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Recovering Sturm-Liouville operators with a discontinuous weight function

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Abstract. In this study, we investigate the inverse problem for Sturm-Liouville differential operators on a finite interval having a discontinuous weight function. We give properties of the spectral characteristics and prove the uniqueness theorem for the inverse problem of recovering operators.

Keywords: Inverse problem, Sturm-Liouville equation, Piece-wise continuous function, Weyl function.
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1. Introduction

Various problems of mathematics, physics, mechanics, geophysics, electronics and other branches of natural sciences can be modeled by Sturm-Liouville equations [3, 19, 22, 26] and inverse problems for these equations are applied for the recovery of parameters. For example, an inverse spectral technique is used to reconstruct some components of the residual stress tensor in the arterial wall [15, 16].

In this work, we consider the boundary valve problem $B(q(x), h, H)$ for the differential equation

$$-y'' + q(x)y = \lambda r(x)y, \quad x \in (0, 1)\,, \tag{1.1}$$

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and the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$

$$V(y) := y'(1) + Hy(1) = 0. \tag{1.3}$$

Let

$$r(x) = \begin{cases} 1, & x < \frac{1}{2}, \\ 16x^2, & x > \frac{1}{2}. \end{cases}$$

Here h and H are real numbers and $\lambda = \rho^2 (\rho = \sigma + i\tau)$ is a spectral parameter. The function $q(x)$ is real-valued and $q(x) \in L^2(0,1)$.

The inverse problem which is the reconstruction of the operator from spectral data is a main research topic. The study of the inverse problems for Sturm-Liouville operators has been begun in 1929 and the results of these researches can be found in the monographs [5, 6, 11, 13, 14, 20, 25, 28, 29]. Some researchers have also investigated the inverse problems for the Sturm-Liouville equation in various forms in recent years. For example, inverse problems for Sturm-Liouville equations with a singularity have been studied in [17, 27, 30]. Discontinuous direct and inverse problems were discussed in [1, 2, 4, 7, 9, 17, 18]. Many scholars have investigated inverse problems for the Sturm-Liouville operator with a piece-wise continuous weight function (see for example [7, 8, 21]). In this work, we investigate the uniqueness solution of the inverse Sturm-Liouville problem with a piece-wise continuous coefficient when the weight-function is a polynomial of second degree in the half of the interval which has not been studied yet. We use the method of the spectral mappings which is an effective and universal method for studying a various class of inverse problems for Sturm-Liouville operators such as with singularities, with piece-wise continuous coefficients and pencils of operators. As a main tool, we define the generalization of the Weyl function for the classical Sturm-Liouville operator that uniquely determines the parameters of the problem. In this study, we introduce the Weyl function and prove the uniqueness theorem for the solution of the inverse problem by the Weyl function. To define the Weyl function, we use the fundamental system of solutions for equation (1.1).

The paper is organized as follows. Properties of the spectral characteristics and the Weyl-type function are introduced and investigated in Sec. 2. In Sec. 3 the uniqueness theorem which is related to the uniqueness of the solution of inverse problems is discussed. Sec. 4 contains some conclusion.

2. Preliminaries

In this section, we establish properties of the spectral characteristics of the BVP(B). Moreover, we formulate the inverse problem of the reconstruction of BVP(B) from the Weyl function. The method employed is similar to those used in Re. [13, 23, 29].

Let $C_0(x,\rho)$, $S_0(x,\rho)$, $\varphi(x,\rho)$ and $\psi(x,\rho)$ be the solution of Eq. (1.1) under the initial conditions

$$\begin{aligned} &C_0(0,\rho) = S_0'(0,\rho) = \varphi(0,\rho) = \psi(1,\rho) = 1, \\ &S_0(0,\rho) = C_0'(0,\rho) = 0, \\ &\varphi'(0,\rho) = h, \quad \psi'(1,\rho) = -H. \end{aligned} \tag{2.1}$$

By regarding to these conditions, it is trivial that the functions $\varphi(x,\rho)$ and $\psi(x,\rho)$ are the solutions of (1.1),(1.2) and (1.1),(1.3), respectively. For each x, the functions $C_0(x,\rho)$, $S_0(x,\rho)$, $\varphi(x,\rho)$ and $\psi(x,\rho)$ are entire in ρ, and

$$< C_0(x,\rho), S_0(x,\rho) >= 1, \tag{2.2}$$

where $< y(x), z(x) >= yz' - z'y$ is the Wronskian of y and z. Denote

$$\Delta(\rho) =:< \psi(x,\rho), \varphi(x,\rho) > . \tag{2.3}$$

This function is called the characteristic function for the boundary value problem B and is entire in ρ. By virtue of Liouville's formula for the Wronskian [10], we know that the Wronskian does not depend on x. So, substituting $x = 0$ and $x = 1$ in (2.3), we get

$$\Delta(\rho) = V(\varphi(x,\rho)) = -U(\psi(x,\rho)). \tag{2.4}$$

Theorem 2.1. *For $|\rho| \longrightarrow \infty$, the characteristic function of the boundary value problem B has the following form*

$$\Delta(\rho) = -\rho\left(3sin2\rho + sin\rho\right) + O\left(exp\left(2|\tau|\right)\right). \tag{2.5}$$

Proof. For sufficiently large ρ, we know that the fundamental system of solutions (FSS) of Eq. (1.1) is of the form

$$\begin{cases} C_0(x,\rho) = cos\rho x + O\left(\frac{1}{\rho}exp(|\tau|x)\right), \\ S_0(x,\rho) = \frac{sin\rho x}{\rho} + O\left(\frac{1}{\rho^2}exp(|\tau|x)\right), \end{cases} \quad x < \frac{1}{2}, \tag{2.6}$$

and

$$\begin{cases} C_0(x,\rho) = cos2\rho x^2 + O\left(\frac{1}{\rho}exp(2|\tau|x^2)\right), \\ S_0(x,\rho) = \frac{sin2\rho x^2}{\rho} + O\left(\frac{1}{\rho^2}exp(2|\tau|x^2)\right), \end{cases} \quad x > \frac{1}{2} \tag{2.7}$$

(see [1, 13]). Taking this FSS, we can write for $x < \frac{1}{2}$,

$$\varphi(x,\rho) = A_1(\rho)C_0(x,\rho) + A_2(\rho)S_0(x,\rho). \tag{2.8}$$

By simple computations and considering the initial conditions at $x = 0$, we get for $|\rho| \longrightarrow \infty$,

$$\varphi(x,\rho) = cos\rho x + O\left(\frac{1}{\rho}exp(|\tau|x)\right), \quad x < \frac{1}{2}. \tag{2.9}$$

Applying this FSS, we can write again for $x > \frac{1}{2}$,

$$\varphi(x,\rho) = B_1(\rho)C_0(x,\rho) + B_2(\rho)S_0(x,\rho). \tag{2.10}$$

Considering the smooth condition $\varphi^{(m)}\left(\frac{1}{2}-0,\rho\right) = \varphi^{(m)}\left(\frac{1}{2}+0,\rho\right)$, $m = 0, 1$, and the relations (2.7), (2.9) and (2.10), we get for sufficiently large ρ,

$$B_1(\rho) = \frac{1}{4}(3+cos\rho) + O\left(\frac{1}{\rho}exp\left(|\tau|\right)\right), \quad B_2(\rho) = \frac{1}{4}\rho sin\rho + O\left(exp\left(|\tau|\right)\right).$$

Now substituting these coefficients and FSS (2.7) in (2.10), we can give for $|\rho| \longrightarrow \infty$,

$$\varphi(x,\rho) = \frac{3}{4}cos2\rho x^2 + \frac{1}{4}cos\rho(1-2x^2) + O\left(\frac{1}{\rho}exp(2|\tau|x^2)\right), \quad x > \frac{1}{2}. \tag{2.11}$$

So

$$\varphi'(x,\rho) = -3\rho x sin2\rho x^2 + \rho x sin\rho(1-2x^2) + O\left(exp(2|\tau|x^2)\right), \quad x > \frac{1}{2}. \tag{2.12}$$

By taking (1.3) and (2.4) and the relations (2.11) and (2.12), we arrive at (2.5). □

Definition 2.2. [13] The values of the parameter λ for which B has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions.

Since the eigenvalues $\lambda_n = \rho_n^2$ of the boundary value problem B coincide with the zeros of the characteristic function $\Delta(\rho)$ [13], by taking Rouche's theorem [12], we can say for sufficiently large n,

$$\rho_n = \rho_n^0 + o\left(1\right), \tag{2.13}$$

where ρ_n^0 is the simple zeros of the function

$$\Delta^0(\rho) = -\rho\left(3sin2\rho + sin\rho\right). \tag{2.14}$$

Therefore the sequence $\{\rho_n^0\} = \{\rho_{n,1}^0\} \cup \{\rho_{n,2}^0\}$ has the form

$$\rho_{n,1}^0 = n\pi, \quad \rho_{n,2}^0 = (2n+1)\pi \pm cos^{-1}\left(\frac{1}{6}\right),$$

and satisfies the relations

$$\lambda_0 < \lambda_1 < \lambda_2 < ... \to \infty.$$

By using the asymptotic formula for $\Delta(\rho)$, we have the following lemma.

Lemma 2.3. *Fixe $\delta > 0$ and define $G_\delta := \{\rho; \mid \rho - \rho_n \mid \geq \delta, \ \forall n \geq 0\}$. We have the inequality*

$$\mid \Delta(\rho) \mid \geq C_\delta \mid \rho \mid exp(2|\tau|), \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*, \tag{2.15}$$

for sufficiently large $\rho^* = \rho^*(\delta)$ and some positive constant C_δ.

Proof. This lemma can be proven by the same methods which can be found in [13, 24]. □

Similar to the proof of Theorem 2.1, we can give the following theorem.

Theorem 2.4. *For $|\rho| \longrightarrow \infty$, the following asymptotic formulae are valid:*

$$\psi(x,\rho) = cos2\rho(1-x^2) + O\left(\frac{1}{\rho}exp(2|\tau|(1-x^2))\right), \qquad x > \frac{1}{2},$$

$$\psi(x,\rho) = \frac{3}{2}cos\rho(2-x) - \frac{1}{2}cos\rho(1+x) + O\left(\frac{1}{\rho}exp(|\tau|(2-x))\right), \quad x < \frac{1}{2}.$$

Now we define the Weyl function which is a main tool for solving inverse problems. We put the meromorphic function

$$\phi(x,\rho) = -\frac{\psi(x,\rho)}{\Delta(\rho)}, \tag{2.16}$$

that is called the Weyl solution of the boundary value problem B. It is trivial that this function satisfies $U(\phi) = 1$ and $V(\phi) = 0$. Denote $M(\rho) := \phi(0,\rho)$ and is called the Weyl function of the boundary value problem B. Taking the initial conditions at $x = 0$, we have

$$\phi(x,\rho) = S_0(x,\rho) + M(\rho)\varphi(x,\rho). \tag{2.17}$$

Clearly,

$$< \varphi(x,\rho), \phi(x,\rho) >= 1. \tag{2.18}$$

Here by regarding to the functions $\varphi(x,\rho)$ and $\psi(x,\rho)$ and analogue methods applied in the works [23, 24], we can give the following lemma.

Lemma 2.5. *We can obtain that for $m = 0, 1$,*

$$\begin{cases} |\varphi^{(m)}(x,\rho)| \le C|\rho|^m exp(|\tau|x), & x < \frac{1}{2}, \\ \\ |\varphi^{(m)}(x,\rho)| \le C|\rho|^m exp(2|\tau|x^2), & x > \frac{1}{2}, \end{cases} \tag{2.19}$$

and

$$\begin{cases} |\psi^{(m)}(x,\rho)| \le C|\rho|^m exp\left(|\tau|\left(2-x\right)\right), & x < \frac{1}{2}, \\ \\ |\psi^{(m)}(x,\rho)| \le C|\rho|^m exp(2|\tau|(1-x^2)), & x > \frac{1}{2}. \end{cases} \tag{2.20}$$

Lemma 2.6. *It follows the following inequalities for $\rho \in G_\delta$,*

$$\begin{cases} |\phi^{(m)}(x,\rho)| \le C_\delta|\rho|^{m-1} exp\left(-|\tau|x\right), & x < \frac{1}{2}, \\ \\ |\phi^{(m)}(x,\rho)| \le C_\delta|\rho|^{m-1} exp\left(-2|\tau|x^2\right), & x > \frac{1}{2}. \end{cases} \tag{2.21}$$

Proof. We have these results from (2.15), (2.16) and (2.20). □

The inverse problem is formulated as follows:

Inverse problem. Given the Weyl function $M(\rho)$, construct the potential function $q(x)$ and the coefficients h, H.

3. Main result

Now we can express the uniqueness theorem that is the main theorem of this paper. For this purpose, together with $B = B(q(x), h, H)$, we will consider a boundary value problem $\widetilde{B} = B(\widetilde{q}(x), \widetilde{h}, \widetilde{H})$ of the form

$$-y'' + \widetilde{q}(x)y = \lambda r(x)y, \quad x \in (0,1), \tag{3.1}$$

and the boundary conditions

$$U(y) := y'(0) - \widetilde{h}y(0) = 0, \tag{3.2}$$

$$V(y) := y'(1) + \widetilde{H}y(1) = 0. \tag{3.3}$$

If a certain symbol denotes an object related to B, then the corresponding symbol with tilde will denote the analogs object related to $\widetilde{B}$.

Theorem 3.1. *Assume that $M(\rho) = \widetilde{M}(\rho)$. Then $B = \widetilde{B}$, i.e., the specification of the Weyl function $M(\rho)$ uniquely determines B.*

Proof. We consider the matrix $P(x,\rho) = \left(P_{j,k}(x,\rho)\right)_{j,k=1,2}$ by the formula

$$P(x,\rho) \begin{pmatrix} \widetilde{\varphi}(x,\rho) & \widetilde{\phi}(x,\rho) \\ \widetilde{\varphi}'(x,\rho) & \widetilde{\phi}'(x,\rho) \end{pmatrix} = \begin{pmatrix} \varphi(x,\rho) & \phi(x,\rho) \\ \varphi'(x,\rho) & \phi'(x,\rho) \end{pmatrix}. \tag{3.4}$$

So

$$\begin{cases} \varphi(x,\rho) = P_{11}(x,\rho)\widetilde{\varphi}(x,\rho) + P_{12}(x,\rho)\widetilde{\varphi}'(x,\rho), \\ \\ \phi(x,\rho) = P_{11}(x,\rho)\widetilde{\phi}(x,\rho) + P_{12}(x,\rho)\widetilde{\phi}'(x,\rho), \end{cases} \tag{3.5}$$

and by virtue of (2.18), we have

$$\begin{cases} P_{11}(x,\rho) = \varphi(x,\rho)\widetilde{\phi}'(x,\rho) - \phi(x,\rho)\widetilde{\varphi}'(x,\rho), \\ \\ P_{12}(x,\rho) = \phi(x,\rho)\widetilde{\varphi}(x,\rho) - \varphi(x,\rho)\widetilde{\phi}(x,\rho). \end{cases} \tag{3.6}$$

Using (2.17) and (3.6), we calculate

$$\begin{cases} P_{11}(x,\rho) = \varphi(x,\rho)\widetilde{S_0}'(x,\rho) - S_0(x,\rho)\widetilde{\varphi}'(x,\rho) \\ +(\widetilde{M}(\rho) - M(\rho))\varphi(x,\rho)\widetilde{\varphi}'(x,\rho), \\ \\ P_{12}(x,\rho) = \widetilde{\varphi}(x,\rho)S_0(x,\rho) - \widetilde{S_0}(x,\rho)\varphi(x,\rho) \\ -(\widetilde{M}(\rho) - M(\rho))\varphi(x,\rho)\widetilde{\varphi}(x,\rho). \end{cases}$$

On the other hand, from the hypothesis $M(\rho) = \widetilde{M}(\rho)$, we get

$$\begin{cases} P_{11}(x,\rho) = \varphi(x,\rho)\widetilde{S_0}'(x,\rho) - S_0(x,\rho)\widetilde{\varphi}'(x,\rho), \\ \\ P_{12}(x,\rho) = \widetilde{\varphi}(x,\rho)S_0(x,\rho) - \widetilde{S_0}(x,\rho)\varphi(x,\rho), \end{cases}$$

and consequently $P_{1k}(x,\rho),\ k = 1,2$ are entire in ρ for each fixed x. It follows from (2.18) and (3.6) that

$$\begin{cases} P_{11}(x,\rho) = 1 + (\varphi(x,\rho) - \widetilde{\varphi}(x,\rho))\,\widetilde{\phi}'(x,\rho) \\ - \left(\phi(x,\rho) - \widetilde{\phi}(x,\rho)\right)\widetilde{\varphi}'(x,\rho), \\ \\ P_{12}(x,\rho) = \left(\phi(x,\rho) - \widetilde{\phi}(x,\rho)\right)\varphi(x,\rho) \\ - (\varphi(x,\rho) - \widetilde{\varphi}(x,\rho))\,\phi(x,\rho). \end{cases} \tag{3.7}$$

Denote $G_\delta^0 = G_\delta \cap \widetilde{G_\delta}$. By virtue of (2.19) and (2.21), we get as $\rho \in G_\delta^0$, and sufficiently large ρ

$$\lim_{|\rho|\longrightarrow\infty} (\varphi(x,\rho) - \widetilde{\varphi}(x,\rho))\,\widetilde{\phi}'(x,\rho) = 0, \quad \lim_{|\rho|\longrightarrow\infty} \left(\phi(x,\rho) - \widetilde{\phi}(x,\rho)\right)\widetilde{\varphi}'(x,\rho) = 0,$$

$$\lim_{|\rho|\longrightarrow\infty} \left(\phi(x,\rho) - \widetilde{\phi}(x,\rho)\right)\varphi(x,\rho) = 0, \quad \lim_{|\rho|\longrightarrow\infty} (\varphi(x,\rho) - \widetilde{\varphi}(x,\rho))\,\phi(x,\rho) = 0.$$

Therefore by using (3.7), we infer $P_{11}(x,\rho) = 1$ and $P_{12}(x,\rho) = 0$. Now together with (3.5), this yields $\widetilde{\varphi}(x,\rho) = \varphi(x,\rho)$ and $\widetilde{\phi}(x,\rho) = \phi(x,\rho)$

for all $x,\ \rho$. Thus $q(x)=\widetilde{q}(x)$ for $x\in(0,1)$, $h=\widetilde{h}$ and $H=\widetilde{H}$. The proof is completed. $\square$

4. Conclusion

In the present work, we developed the spectral mappings method for solving a class of inverse Sturm-Liouville problems. We show that the boundary value problem B is uniquely determined by the Weyl function.

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