3.19.** *Let $A$ be a $S_\beta$-connected subset of a topological space $X$. Then $S_\beta Cl(A)$ is $S_\beta$-connected.*

**Theorem 3.20.** *Let $A$ and $B$ be subsets of a topological space $X$. If $A$ and $B$ are $S_\beta$-connected and not $S_\beta$-separated in $X$, then $A \cup B$ is $S_\beta$-connected.*

*Proof.* Suppose $A \cup B$ is not $S_\beta$-connected. Then there exist $S_\beta$-separated sets $C$, $D$ in $X$ such that $A \cup B = C \cup D$ then $A \subseteq C \cup D$. From Theorem 3.17, either $A \subseteq C$ or $A \subseteq D$. Then either $B \subseteq C$ or $B \subseteq D$. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$ and $D = \emptyset$, a contradiction. Therefore $A \subseteq C$ and $B \subseteq D$. Similarly $A \subseteq D$ and $B \subseteq C$. Thus we obtain $S_\beta Cl(A) \cap B \subseteq S_\beta Cl(C) \cap D = \emptyset$ and $S_\beta Cl(B) \cap A \subseteq S_\beta Cl(C) \cap D = \emptyset$. Hence $A$, $B$ are $S_\beta$-separated in $X$. This is a contradiction. Therefore, $A \cup B$ is $S_\beta$-connected. □

**Theorem 3.21.** *If $\{B_\gamma |\ \gamma \in \Gamma\}$ is a nonempty family of $S_\beta$-connected subsets of a topological space $X$ such that $\bigcap_{\gamma\in\Gamma} B_\gamma \neq \emptyset$, then $\bigcup_{\gamma\in\Gamma} B_\gamma$ is $S_\beta$-connected.*

*Proof.* Suppos $N = \bigcup_{\gamma\in\Gamma} B_\gamma$ and $N$ is not $S_\beta$-connected. Then $N = U \cup V$, where $U$ and $V$ are $S_\beta$-separated sets in $X$. Since $\bigcap_{\gamma\in\Gamma} B_\gamma \neq \emptyset$, we can choose $x \in \bigcap_{\gamma\in\Gamma} B_\gamma$. Since $x \in N$, either $x \in U$ or $x \in V$.

(i) Suppose $x \in U$. Since $x \in B_\gamma$ for each $\gamma \in \Gamma$, $B_\gamma$ and $U$ intersect for each $\gamma \in \Gamma$. Then by Theorem 3.17, either $B_\gamma \subseteq U$ or $B_\gamma \subseteq V$. Since $U$ and $V$ are disjoint, $B_\gamma \subseteq U$ for all $\gamma \in \Gamma$ and hence $N \subseteq U$. This means that $V$ is empty which is a contradiction.

(ii) Suppose $x \in V$. Then similarly we obtain that $U$ is empty which is a contradiction. Hence $\bigcup_{\gamma\in\Gamma} B_\gamma$ is $S_\beta$-connected. □

**Theorem 3.22.** *If $\{A_n |\ n \in \mathbb{N}\}$ is an infinite sequence of $S_\beta$-connected subsets of a topological space $X$ and $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n$ is $S_\beta$-connected.*

*Proof.* Obvious. □

**Theorem 3.23.** *For $S_\beta$-connected spaces $X$ and $Y$, $X \times Y$ is $S_\beta$-connected.*

*Proof.* Proof is obvious from Theorem 2.31 of [9]. □

**Definition 3.24.** Let $X$ be a topological space and $x \in X$. The $S_\beta$-component of $X$ containing $x$ is the union of all $S_\beta$-connected subsets of $X$ containing $x$.

**Theorem 3.25.** *For a topological space $X$, the followings hold:*

(1) *Each $S_\beta$-component of $X$ is a maximal $S_\beta$-connected subset of $X$.*
(2) *The set of all distinct $S_\beta$-components of $X$ forms a partition of $X$.*
(3) *Each $S_\beta$-component of $X$ is $S_\beta$-closed in $X$.*

*Proof.* Obvious. □

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