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On category of co-coverings

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Abstract. In this article, we introduce the concept of co-covering which is the dual of covering concept. Then, we prove several theorems being similar to the theorems that have been developed for the covering concept. For example, we provide the lifting criterion for co-coverings of a topological space X, which helps us to classify them by subgroups of the group of all homeomorphisms of X.

Keywords: Covering map, Covering space, Co-covering.

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1. Introduction

Covering spaces play an important role in the homotopy theory, harmonic analysis, and differential topology, and many researchers are interested in them. This became even more interesting, when the well-known one-to-one correspondences between the equivalent classes of connected covering spaces and the conjugacy classes of some subgroups of its fundamental group were found; see [2, 3]. The dual concept of covering spaces of a topological space has not been explored in previous studies. In this paper, we intend to introduce the category of *co-covering spaces* of a topological space X, which is denoted by $CoCov(X)$. It is clear that if $p : Y \to X$ is a covering map, then (Y, p) belongs in the category of covering spaces of X, denoted by $Cov(X)$. As a dual notion, one can

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say that (p, X) is a co-covering space of Y and belongs to $CoCov(Y)$. Therefore, similar to covering spaces, the study of co-covering spaces of a space could be beneficial, however no effort has been made in this field so far. At first, in Section 2, some examples are given for showing that the category of co-covering spaces may not have the terminal object . Moreover, we present some results for co-coverings of spacial types of topological spaces such as path connected compact spaces, Euclidean spaces, and topological groups. In Section 3, a necessary and sufficient condition for existence of the morphism between two co-covering spaces of a space are presented. Using this notion, we can find some equivalence conditions for equivalenty of co-covering spaces (Proposition 3.5). Finally, we intend to give a classification for co-covering spaces of an arbitrary topological space by its covering transformation group.

2. Co-coverings

Recall that a continuous map $p : \widetilde{X} \to X$ is a covering map if every $x \in X$ has an open neighborhood U_x that is evenly covered by p, that is, $p^{-1}(U_x)$ is a disjoint union of open subsets of $\widetilde{X}$, each of which is mapped homeomorphically onto U_x by p. In this case, p is called a covering of X and $(\widetilde{X}, p)$ or $\widetilde{X}$ is called a covering space of X. We define the dual concept of covering space of a topological space as follows.

Definition 2.1. If X is a topological space, then a *co-covering space* of X is a pair $(p, \widetilde{X})$ such that $p : X \to \widetilde{X}$ is a covering map.

The category of all co-covering spaces of X is denoted by $CoCov(X)$. The morphism between co-coverings $(p, \widetilde{X})$ and $(q, \widetilde{Y})$ of X, is a continuous map $h : \widetilde{X} \to \widetilde{Y}$ that $h \circ p = q$. Clearly, for every space X, the category $Cov(X)$ has the terminal object (X, id_X). Also if X is connected, locally path connected, and semi locally simply connected, $Cov(X)$ has the (universal) initial object; see [2]. Indeed $CoCov(X)$ has the initial object(id_X, X) for every space X. But in general, the terminal object may not exist in $CoCov(X)$ even X is semi locally simply connected (see Example 2.2).

Example 2.2. As a famous example, one can investigate the category of co-covering spaces of the real line. For every $n \in \mathbb{N}$, clearly, $(\alpha_n, \mathbb{S}^1)$ with $\alpha_n(t) = e^{2n\pi ti}$ are some examples of co-covering spaces of $\mathbb{R}$. Clearly, for any $n \in \mathbb{N}$, the map $q_n : \mathbb{S}^1 \to \mathbb{S}^1$ with $q_n(e^{2\pi ti}) = e^{2n\pi ti}$, for every $t \in \mathbb{R}$, is a covering map. Moreover, if m is n divisible, then the map $q_{n,m} : \mathbb{S}^1 \to \mathbb{S}^1$ with $q_{n,m}(e^{2n\pi ti}) = e^{2m\pi ti}$ is also a covering map such that $q_{n,m} \circ q_n = q_m$ and $q_{n,m} \circ \alpha_n = \alpha_m$. This fact shows that $q_{n,m}$ is a morphism in $CoCov(\mathbb{S}^1)$ and $CoCov(\mathbb{R})$. Now we show that $CoCov(\mathbb{S}^1)$ does not have a terminal object. Let $CoCov(\mathbb{S}^1)$ have the terminal object $f:\mathbb{S}^1 \to Y$. Since f is a covering map and $\mathbb{S}^1$ is Hausdorff, then Y is Hausdorff. Let $y_0 \in Y$ and let $n \in \mathbb{N}$. Since f is the terminal object in $CoCov(\mathbb{S}^1)$, then there is a continuous map $h:\mathbb{S}^1 \to Y$ such that $h \circ q_n = f$. Hence $q_n^{-1}(h^{-1}(y_0)) = f^{-1}(y_0)$, which implies that $f^{-1}(y_0)$ has at least n elements, since every fiber of q_n has n elements. Therefore f is an infinite sheeted covering map which is a contradiction (see Proposition 2.3).

For a co-covering space $(p, \widetilde{X})$ of X, since $p: X \to \widetilde{X}$ is a covering map, the fiber $p^{-1}(\tilde{x})$ is a discrete subset of X. This fact implies the following results.

Proposition 2.3. *Every compact space has no infinite sheeted T_1 path connected co-covering space.*

Proof. Let $(p, \widetilde{X})$ be a co-covering space of a compact space X. If $\widetilde{X}$ is T_1 and $\tilde{x} \in \widetilde{X}$, then $\{\tilde{x}\}$ is closed. Continuity of p implies that $p^{-1}(\tilde{x})$ is a closed subset of X, and hence it is compact. Moreover, the fiber $p^{-1}(\tilde{x})$ is a discrete subset of X, since $p: X \to \widetilde{X}$ is a covering map. Therefore $p^{-1}(\tilde{x})$ is finite since every discrete compact space is finite. □

It is well-known that if G is a topological group and (X, p) is a covering space of G, then X is a topological group and $p: G \to H$ is a group homomorphism. The converse statement is not true, in general, that is, if $p: G \to X$ is a covering map and G is a topological group, p may not transfer the group structure to make X a topological group (see Example 2.7). Of course, it needs an extra condition which we call it *compatibility.*

Definition 2.4. Let G be a topological group and let X be an arbitrary topological space. A map $p: G \to X$ is called *compatible* if $p(a) = p(c)$ and $p(b) = p(d)$ implies $p(a.b) = p(c.d)$ for every $a, b, c, d \in G$.

Proposition 2.5. *Let (G, m) be a topological group and let (p, X) be a co-covering space of G. Therefore X is also a topological group and $p: G \to X$ is a group homomorphism if and only if p is compatible.*

Proof. It is clear that if p is compatible, then there exists one way to define a group structure on X that makes p a group homomorphism. For every $x, y \in X$, put

$$\mu(x, y) = p(m(a, b)),$$

where $a \in p^{-1}(x)$ and $b \in p^{-1}(y)$. Thus it is enough to show that the multiplication μ and the operation taking inverse, are continuous. Consider the following commutative digram:

Since p is a covering map, then $p \times p$ is a covering map which implies that it is an open map. Therefore μ is continuous, since m is continuous.

$$\begin{array}{ccc} G \times G & \xrightarrow{p \times p} & X \times X \\ \downarrow m & \circlearrowleft & \downarrow \mu \\ G & \xrightarrow[p]{} & X \end{array}$$

One can proceed similarly with the inverse map to show that it is also continuous. Therefore, compatibility is sufficient to make X a topological group.
For the converse, let (X, μ) be a topological group and let $p : G \to X$ be a homomorphism. If $a, b, c, d \in G$ such that $p(a) = p(c)$ and $p(b) = p(d)$, then $p(m(a, b)) = \mu(p(a), p(b)) = \mu(p(c), p(d)) = p(c.d)$, which implies that p is compatible.

□

Corollary 2.6. *Let G be a topological group and let (p, X) be a co-covering space of G. If p is compatible, then X is isomorphic to $\frac{G}{ker(p)}$.*

In the following example, we show that there is a covering map from a topological group which is not compatible.

Example 2.7. It is well-known that the Torus is a double covering of the Kline bottle. Clearly, $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ is a topological group by its natural action. On the other hand, there is no action to make the Kline bottle as a topological group. It shows that the covering map $p : \mathbb{T} \to \mathbb{K}$ is not compatible.

Example 2.8. Let $p : \mathbb{S}^1 \to \mathbb{S}^1$ be a map with $p(e^{2\pi ti}) = e^{4\pi ti}$ if $0 \leq t \leq \frac{1}{2}$ and $p(e^{2\pi ti}) = e^{8\pi ti}$ if $\frac{1}{2} \leq t \leq 1$. Clearly, p is a covering map which is not a compatible map, since $p(e^{0\times\pi i}) = p(e^{\pi i})$ and $p(e^{\frac{1}{2}\pi i}) = e^{\pi i} \neq e^{0\times\pi i} = p(e^{\frac{3}{2}\pi i})$.

3. Classification of co-coverings

In the following, we introduce a necessary and sufficient condition for existence the morphism between two co-covering spaces of X.

Definition 3.1. Let X be a topological space and let $p : (X, x_0) \to (\widetilde{X}, \tilde{x}_0)$ be a map. A co-lifting of a continuous map $q : (X, x_0) \to (\widetilde{Y}, \tilde{y}_0)$ with respect to p, is a continuous map $\tilde{q} : (\widetilde{X}, \tilde{x}_0) \to (\widetilde{Y}, \tilde{y}_0)$ such that $q = \tilde{q} \circ p$.

Theorem 3.2. *Let X be a path connected topological space and let $q : (X, x_0) \to (\widetilde{Y}, \tilde{y}_0)$ be a continuous map. If $(p, \widetilde{X})$ is a co-covering space*

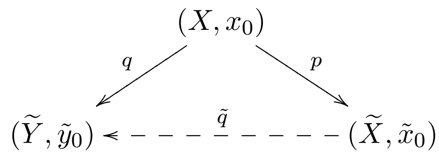


FIGURE 1. Diagram of co-lifting property

of X, then q has the unique co-lifting $\tilde{q}$ with respect to p if and only if for every $\tilde{x} \in \widetilde{X}$, there is $\tilde{y} \in \widetilde{Y}$ such that $p^{-1}(\tilde{x}) \subseteq q^{-1}(\tilde{y})$.

Proof. For every $\tilde{x} \in \widetilde{X}$ take $f : I \to \widetilde{X}$ as a path starting at $\tilde{x}_0$ and ending at $\tilde{x}$. Since p is a covering map, there exists a unique lifting $\tilde{f} : I \to X$ starting at x_0 such that $p \circ \tilde{f} = f$. Define $\tilde{q} : \widetilde{X} \to \widetilde{Y}$ with $\tilde{q}(\tilde{x}) = q \circ \tilde{f}(1)$. We show that $\tilde{q}$ is the unique well-defined continuous map which commutes the diagram of Figure 1.

Let $g : I \to \widetilde{X}$ be another path starting at $\tilde{x}_0$ and ending at $\tilde{x}$. Take $\tilde{g} : I \to X$ as the unique lifting of g starting at x_0. Clearly $\tilde{f}(1)$ and $\tilde{g}(1)$ are in $p^{-1}(\tilde{x})$. If $\tilde{y} = \tilde{q}(\tilde{x}) = q \circ \tilde{f}(1)$, then by hypothesis $\tilde{f}(1)$ and $\tilde{g}(1)$ belong to $q^{-1}(\tilde{y})$ for some $\tilde{y} \in \widetilde{Y}$, that is, $q(\tilde{f}(1)) = q(\tilde{g}(1)) = \tilde{y}$. Therefore, $\tilde{q}$ is well-defined.

For any $x \in X$ put $\tilde{x} = p(x)$ and $\tilde{y} = \tilde{q}(\tilde{x})$. By hypothesis, $x \in p^{-1}(\tilde{x}) \subseteq q^{-1}(\tilde{y})$, that is,

$$q(x) = \tilde{y} = \tilde{q}(\tilde{x}) = \tilde{q}(p(x)) = \tilde{q} \circ p(x),$$

which shows that the diagram is commute. Moreover, $\tilde{q}$ is continuous because q is continuous and p is open. Indeed, for any map $\tilde{r} : (\widetilde{X}, \tilde{x}_0) \to (\widetilde{Y}, \tilde{y}_0)$ with $\tilde{r} \circ p = q$, let $\tilde{x} \in \widetilde{X}$ and pick $x \in p^{-1}(\tilde{x})$. Then $p(x) = \tilde{x}$ and

$$\tilde{r}(\tilde{x}) = \tilde{r}(p(x)) = \tilde{r} \circ p(x) = q(x) = \tilde{q} \circ p(x) = \tilde{q}(p(x)) = \tilde{q}(\tilde{x}).$$

Conversely, if $\tilde{q}$ exists and $\tilde{y} = \tilde{q}(\tilde{x})$, then for every $x \in p^{-1}(\tilde{x})$

$$\tilde{y} = \tilde{q}(\tilde{x}) = \tilde{q}(p(x)) = q(x),$$

which shows that $p^{-1}(\tilde{x}) \subseteq q^{-1}(\tilde{y})$. □

Corollary 3.3. *Let X be a path connected topological space. Then there exists a morphism $\tilde{q} : (\widetilde{X}, \tilde{x}) \to (\widetilde{Y}, \tilde{x})$ between co-coverings $(p, \widetilde{X})$ and $(q, \widetilde{Y})$ of X if and only if for every $\tilde{x} \in \widetilde{X}$, there is $\tilde{y} \in \widetilde{Y}$ such that $p^{-1}(\tilde{x}) \subseteq q^{-1}(\tilde{y})$. Moreover, $\tilde{q}$ is a semicovrering map.*

Proof. If every fiber of p is a subset of a fiber of q, then by Theorem 3.2, there is a continuous map $\tilde{q}$ such that $q = \tilde{q} \circ p$. Now since every covering map is a semicovering map, then [1, Lemma 3.4] implies that $\tilde{q} : (\widetilde{X}, \tilde{x}) \to (\widetilde{Y}, \tilde{x})$ is a semicovering map. □

It is well-known that in the case of semi locally simply connected spaces every semicovering map is a covering map; see [1]. The following corollary is obtained from this fact and Corollary 3.3.

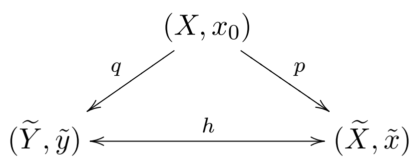
Corollary 3.4. *Let X be a path connected topological space and let $(p, \widetilde{X}), (q, \widetilde{Y}) \in CoCov(X)$. If for every $\tilde{x} \in \widetilde{X}$, there is $\tilde{y} \in \widetilde{Y}$ such that $p^{-1}(\tilde{x}) \subseteq q^{-1}(\tilde{y})$ and $\widetilde{Y}$ is a semilocally simply connected space, then $\tilde{q} : \widetilde{X} \to \widetilde{Y}$ is a covering map.*

As an equivalence in a category, two co-covering space $(p, \widetilde{X}), (q, \widetilde{Y})$ of a space X are equivalent if there exist two continuous maps $h : (\widetilde{X}, \tilde{x}_0) \to (\widetilde{Y}, \tilde{y}_0)$ and $f : (\widetilde{Y}, \tilde{y}_0) \to (\widetilde{X}, \tilde{x}_0)$ such that $h \circ p = q$ and $f \circ q = p$. Combination of this fact with Theorem 3.2 implies the following proposition.

Proposition 3.5. *For a path connected pointed space (X, x_0) the following statements are equivalent:*

i) *Two co-covering space $(p, \widetilde{X}), (q, \widetilde{Y})$ of X are equivalent.*

ii) *There exists a homeomorphism $h : \widetilde{X} \to \widetilde{Y}$ making the following diagram commutative:*

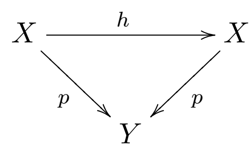


iii) *For every $\tilde{x} \in \widetilde{X}$, there is $\tilde{y} \in \widetilde{Y}$ such that $p^{-1}(\tilde{x}) = q^{-1}(\tilde{y})$.*

Proof. $i) \Leftrightarrow ii)$ is clear.

$i) \Leftrightarrow iii)$ For every $\tilde{x} \in \widetilde{X}$, there is $\tilde{y} \in \widetilde{Y}$ such that $p^{-1}(\tilde{x}) = q^{-1}(\tilde{y})$. Using Corollary 3.3 twice , one can conclude the existence of maps $\tilde{q} : (\widetilde{X}, \tilde{x}) \to (\widetilde{Y}, \tilde{y})$ and $\tilde{p} : (\widetilde{Y}, \tilde{y}) \to (\widetilde{X}, \tilde{x})$ with $\tilde{q} \circ p = q$ and $\tilde{p} \circ q = p$, respectively. Therefore, $(p, \widetilde{X})$ and $(q, \widetilde{Y})$ are equivalent co-covering spaces of X. The converse statement is the trivial result of the definition. □

Recall from [2] that if (X, p) is a covering space of a space Y, then a *covering transformation* (or *deck transformation*) is a homeomorphism $h : X \to X$ with $p \circ h = p$, that is, the following diagram commutes:



The set of all covering transformation of X which commutes by p is denoted by $COV(X/Y)$ and is a subgroup of, $HOM(X)$, the group of all homeomorphisms of X, clearly.

Theorem 3.6. *Let X be a path connected space and let $(p, \widetilde{X})$ and $(q, \widetilde{Y})$ be two co-covering spaces of X. Then $Cov(X/\widetilde{X}) \leq Cov(X/\widetilde{Y})$ if for every $\tilde{x} \in \widetilde{X}$, there is $\tilde{y} \in \widetilde{Y}$ such that $p^{-1}(\tilde{x}) \subseteq q^{-1}(\tilde{y})$. The converse is true if (X, p) is a regular covering of $\widetilde{X}$.*

Proof. If every fiber of p is a subset of a fiber of q, then by Corollary 3.3, there exists a co-lifting $\tilde{q} : (\widetilde{X}, x_0) \to (\widetilde{Y}, y_0)$ such that $\tilde{q} \circ p = q$. If $h \in Cov(X/\widetilde{X})$, then $p \circ h = p$, which implies that

$$q = \tilde{q} \circ p = \tilde{q} \circ (p \circ h) = (\tilde{q} \circ p) \circ h = q \circ h$$

Therefore, $h \in Cov(X/\widetilde{Y})$ and so $Cov(X/\widetilde{X}) \leq Cov(X/\widetilde{Y})$.

Conversely, let $\tilde{x}_0 \in \widetilde{X}$, let $x, x_0 \in p^{-1}(\tilde{x}_0)$, and let $\tilde{y}_0 = q(x_0)$. Using [2, Theorem 10.18] implies that $Cov(X/\widetilde{X})$ acts transitively on $p^{-1}(\tilde{x}_0)$. Thus there is $h \in Cov(X/\widetilde{X})$ such that $h(x_0) = x$. Since $Cov(X/\widetilde{X}) \leq Cov(X/\widetilde{Y})$, $h \in Cov(X/\widetilde{Y})$, that is, $q \circ h = q$. Hence $\tilde{y}_0 = q(x_0) = q \circ h(x_0) = q(x)$. Therefore, $x \in q^{-1}(\tilde{y}_0)$, which implies that $p^{-1}(\tilde{x}_0) \subseteq q^{-1}(\tilde{y}_0)$.

□

By Proposition 3.5 and Theorem 3.6 we have the following corollary.

Corollary 3.7. *Let $(p, \widetilde{X}), (q, \widetilde{Y})$ be co-covering spaces of a path connected space X. If p and q are regular coverings, then $(p, \widetilde{X})$ and $(q, \widetilde{Y})$ are equivalent if and only if $Cov(X/\widetilde{X}) = Cov(X/\widetilde{Y})$.*

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