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Adomian Polynomial and Elzaki Transform Method of Solving Fifth Order Korteweg-De Vries Equations

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Abstract. Elzaki transform and Adomian polynomial is used to obtain the exact solutions of nonlinear fifth order Korteweg-de Vries (KdV) equations. In order to investigate the effectiveness of the method, three fifth order KdV equations were considered. Adomian polynomial is introduced as an essential tool to linearize all the nonlinear terms in any given equation because Elzaki transform cannot handle nonlinear functions on its own. In all the three problems considered, the series solutions obtained converges to the exact solutions. Three dimensional graphs were also plotted to give the shape of the solutions of some KdV equations considered. Hence, Elzaki transform and Adomian polynomial together gives a very powerful and effective method for solving nonlinear partial differential equations.

Keywords: Elzaki transform method, Adomian polynomial, Fifth order KdV equations.

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1. Introduction

In 1895, D.J. Korteweg and G. de Vries derived Korteweg-de Vries (KdV) equation, they proposed that KdV equation describes the propagation of shallow water wave. Several attempts have been made by scientists to obtain the solution of KdV equation which is called soliton after its discovery. The word soliton which is a solution to a nonlinear partial differential equation was used for the first time by Zabusky and Kruskal [30]. The KdV equation was categorized as a typical nonlinear partial differential equation that results to soliton solution.

The Fifth order Korteweg-de Vries (KdV) equation is of the form [5, 26]:

$$\phi_t - a\phi_{xxxxx} = F(x,t,\phi,\phi^2,\phi_x,\phi_{xx},\phi_{xxx}), \tag{1.1}$$

with the initial conditions

$$\phi(x,0) = f(x). \tag{1.2}$$

where a is a constant, this equation occur in the theory of magnetoacoustic waves in plasmas [1] as well as shallow water waves with surface tension [18]. Over the years, the fifth order KdV equation has been studied extensively. Previous studies showed that the travelling-wave solutions of this equation do not vanish at infinity [3, 4].

So many methods have been applied to find the approximate analytical solutions and numerical solutions of KdV equations and some nonlinear differential equations. These methods are Homotopy Perturbation method using Elzaki Transform [6], Homotopy Perturbation method [9], Numerical solutions to a linearized KdV equation on unbounded domain [31], the numerical solutions of KdV equation using radial basis functions [7], numerical solution of separated solitary waves for KdV equation through finite element technique [23]. Moreover, several other methods have also been used to solve KdV equations, which are given in [2, 19, 22, 24, 25, 27, 28, 29].

The solutions of nonlinear KdV equations by Elzaki transform method (ETM) and Adomian Polynomial is obtained in this paper. This method gives the solutions as an approximate analytical solutions in series form, most of the time the series solutions converge to the exact solutions.

This article is structured as follows. Section 2 contain the basic definitions and the properties of the proposed method. Section 3 shows the theoretical approach of the proposed method on KdV equation. In section 4, we applied the Elzaki transform method and Adomian polynomial to solve three problems in order to show its efficiency.

2. Properties of Elzaki transform

Elzaki transform [8, 10, 11, 12, 13, 14, 15] is defined for function of exponential order as

$$A = \left\{ f(t) : \exists M, c_1, c_2 > 0, |f(t)| < Me^{\frac{|t|}{c_j}},\ \text{if } t \in (-1)^j \times [0, \infty) \right\},$$

for any given function in the set A defined above, the constant c_1, c_2 may either be finite or infinite, but M must definitely be infinite.

According to Tarig Elzaki [11], Elzaki transform is defined as:

$$E[f(t)] = u^2 \int_0^\infty f(ut)e^{-t}dt = T(u),\ t \geq 0,\ u \in (c_1, c_2),$$

or

$$E[f(t)] = u \int_0^\infty f(t)e^{-\frac{t}{u}}dt = T(u),\ t \geq 0,\ u \in (c_1, c_2). \tag{2.1}$$

where u in the above definition is used to factor t in the analysis of function f.

Let $T(u)$ be the Elzaki transform of f(t) that is, $E[f(t)] = T(u)$, then:

(i) $E[f'(t)] = \frac{T(u)}{u} - uf(0)$.
(ii) $E[f''(t)] = \frac{T(u)}{u^2} - f(0) - uf'(0)$.
(iii) $E[f^{(n)}(t)] = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0)$.

$E[f(t)] = T(u)$ means that $T(u)$ is the Elzaki transform of $f(t)$, and $f(t)$ is the inverse Elzaki transform of $T(u)$. that is,

$$f(t) = E^{-1}[T(u)].$$

In order to obtain the Elzaki transform of partial derivative, we used the integration by part on the definition of Elzaki transform and the resulting expressions is given by [16]

$$E\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{T(x,v)}{v} - vf(x,0),$$

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{T(x,v)}{v^2} - f(x,0) - v\frac{\partial f(x,0)}{\partial t},$$

$$E\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}[T(x,v)],$$

$$E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)],$$

$$E\left[\frac{\partial^3 f(x,t)}{\partial x^3}\right] = \frac{d^3}{dx^3}[T(x,v)].$$

3. Theoretical Approach: Elzaki transform on fifth Order KdV equation

The main focus is to solve the fifth order nonlinear KdV equations considered in this article. According to [20, 21, 33], let us consider;

$$\frac{\partial^w \phi(x,t)}{\partial t^w} + R\phi(x,t) + N\phi(x,t) = f(x,t), \tag{3.1}$$

where $w = 1, 2, 3$, and the initial conditions is given as

$$\left.\frac{\partial^{w-1}\phi(x,t)}{\partial t^{w-1}}\right|_{t=0} = g_{w-1}(x),$$

The partial derivative of the function $\phi(x,t)$ of w^{th} order is the one given as $\frac{\partial^w \phi(x,t)}{\partial t^w}$, R represents the linear differential operator, N indicates the nonlinear term of the differential equation, and $f(x,t)$ is the non-homogeneous/source term.

By applying the Elzaki transform on equation (3.1) we have;

$$E\left[\frac{\partial^w \phi(x,t)}{\partial t^w}\right] + E\left[R\phi(x,t)\right] + E\left[N\phi(x,t)\right] = E\left[f(x,t)\right]. \tag{3.2}$$

where

$$E\left[\frac{\partial^w \phi(x,t)}{\partial t^w}\right] = \frac{E[\phi(x,t)]}{v^w} - \sum_{k=0}^{w-1} v^{2-w+k}\frac{\partial^k \phi(x,0)}{\partial t^k}. \tag{3.3}$$

Substituting equation (3.3) into equation (3.2) gives;

$$\begin{aligned}\frac{E[\phi(x,t)]}{v^w} - \sum_{k=0}^{w-1} v^{2-w+k}\frac{\partial^k \phi(x,0)}{\partial t^k} + E\left[R\phi(x,t)\right] + E\left[N\phi(x,t)\right]\\ = E\left[f(x,t)\right].\end{aligned}$$

This is the same as

$$\begin{aligned}\frac{E[\phi(x,t)]}{v^w} &= E\left[f(x,t)\right] + \sum_{k=0}^{w-1} v^{2-w+k}\frac{\partial^k \phi(x,0)}{\partial t^k}\\ &\quad - \left\{E\left[R\phi(x,t)\right] + E\left[N\phi(x,t)\right]\right\}. \end{aligned}\tag{3.4}$$

Simplifying equation (3.4) yields;

$$\begin{aligned}E[\phi(x,t)] &= v^w E\left[f(x,t)\right] + \sum_{k=0}^{w-1} v^{2+k}\frac{\partial^k \phi(x,0)}{\partial t^k}\\ &\quad - v^w\left\{E\left[R\phi(x,t)\right] + E\left[N\phi(x,t)\right]\right\}. \end{aligned}\tag{3.5}$$

Applying the inverse Elzaki transform to equation (3.5), we have

$$\begin{aligned}\phi(x,t) =& E^{-1}\left[v^w E\left[f(x,t)\right] + \sum_{k=0}^{w-1} v^{2+k}\frac{\partial^k \phi(x,0)}{\partial t^k}\right] \\ &- E^{-1}\left[v^w\left\{E\left[R\phi(x,t)\right] + E\left[N\phi(x,t)\right]\right\}\right],\end{aligned}$$

this is rewrite as;

$$\phi(x,t) = F(x,t) - E^{-1}\left[v^w\left\{E\left[R\phi(x,t)\right] + E\left[N\phi(x,t)\right]\right\}\right], \tag{3.6}$$

where $F(x,t)$ denotes the expression that arises from the given initial conditions and the source terms after simplification.

The solution is given in the form of infinite series as

$$\phi(x,t) = \sum_{n=0}^{\infty} \phi_n(x,t). \tag{3.7}$$

The nonlinear terms can be decompose as

$$N\phi(x,t) = \sum_{n=0}^{\infty} A_n, \tag{3.8}$$

where A_n is defined as the Adomian polynomials which can be computed by using the formula [32]

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i \phi_i\right)\right]_{\lambda=0}, \ n = 0, 1, \cdots$$

Substituting equations (3.7) and (3.8) into equation (3.6), this gives

$$\sum_{n=0}^{\infty}\phi_n(x,t) = F(x,t) - E^{-1}\left[v^w\left\{E\left[R\sum_{n=0}^{\infty}\phi_n(x,t)\right] + E\left[\sum_{n=0}^{\infty}A_n\right]\right\}\right]. \tag{3.9}$$

Then from equation (3.9), we have

$$\phi_0(x,t) = F(x,t), \tag{3.10}$$

and the recursive relation is given by

$$\phi_{n+1} = -E^{-1}\left[v^w\left\{E\left[R\phi_n(x,t)\right] + E\left[A_n\right]\right\}\right],$$

here $w = 1, 2, 3$ and $n \geq 0$. The analytical solution $\phi(x,t)$ can be approximated by a truncated series:

$$\phi(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N}\phi_n(x,t).$$

4. Applications

The effectiveness of the Elzaki transform and Adomian polynomial are demonstrated by solving the following fifth order Korteweg-De Vries (KdV) equations.

Example 4.1: Consider the homogeneous KdV equation [17]

$$\phi_t + \phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx} = 0, \tag{4.1}$$

with the initial condition

$$\phi(x,0) = e^x.$$

Equation (4.1) can be written as

$$\phi_t = -\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]. \tag{4.2}$$

Applying the Elzaki transform to both sides of equation (4.2) gives

$$E[\phi_t] = -E\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]. \tag{4.3}$$

Since

$$E\left[\phi_t\right] = \frac{\Phi(x,v)}{v} - v\phi(x,0),$$

So equation (4.3) becomes;

$$\frac{\Phi(x,v)}{v} - v\phi(x,0) = -E\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]. \tag{4.4}$$

Applying the given initial condition on equation (4.4) and simplifying, we obtain;

$$\Phi(x,v) = v^2 e^x - vE\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]. \tag{4.5}$$

Applying the inverse Elzaki transform to equation (4.5), we have;

$$\phi(x,t) = E^{-1}\left\{v^2 e^x\right\} - E^{-1}\left\{vE\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]\right\}.$$

The resulting expression is

$$\phi(x,t) = e^x - E^{-1}\left\{vE\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]\right\}. \tag{4.6}$$

From equation (4.6), let

$$\phi_0 = e^x.$$

The recursive relation is given as:

$$\phi_{n+1} = -E^{-1}\left\{vE\left[A_n + \frac{\partial^5 \phi_n}{\partial x^5}\right]\right\}, \tag{4.7}$$

where A_n is the Adomian polynomial to decompose the nonlinear terms using the relation:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f \left[\sum_{i=0}^{\infty} \lambda^i \phi_i \right]_{\lambda=0}. \tag{4.8}$$

The nonlinear term is represented by

$$f(\phi) = \phi \frac{\partial \phi}{\partial x} - \phi \frac{\partial^3 \phi}{\partial x^3}. \tag{4.9}$$

By using equation (4.9) in equation (4.8), we obtain;

$$\begin{aligned}
A_0 &= \phi_0 \left[\frac{\partial \phi_0}{\partial x} - \frac{\partial^3 \phi_0}{\partial x^3} \right], \\
A_1 &= \phi_1 \left[\frac{\partial \phi_0}{\partial x} - \frac{\partial^3 \phi_0}{\partial x^3} \right] + \phi_0 \left[\frac{\partial \phi_1}{\partial x} - \frac{\partial^3 \phi_1}{\partial x^3} \right], \\
A_2 &= \phi_2 \left[\frac{\partial \phi_0}{\partial x} - \frac{\partial^3 \phi_0}{\partial x^3} \right] + \phi_1 \left[\frac{\partial \phi_1}{\partial x} - \frac{\partial^3 \phi_1}{\partial x^3} \right] + \phi_0 \left[\frac{\partial \phi_2}{\partial x} - \frac{\partial^3 \phi_2}{\partial x^3} \right], \cdots
\end{aligned}$$

From Equation (4.7), when n=0, we have

$$\phi_1 = -E^{-1} \left\{ vE \left[A_0 + \frac{\partial^5 \phi_0}{\partial x^5} \right] \right\}.$$

$$\phi_1 = -E^{-1} \left\{ vE \left[\phi_0 \left[\frac{\partial \phi_0}{\partial x} - \frac{\partial^3 \phi_0}{\partial x^3} \right] + \frac{\partial^5 \phi_0}{\partial x^5} \right] \right\}.$$

Since $\phi_0 = e^x$, then

$$\phi_1 = -E^{-1} \left\{ vE \left[e^x \right] \right\}. \tag{4.10}$$

By simplifying equation (4.10) we have;

$$\phi_1 = -te^x. \tag{4.11}$$

When $n = 1$, we have;

$$\phi_2 = -E^{-1} \left\{ vE \left[A_1 + \frac{\partial^5 \phi_1}{\partial x^5} \right] \right\}.$$

Since $\phi_1 = -te^x$, then

$$\phi_2 = -E^{-1} \left\{ vE \left[\phi_1 \left[\frac{\partial \phi_0}{\partial x} - \frac{\partial^3 \phi_0}{\partial x^3} \right] + \phi_0 \left[\frac{\partial \phi_1}{\partial x} - \frac{\partial^3 \phi_1}{\partial x^3} \right] \right] + \frac{\partial^5 (-te^x)}{\partial x^5} \right\}. \tag{4.12}$$

Simplifying equation (4.12) we obtain;

$$\phi_2 = \frac{t^2}{2!} e^x.$$

When $n = 2$:

$$\phi_3 = -E^{-1}\left\{vE\left[A_2 + \frac{\partial^5 \phi_2}{\partial x^5}\right]\right\}.$$

Since $\phi_2 = \frac{t^2}{2!}e^x$, we have

$$\phi_3 = -E^{-1}\left\{v\left[\phi_2\left[\frac{\partial\phi_0}{\partial x} - \frac{\partial^3\phi_0}{\partial x^3}\right] + \phi_1\left[\frac{\partial\phi_1}{\partial x} - \frac{\partial^3\phi_1}{\partial x^3}\right] + \phi_0\left[\frac{\partial\phi_2}{\partial x} - \frac{\partial^3\phi_2}{\partial x^3}\right]\right]\right\} - E^{-1}\left\{\frac{\partial^5(\frac{t^2}{2!}e^x)}{\partial x^5}\right\}. \tag{4.13}$$

Simplifying equation (4.13) we obtain;

$$\phi_3 = -\frac{t^3}{3!}e^x.$$

The approximate series solution is given as

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \cdots,$$

substituting all the values computed above

$$\phi(x,t) = e^x - te^x + \frac{t^2}{2!}e^x - \frac{t^3}{3!}e^x + \cdots,$$

we can rewrite this as;

$$\phi(x,t) = e^x\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right]$$

Note that from the Taylor's series expansion of exponential function we have:

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots. \tag{4.14}$$

Therefore;

$$\phi(x,t) = e^x e^{-t} = e^{x-t}. \tag{4.15}$$

The closed form solution of the equation (4.1) is in agreement with the one obtained by the Laplace decomposition method [17]

Figure 1 below shows the 3D graph of the solution of equation (4.1).

Example 4.2: Consider the homogeneous KdV equation [17]

$$\phi_t + \phi_x + \phi^2\phi_{xx} + \phi_x\phi_{xx} - 20\phi^2\phi_{xxx} + \phi_{xxxxx} = 0, \tag{4.16}$$

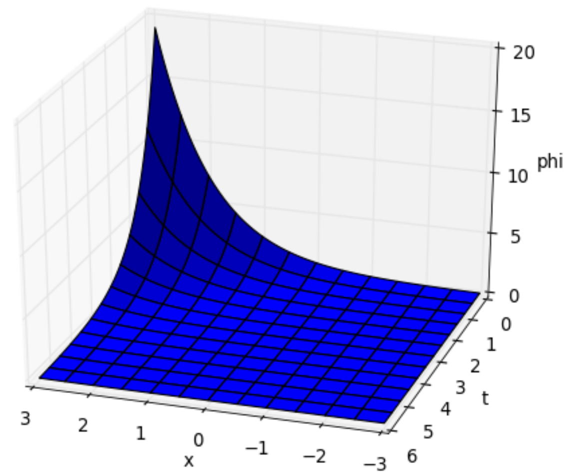

FIGURE 1. The solution of the first fifth order KdV equation in (4.1) by ETM.

with the initial condition

$$\phi(x,0)=\frac{1}{x}.$$

Equation (4.16) could be written as:

$$\phi_t+\phi_x=-\left[\phi^2\phi_{xx}+\phi_x\phi_{xx}-20\phi^2\phi_{xxx}+\phi_{xxxxx}\right]. \tag{4.17}$$

Applying the Elzaki transform to both sides of equation (4.17), this gives

$$E[\phi_t]+E[\phi_x]=-E\left[\phi^2\phi_{xx}+\phi_x\phi_{xx}-20\phi^2\phi_{xxx}+\phi_{xxxxx}\right], \tag{4.18}$$

where

$$E\left[\phi_t\right]=\frac{\Phi(x,v)}{v}-v\phi(x,0),$$
$$E\left[\phi_x\right]=\frac{d}{dx}E[\phi].$$

So equation (4.18) becomes;

$$\frac{\Phi(x,v)}{v}-v\phi(x,0)+\frac{d}{dx}E[\phi]=-E\left[\phi^2\phi_{xx}+\phi_x\phi_{xx}-20\phi^2\phi_{xxx}+\phi_{xxxxx}\right]. \tag{4.19}$$

Applying the given initial condition to equation (4.19) and simplifying, we obtain;

$$\Phi(x,v)=v^2\frac{1}{x}-v\frac{d}{dx}E[\phi]-vE\left[\phi^2\phi_{xx}+\phi_x\phi_{xx}-20\phi^2\phi_{xxx}+\phi_{xxxxx}\right]. \tag{4.20}$$

Applying the inverse Elzaki transform to equation (4.20), we have;

$$\phi(x,t) = E^{-1}\left\{v^2\frac{1}{x}\right\} - E^{-1}\left\{v\frac{d}{dx}E[\phi] + vE\left[\phi^2\phi_{xx} + \phi_x\phi_{xx} - 20\phi^2\phi_{xxx} + \phi_{xxxxx}\right]\right\}.$$

The resulting expression is

$$\phi(x,t) = \frac{1}{x} - E^{-1}\left\{v\frac{d}{dx}E[\phi] + vE\left[\phi^2\phi_{xx} + \phi_x\phi_{xx} - 20\phi^2\phi_{xxx} + \phi_{xxxxx}\right]\right\}. \tag{4.21}$$

From equation (4.21), let

$$\phi_0 = \frac{1}{x},$$

and the recursive relation is given as:

$$\phi_{n+1} = -E^{-1}\left\{v\frac{d}{dx}E[\phi_n] + vE\left[A_n + \frac{\partial^5\phi_n}{\partial x^5}\right]\right\}, \tag{4.22}$$

A_n is the Adomian polynomial to decompose the nonlinear terms by using the relation:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}f\left[\sum_{i=0}^{\infty}\lambda^i\phi_i\right]_{\lambda=0}. \tag{4.23}$$

The nonlinear term is represented by

$$f(\phi) = \phi^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} - 20\phi^2\frac{\partial^3\phi}{\partial x^3}. \tag{4.24}$$

By using equation (4.24) in equation (4.23), we obtain;

$$A_0 = \phi_0^2\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial\phi_0}{\partial x}\frac{\partial^2\phi_0}{\partial x^2} - 20\phi_0^2\frac{\partial^3\phi}{\partial x^3},$$
$$A_1 = 2\phi_0\phi_1\frac{\partial^2\phi_0}{\partial x^2} + \phi_0^2\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial\phi_0}{\partial x}\frac{\partial^2\phi_1}{\partial x^2} - 40\phi_0\phi_1\frac{\partial^3\phi_0}{\partial x^3} - 20\phi_0^2\frac{\partial^3\phi_1}{\partial x^3},$$
$$A_2 = (\phi_1^2 + 2\phi_0\phi_2)\frac{\partial^2\phi_0}{\partial x^2} + 2\phi_0\phi_1\frac{\partial^2\phi_1}{\partial x^2} + \phi_0^2\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial\phi_2}{\partial x}\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial\phi_0}{\partial x}\frac{\partial^2\phi_2}{\partial x^2} - 20(\phi_1^2 + 2\phi_0\phi_2)\frac{\partial^3\phi_0}{\partial x^3} - 40\phi_0\phi_1\frac{\partial^3\phi_1}{\partial x^3} - 20\phi_0^2\frac{\partial^3\phi_2}{\partial x^3}, \cdots$$

From equation (4.22), when n=0, we have

$$\phi_1 = -E^{-1}\left\{v\frac{d}{dx}E[\phi_0] + vE\left[A_0 + \frac{\partial^5\phi_0}{\partial x^5}\right]\right\}.$$

Since $\phi_0 = \dfrac{1}{x}$, then

$$\phi_1 = -E^{-1}\left\{v\frac{d}{dx}E\left[\frac{1}{x}\right] + vE\left[A_0 + \frac{\partial^5}{\partial x^5}\left[\frac{1}{x}\right]\right]\right\}.$$

And A_0 is computed as

$$A_0 = \frac{120}{x^6},$$

so that

$$\phi_1 = -E^{-1}\left\{-\frac{v^3}{x^2} + vE\left[\frac{120}{x^6} + \frac{\partial^5}{\partial x^5}\left[\frac{1}{x}\right]\right]\right\}. \tag{4.25}$$

By simplifying equation (4.25), we have;

$$\phi_1 = \frac{t}{x^2}. \tag{4.26}$$

When n=1,

$$\phi_2 = -E^{-1}\left\{v\frac{d}{dx}E[\phi_1] + vE\left[A_1 + \frac{\partial^5\phi_1}{\partial x^5}\right]\right\}.$$

Since $\phi_1 = \dfrac{t}{x^2}$, then

$$\phi_2 = -E^{-1}\left\{v\frac{d}{dx}E\left[\frac{t}{x^2}\right] + vE\left[A_1 + \frac{\partial^5}{\partial x^5}\left[\frac{t}{x^2}\right]\right]\right\}.$$

A_1 is computed as:

$$A_1 = \frac{720t}{x^7},$$

so that

$$\phi_2 = -E^{-1}\left\{-\frac{2v^4}{x^3} + vE\left[\frac{720t}{x^7} + \frac{\partial^5}{\partial x^5}\left[\frac{t}{x^2}\right]\right]\right\}. \tag{4.27}$$

Simplifying equation (4.27) yields;

$$\phi_2 = \frac{t^2}{x^3}. \tag{4.28}$$

When $n = 2$,

$$\phi_3 = -E^{-1}\left\{v\frac{d}{dx}E[\phi_2] + vE\left[A_2 + \frac{\partial^5\phi_2}{\partial x^5}\right]\right\}.$$

Since $\phi_2 = \dfrac{t^2}{x^3}$, we have

$$\phi_3 = -E^{-1}\left\{v\frac{d}{dx}E\left[\frac{t^2}{x^3}\right] + vE\left[A_2 + \frac{\partial^5}{\partial x^5}\left[\frac{t^2}{x^3}\right]\right]\right\}.$$

And A_2 is computed as

$$A_2 = \frac{2520t^2}{x^8},$$

so that,

$$\phi_3 = -E^{-1}\left\{-\frac{6v^5}{x^4} + vE\left[\frac{2520t^2}{x^8} + \frac{\partial^5}{\partial x^5}\left[\frac{t^2}{x^3}\right]\right]\right\}. \tag{4.29}$$

Simplifying equation (4.29), we get;

$$\phi_3 = \frac{t^3}{x^4}. \tag{4.30}$$

The approximate series solution is given by

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \cdots,$$

substituting the values of $\phi_0,\ \phi_1,\ \phi_2,\ \phi_3,\ \phi_4$

$$\phi(x,t) = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4} + \cdots,$$

and can be written as;

$$\phi(x,t) = \frac{1}{x}\left[1 + \left(\frac{t}{x}\right) + \left(\frac{t}{x}\right)^2 + \left(\frac{t}{x}\right)^3 + \cdots\right].$$

Therefore, the closed form solution is

$$\phi(x,t) = \frac{1}{x-t}. \tag{4.31}$$

The closed form solution of the equation (4.16) is in agreement with the one obtained by the Laplace decomposition method [17]

Figure 2 below shows the 3D graph of the solution of equation (4.16).

Example 4.3: Consider the homogeneous KdV equation [17]

$$\phi_t + \phi\phi_x - \phi_{xxx} + \phi_{xxxxx} = 0, \tag{4.32}$$

with the initial condition

$$\phi(x,0) = \frac{105}{169}\text{sech}^4\left[\frac{1}{2\sqrt{13}}(x - x_0)\right].$$

Equation (4.32) could be written as:

$$\phi_t = -\left[\phi\phi_x - \phi\phi_{xxx} + \phi_{xxxxx}\right]. \tag{4.33}$$

Applying Elzaki transform to both sides of equation (4.33), this gives

$$E[\phi_t] = -E\left[\phi\phi_x - \phi_{xxx} + \phi_{xxxxx}\right], \tag{4.34}$$

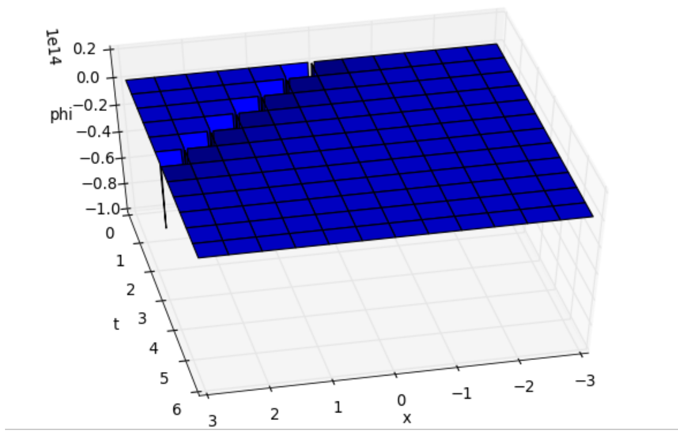

FIGURE 2. The solution of the second fifth order KdV equation in (4.16) by ETM

where

$$E\left[\phi_t\right]=\frac{\Phi(x,v)}{v}-v\phi(x,0),$$

so equation (4.34) becomes;

$$\frac{\Phi(x,v)}{v}-v\phi(x,0)=-E\left[\phi\phi_x-\phi_{xxx}+\phi_{xxxxx}\right]. \tag{4.35}$$

Applying the given initial condition to equation (4.35) and simplifying, to obtain;

$$\Phi(x,v)=v^2\frac{105}{169}\operatorname{sech}^4\left[\frac{1}{2\sqrt{13}}(x-x_0)\right]-vE\left[\phi\phi_x-\phi_{xxx}+\phi_{xxxxx}\right]. \tag{4.36}$$

Applying the inverse Elzaki transform to equation (4.36), we have;

$$\begin{aligned}\phi(x,t)=&E^{-1}\left\{v^2\frac{105}{169}\operatorname{sech}^4\left[\frac{1}{2\sqrt{13}}(x-x_0)\right]\right\}\\&-E^{-1}\left\{vE\left[\phi\phi_x-\phi_{xxx}+\phi_{xxxxx}\right]\right\}.\end{aligned}$$

The resulting expression is

$$\phi(x,t)=\frac{105}{169}\operatorname{sech}^4\left[\frac{1}{2\sqrt{13}}(x-x_0)\right]-E^{-1}\left\{vE\left[\phi\phi_x-\phi_{xxx}+\phi_{xxxxx}\right]\right\}. \tag{4.37}$$

From equation (4.37), let

$$\phi_0 = \frac{105}{169}\text{sech}^4\left[\frac{1}{2\sqrt{13}}(x - x_0)\right].$$

The recursive relation is given by

$$\phi_{n+1} = -E^{-1}\left\{vE\left[A_n + \frac{\partial^3\phi_n}{\partial x^3} - \frac{\partial^5\phi_n}{\partial x^5}\right]\right\}, \tag{4.38}$$

A_n is the Adomian polynomial to decompose the nonlinear terms by using the relation:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}f\left[\sum_{i=0}^{\infty}\lambda^i\phi_i\right]_{\lambda=0}. \tag{4.39}$$

The nonlinear term is represented by

$$f(\phi) = \phi\frac{\partial\phi}{\partial x}. \tag{4.40}$$

By substituting equation (4.40) into equation (4.39), we obtain;

$$\begin{aligned}
A_0 &= \phi_0\frac{\partial\phi_0}{\partial x},\\
A_1 &= \phi_1\frac{\partial\phi_0}{\partial x} + \phi_0\frac{\partial\phi_1}{\partial x},\\
A_2 &= \phi_2\frac{\partial\phi_0}{\partial x} + \phi_1\frac{\partial\phi_1}{\partial x} + \phi_0\frac{\partial\phi_2}{\partial x}, \cdots
\end{aligned}$$

From equation (4.38), when n=0, we have

$$\phi_1 = -E^{-1}\left\{vE\left[A_0 + \frac{\partial^3\phi_0}{\partial x^3} - \frac{\partial^5\phi_0}{\partial x^5}\right]\right\}.$$

Since $\phi_0 = \frac{105}{169}\text{sech}^4\left[\frac{1}{2\sqrt{13}}(x - x_0)\right]$, we obtain

$$\phi_1 = \frac{7560}{28561\sqrt{13}}t\text{sech}^4\left[\frac{1}{2\sqrt{13}}(x - x_0)\right]\tanh^4\left[\frac{1}{2\sqrt{13}}(x - x_0)\right]. \tag{4.41}$$

When $n = 1$, we have;

$$\phi_2 = -E^{-1}\left\{vE\left[A_1 + \frac{\partial^3\phi_1}{\partial x^3} - \frac{\partial^5\phi_1}{\partial x^5}\right]\right\},$$

which gives,

$$\begin{aligned}
\phi_2 =& \frac{68040}{62748517\sqrt{13}}t^2\text{sech}^6\left[\frac{1}{2\sqrt{13}}(x - x_0)\right]\times\\
&\left[-3 + 2\cosh\left[\frac{1}{2\sqrt{13}}(x - x_0)\right]\right].
\end{aligned}$$

When $n = 2$

$$\phi_3 = -E^{-1}\left\{vE\left[A_2 + \frac{\partial^3\phi_2}{\partial x^3} - \frac{\partial^5\phi_2}{\partial x^5}\right]\right\}.$$

Therefore,

$$\begin{aligned}\phi_3 =& \frac{816480}{10604499373\sqrt{13}}t^3\mathrm{sech}^7\left[\frac{1}{2\sqrt{13}}(x-x_0)\right]\left[-13\sinh\left[\frac{1}{2\sqrt{13}}(x-x_0)\right]\right] \\ &+ 2\sinh\left[\frac{1}{2\sqrt{13}}(x-x_0)\right].\end{aligned}$$

The approximate series solution is given by

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \cdots,$$

substituting ϕ_0, ϕ_1, ϕ_2, and ϕ_3 into $\phi(x,t)$ gives the solution in a series form. However, the closed form solution is

$$\phi(x,t) = \frac{105}{169}\mathrm{sech}^4\left[\frac{1}{2\sqrt{13}}(x - \frac{36t}{169} - x_0)\right]. \tag{4.42}$$

The closed form solution of the equation (4.32) is in agreement with the one obtained by the Laplace decomposition method [17]

5. Conclusion

In this paper, Elzaki transform and Adomian polynomial have been effectively used to solve nonlinear fifth order Korteweg-de Vries (KdV) equations. The solutions obtained were presented in series form and converges to the exact solutions in all the three problems considered. These solutions also agree with the solutions obtained when Laplace decomposition method is used as provided in the reference. Three dimensional graphs were also plotted to give the shape of the solutions to KdV equations considered. Furthermore, the combination of the Elzaki transform and Adomian polynomial has overcome the hurdle of nonlinearity which Elzaki transform cannot handle independently as well as other linear transforms like Sumudu transforms and Laplace transforms. However, it is not in all cases the series solutions converges directly to the exact solution and so the solution obtained in this scenario would be approximate analytical solution.

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