Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **8**(1)(2019), 43-50

Existence of multiple solutions to a two-point boundary value system via variational method

Armin Hadjian[1] and Mohsen Rostamian Delavar

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

Abstract. In this paper, we prove the existence of an open interval $]\lambda^{'},\lambda^{''}[$ for each λ of which a class of two-point boundary value equations depending on λ admits at least three solutions. Our main tool is a three critical points theorem of Averna and Bonanno.

Keywords: Three solutions, critical points, two-point boundary value system.

2010 Mathematics subject classification: 34B15; Secondary 58E05.

1. Introduction

Let us consider the following quasilinear elliptic system

$$\begin{cases} -{u_i}''+a_i(x)u_i = \lambda F_{u_i}(x,u_1,\ldots,u_n) & \text{in } (0,1), \\ {u_i}'(0) ={u_i}'(1) = 0, \end{cases} \tag{1.1}$$

for $1 \leq i \leq n$, where λ is a positive parameter, $F\colon [0,1]\times\mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1,t_2,\ldots,t_n) \to F(x,t_1,t_2,\ldots,t_n)$ is measurable in $[0,1]$ for all $(t_1,\ldots,t_n)\in\mathbb{R}^n$ and is C^1 in $\mathbb{R}^n$ for a.e. $x\in[0,1]$ satisfying the condition

$$\sup_{\sum_{i=1}^n |t_i|^2\leq\rho} |F(\cdot,t_1,\ldots,t_n)| \in L^1([0,1])$$

for every $\rho>0$, F_{u_i} denotes the partial derivative of F with respect to u_i, and $a_i\in L^\infty([0,1])$ with $\operatorname{ess\,inf}_{(0,1)} a_i > 0$ for $1\leq i\leq n$.

[1]Corresponding author: hadjian83@gmail.com, a.hadjian@ub.ac.ir
Received: 30 October 2018
Accepted: 08 December 2018

Throughout this paper, we let X be the Cartesian product of n copies of $W^{1,2}([0,1])$, i.e., $X = (W^{1,2}([0,1]))^n$ equipped with the norm

$$\|(u_1,\ldots,u_n)\| := \sum_{i=1}^{n} \|u_i\|,$$

where

$$\|u_i\| := \left(\int_0^1 \left(|u_i{}'(x)|^2 + a_i(x)|u_i(x)|^2 \right) dx \right)^{1/2}$$

for $1 \le i \le n$, which is equivalent to the usual one.

Put

$$c := \max \left\{ \sup_{u_i \in W^{1,2}([0,1]) \setminus \{0\}} \frac{\max_{x\in[0,1]} |u_i(x)|^2}{\|u_i\|^2} \ : \ \text{for } 1 \le i \le n \right\}. \tag{1.2}$$

Note that X is compactly embedded in $(C^0([0,1]))^n$, so $c < +\infty$. It follows from Proposition 4.1 of [1] that

$$\sup_{u_i \in W^{1,2}([0,1]) \setminus \{0\}} \frac{\max_{x\in[0,1]} |u_i(x)|^2}{\|u_i\|^2} > \frac{1}{\|a_i\|_1} \quad \text{for} \quad 1 \le i \le n,$$

where $\|a_i\|_1 := \int_0^1 |a_i(x)| dx$ for $1 \le i \le n$, and so $\frac{1}{\|a_i\|_1} \le c$ for $1 \le i \le n$.

By a *(weak) solution* of system (1.1), we mean any $u = (u_1,\ldots,u_n) \in X$ such that

$$\int_0^1 \sum_{i=1}^{n} u_i{}'(x) v_i{}'(x) dx + \int_0^1 \sum_{i=1}^{n} a_i(x) u_i(x) v_i(x) dx - \lambda \int_0^1 \sum_{i=1}^{n} F_{u_i}(x, u_1(x),\ldots,u_n(x)) v_i(x) dx = 0$$

for all $v = (v_1,\ldots,v_n) \in X$.

We shall establish the existence of a definite interval, in which λ lies, system (1.1) admits at least three weak solutions in X, by means of a recent abstract critical points result of Averna and Bonanno [2] which is actually a refinement of a general principle of Ricceri [8]. The existence and multiplicity of solutions for two-point boundary value problems have been widely investigated (see, for instance, [3, 4, 5, 6, 7] and references therein). For other basic notations and definitions we refer to [9].

2. Main result

First we here recall for the reader's convenience the three critical points theorem of [2] which is our main tool to prove the results. Here, Y^* denotes the dual space of Y.

Theorem 2.1 (Theorem B of [2]). *Let Y be a real reflexive Banach space; $\Phi : Y \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on Y^*; $\Psi : Y \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u)) = +\infty$ *for all* $\lambda \in [0,+\infty[$;

(ii) *there is* $r \in \mathbb{R}$ *such that:*

$$\inf_{Y} \Phi < r,$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi}{r-\Phi(u)},$$

$$\varphi_2(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \sup_{v\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)},$$

and $\overline{\Phi^{-1}(]-\infty,r[)}^{w}$ *is the closure of* $\Phi^{-1}(]-\infty,r[)$ *in the weak topology.*

Then, for each $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$ *the functional* $\Phi+\lambda\Psi$ *has at least three critical points in* Y.

For all $\gamma > 0$ we denote by $K(\gamma)$ the set

$$\left\{(t_1,\ldots,t_n)\in\mathbb{R}^n \ : \ \sum_{i=1}^{n}|t_i|^2 \le \gamma\right\}. \tag{2.1}$$

We formulate our main result as follows.

Theorem 2.2. *Assume that there exist two positive constants* γ *and* δ *with* $\delta^2 > \frac{\gamma}{2^{n-1}}$ *such that*

(j) $\dfrac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\gamma} < \dfrac{\int_0^1 F(x,\delta,\ldots,\delta)dx}{2^n c\delta^2 \sum_{i=1}^{n}\|a_i\|_1}$, *where* c *and* $K(\frac{\gamma}{2^{n-1}})$ *are given by* (1.2) *and* (2.1)*;*

(jj) $\limsup_{|t_1|\to+\infty,\ldots,|t_n|\to+\infty} \dfrac{F(x,t_1,\ldots,t_n)}{\sum_{i=1}^{n}\frac{|t_i|^2}{2}} \le 0$ *uniformly with respect to* $x \in [0,1]$;

(jjj) $F(x,0,\ldots,0) = 0$ *for every* $x \in [0,1]$.

Then, setting

$$\lambda^{'}:=\frac{\frac{\delta^2}{2}\sum_{i=1}^{n}\|a_i\|_1}{\int_0^1 F\left(x,\delta,\ldots,\delta\right)\ -\ \int_0^1 \sup\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}$$

and

$$\lambda^{''}:=\frac{\gamma}{2^n c\int_0^1 \sup\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx},$$

for each $\lambda\in\,]\lambda^{'},\lambda^{''}[$ *system* (1.1) *admits at least three weak solutions in* X.

Proof. For each $u=(u_1,\ldots,u_n)\in X$, put

$$\Phi\left(u\right):=\sum_{i=1}^{n}\frac{\|u_i\|^2}{2}\quad\text{and}\quad\Psi(u):=-\int_0^1 F(x,u_1(x),\ldots,u_n(x))dx.$$

It is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable functionals with

$$\Phi'(u)(v)=\int_0^1\sum_{i=1}^{n}u_i{}'(x)v_i{}'(x)dx+\int_0^1\sum_{i=1}^{n}a_i(x)u_i(x)v_i(x)dx$$

and

$$\Psi'(u)(v)=-\int_0^1\sum_{i=1}^{n}F_{u_i}(x,u_1(x),\ldots,u_n(x))v_i(x)dx$$

for every $u=(u_1,\ldots,u_n)$,$v=(v_1,\ldots,v_n)\in X$, as well as $\Psi':X\to X^*$ is continuous and compact operator, and $\Phi':X\to X^*$ admits a continuous inverse on X^*. Furthermore, by Proposition 25.20 of [9], Φ is sequentially weakly lower semicontinuous. Thanks to the assumption (jj), for each $\lambda>0$ one has

$$\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u))=+\infty.$$

Put $r:=\frac{\gamma}{2^n c}$. From the hypothesis (j), we get

$$\frac{\int_0^1 \sup\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\gamma}<\frac{\int_0^1 F(x,\delta,\ldots,\delta)dx}{2^{n-1}c\delta^2\sum_{i=1}^{n}\|a_i\|_1}-\frac{\int_0^1 \sup\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\gamma}.$$

Thus, since $\delta^2 > \frac{\gamma}{2^{n-1}}$, and $c\|a_i\|_1 \geq 1$ for $1 \leq i \leq n$, we have

$$\frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\gamma} < \frac{\int_0^1 F(x,\delta,\ldots,\delta)\,dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{2^{n-1}c\delta^2 \sum_{i=1}^n \|a_i\|_1},$$

from which, multiplying by $2^n c$, we obtain

$$\frac{2^n c\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\gamma} < \frac{\int_0^1 F(x,\delta,\ldots,\delta)\,dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\frac{\delta^2}{2} \sum_{i=1}^n \|a_i\|_1}. \tag{2.2}$$

We claim that

$$\varphi_1(r) \leq \frac{2^n c\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\gamma} \tag{2.3}$$

and

$$\varphi_2(r) \geq \frac{\int_0^1 F(x,\delta,\ldots,\delta)\,dx \;-\; \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\frac{\delta^2}{2} \sum_{i=1}^n \|a_i\|_1}, \tag{2.4}$$

from which (ii) of Theorem 2.1 follows.

In fact, taking into account that the function identically 0 obviously belongs to $\Phi^{-1}(]-\infty,r[)$, and that $\Psi(0)=0$, we get

$$\varphi_1(r) \leq \frac{\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{r},$$

and, since $\overline{\Phi^{-1}(]-\infty,r[)}^w = \Phi^{-1}(]-\infty,r])$, we have

$$\frac{\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{r} = \frac{\sup_{\Phi^{-1}(]-\infty,r])} \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{r}.$$

Since for each $u_i \in W^{1,2}([0,1])$

$$\sup_{x\in[0,1]} |u_i(x)|^2 \leq c\|u_i\|^2$$

for $1 \le i \le n$ (see (1.2)), we have that

$$\sup_{x\in[0,1]} \sum_{i=1}^{n} \frac{|u_i(x)|^2}{2} \le c \sum_{i=1}^{n} \frac{\|u_i\|^2}{2} = c\Phi(u)$$

for every $u = (u_1,\ldots,u_n) \in X$. Thus, taking into account that $\sum_{i=1}^{n} |u_i(x)|^2 \le \frac{\gamma}{2^{n-1}}$, for every $u = (u_1,\ldots,u_n) \in X$ such that $\Phi(u) \le r$ and for each $x \in [0,1]$, we obtain

$$\frac{\sup_{\Phi^{-1}(]-\infty,r])} \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{r} \le \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{r}.$$

So, (2.3) follows at once by the definition of r.

Moreover, for each $v = (v_1,\ldots,v_n) \in X$ such that $\Phi(v) \ge r$, we have

$$\varphi_2(r) \ge \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_0^1 F(x,v_1(x),\ldots,v_n(x))\,dx \;-\; \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{\Phi(v)-\Phi(u)},$$

thus, from $\sum_{i=1}^{n} |u_i(x)|^2 \le \frac{\gamma}{2^{n-1}}$, for every $u = (u_1,\ldots,u_n) \in X$ such that $\Phi(u) < r$ and for each $x \in [0,1]$, we obtain

$$\inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_0^1 F(x,v_1(x),\ldots,v_n(x))\,dx \;-\; \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{\Phi(v)-\Phi(u)}$$

$$\ge \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_0^1 F(x,v_1(x),\ldots,v_n(x))\,dx \;-\; \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\Phi(v)-\Phi(u)},$$

from which, being $0 < \Phi(v)-\Phi(u) \le \Phi(v)$ for every $u \in \Phi^{-1}(]-\infty,r[)$, and under further condition

$$\int_0^1 F(x,v_1(x),\ldots,v_n(x))dx \ge \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx, \tag{2.5}$$

we can write

$$\inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_0^1 F(x,v_1(x),\ldots,v_n(x))dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\Phi(v)-\Phi(u)}$$

$$\ge \frac{\int_0^1 F(x,v_1(x),\ldots,v_n(x))dx - \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x,t_1,\ldots,t_n)dx}{\sum_{i=1}^{n} \frac{\|v_i\|^2}{2}}.$$

If we put $v(x) := (\delta,\ldots,\delta)$, for each $x \in [0,1]$, we have $\|v_i\| = \|a_i\|_1^{1/2}\delta$ for $1 \le i \le n$.

Now since $\delta^2 > \frac{\gamma}{2^{n-1}}$, bearing in mind that $\frac{1}{\|a_i\|_1} \leq c$ for $1 \leq i \leq n$, we get $\Phi(v) = \frac{\delta^2}{2} \sum_{i=1}^n \|a_i\|_1 > r$. Moreover, with this choice of v, (2.2) ensures (2.5), thus (2.4) is also proved.

Taking into account that the weak solutions of system (1.1) are exactly the solutions of the equation $\Phi'(u) + \lambda\Psi'(u) = 0$, we have the conclusion by using of Theorem 2.1. Namely, by observing that

$$\frac{1}{\varphi_2(r)} \leq \frac{\frac{\delta^2}{2} \sum_{i=1}^n \|a_i\|_1}{\int_0^1 F(x, \delta, \ldots, \delta)\, dx \;-\; \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x, t_1, \ldots, t_n) dx}$$

and

$$\frac{1}{\varphi_1(r)} \geq \frac{\gamma}{2^n c \int_0^1 \sup\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(x, t_1, \ldots, t_n) dx},$$

for each $\lambda \in]\lambda', \lambda''[$ system (1.1) admits at least three weak solutions in X.

□

It is of interest to list some special cases of Theorem 2.2.

Theorem 2.3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and assume that there exist two positive constants γ and δ with $\delta^2 > \frac{\gamma}{2^{n-1}}$ such that*

(j$'$) $\dfrac{\max\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(t_1,\ldots,t_n)}{\gamma} < \dfrac{F(\delta,\ldots,\delta)}{2^n c \delta^2 \sum_{i=1}^n \|a_i\|_1}$, *where c and $K(\frac{\gamma}{2^{n-1}})$ are given by* (1.2) *and* (2.1)*;*

(jj$'$) $\limsup\limits_{|t_1|\to+\infty,\ldots,|t_n|\to+\infty} \dfrac{F(t_1,\ldots,t_n)}{\sum_{i=1}^n \frac{|t_i|^2}{2}} \leq 0;$

(jjj$'$) $F(0, \ldots, 0) = 0.$

Then, setting

$$\lambda' := \frac{\frac{\delta^2}{2} \sum_{i=1}^n \|a_i\|_1}{F(\delta, \ldots, \delta) \;-\; \max\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(t_1, \ldots, t_n)}$$

and

$$\lambda'' := \frac{\gamma}{(2^n c) \max\limits_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{2^{n-1}})} F(t_1, \ldots, t_n)},$$

for each $\lambda \in]\lambda', \lambda''[$ the system

$$\begin{cases} -u_i'' + a_i(x)u_i = \lambda F_{u_i}(u_1, \ldots, u_n) & \text{in } (0, 1), \\ u_i'(0) = u_i'(1) = 0, \end{cases}$$

for $1 \leq i \leq n$, admits at least three weak solutions in X.

Corollary 2.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Put $F(t)=\int_0^t f(\xi)d\xi$ for each $t\in\mathbb{R}$ and assume that there exist two positive constants γ and δ with $\delta^2>\gamma$ such that*

(j$''$) $\frac{\max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t)}{\gamma} < \frac{F(\delta)}{2c\delta^2\|a\|_1}$, *where* $c := \sup_{u\in W^{1,2}([0,1])\setminus\{0\}} \left(\frac{\|u\|_\infty}{\|u\|}\right)^2$;

(jj$''$) $\limsup_{|t|\to+\infty} \frac{F(t)}{|t|^2} \le 0$.

Then, setting

$$\lambda' := \frac{\delta^2\|a\|_1}{2(F(\delta) - \max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t))}$$

and

$$\lambda'' := \frac{\gamma}{(2c)\max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t)},$$

for each $\lambda\in]\lambda',\lambda''[$ the problem

$$\begin{cases} -u'' + a(x)u = \lambda f(u) & in\ (0,1),\\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions in $W^{1,2}([0,1])$.

Acknowledgment. This research was in part supported by a grant from University of Bojnord (No. 97/367/4878).

References

[1] G. Anello and G. Cordaro, An existence theorem for the Neumann problem involving the p-Laplacian, *J. Convex Anal.* **10**(2003), 185-198.

[2] D. Averna and G. Bonanno, A three critical points theorem and its applications to the ordinary Dirichlet problem, *Topol. Methods Nonlinear Anal.* **22**(2003), 93-103.

[3] J.R. Graef, S. Heidarkhani and L. Kong, A critical points approach to multiplicity results for multi-point boundary value problems, *Appl. Anal.* **90**(2011), 1909-1925.

[4] J.R. Graef and L. Kong, Existence of solutions for nonlinear boundary value problems, *Comm. Appl. Nonlinear Anal.* **14**(2007), 39-60.

[5] S. Heidarkhani and D. Motreanu, Multiplicity results for a two-point boundary value problem, *Panamer. Math. J.* **19**(2009), 69-78.

[6] S. Heidarkhani and Y. Tian, Multiplicity results for a class of gradient systems depending on two parameters, *Nonlinear Anal.* **73**(2010), 547-554.

[7] R. Livrea, Existence of three solutions for a quasilinear two point boundary value problem, *Arch. Math.* **79**(2002), 288-298.

[8] B. Ricceri, On a three critical points theorem, *Arch. Math.* **75**(2000), 220-226.

[9] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol. II/B, Berlin-Heidelberg-New York, 1990.