Fundamental Commutative (m, n)-rings Obtained from (m, n)-hyperrings

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ABSTRACT. We introduce the fundamental relation α^* on an (m, n)-hyperring R and prove that it is the smallest strongly regular equivalence relation on R, such that the quotient R/α^* is a commutative (m, n)-ring. We use α^* to construct the fundamental functor from category of (m, n)-hyperrings to category of commutative (m, n)-rings, which assign to each (m, n)-hyperring its fundamental (m, n)-ring. Finally, some necessary and sufficient conditions for transitivity of α are given.

Keywords: n-ary hypergroup; (m, n)-hyperring; strongly regular relation.

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1. INTRODUCTION

The first paper about n-ary groups was written by Dörnte ([12]) in 1928. This concept was extended to algebraic hyperstructures with n-ary hypergroups, defined by Davvaz and Vougiouklis ([10]) as a generalization of hypergroups (Marty [17]). (m, n)-hyperrings are other type of n-ary algebraic hyperstructures, which are as an extension of (m, n)-rings ([6], [7]) in the framework of hyperstructures. Some applications of n-hypergroups and (m, n)-hyperrings can be seen in [3] (hyperideals),

[11], [20] (fundamental relations), [15], [16], [19] (binary relations) and also other concepts in [7].

On the other hands, one of the most important tools in algebraic hyperstructures is represented by strongly regular relations, in particular fundamental relations, which connect an algebraic hyperstructure to the associated algebraic structure. The fundamental relation Γ^* were studied on hyperrings by Vougiouklis and Spartalis in [22], [23] and [24]. After defining (m, n)-hyperrings by Mirvakili and Davvaz in [20], they introduced the concept of strongly regular relations on (m, n)-hyperrings and were able to obtain (m, n)-rings from (m, n)-hyperrings, using the fundamental relation Γ^* defined on (m, n)-hyperrings. N. Jafarzadeh and R. Ameri introduced and studied the category of (m, n)-hypermodules and shown this category is exact [1].

The α^*-relation is another fundamental relation on hyperrings which was introduced by Davvaz and Vougiouklis in [9] (for more details see [8]). In [21], Pelea applied α^*-relation for general hyperstructures and arbitrary identities and also for (m, n)-hyperrings, in framework of multialgebras theory and universial algebras. Now, in this paper, we provide α-relation for (m, n)-hyperrings, with its usual and conventional view in n-ary hyperstructures, similar to what happens for Γ^*-relation on (m, n)-hyperrings ([20]). Hence, we prove that α^* as transitive closure of α is a commutative fundamental relation which commutative (m, n)-rings can be derived by it. Moreover, the connection between categories of (m, n)-rings and (m, n)-hyperrings is investigated by using functors and the fundamental relation α^*. Finally, some necessary and sufficient conditions for transitivity of the relation α are stated, and some results are obtained regarding them.

2. Preliminaries

In this section we give some definitions and results of n-ary hyperstructures which we need in what follows.

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \longrightarrow \mathcal{P}^*(H)$ is called an *n-ary hyperoperation*, where $\mathcal{P}^*(H)$ is the set of all the nonempty subsets of H. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid*.

For abbreviation, we denote

$$f(x_1, ..., x_i, y_{i+1}, ..., y_j, z_{j+1}, ..., z_n)$$

as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. Also, if $y_{i+1} = \cdots = y_j = y$, then it will be written as $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$. Moreover, if f is an n-ary hyperoperation and $t = l(n - 1) + 1$, for some $l \geq 0$, then t-ary hyperoperation $f_{(l)}$ is given

by

$$f_{(l)}(x_1^{l(n-1)+1}) = \underbrace{f(f(..., f(f(x_1^n), x_{n+1}^{2n-1}), ...),}_{l} x_{(l-1)(n-1)+1}^{l(n-1)+1})$$

and $f_{(0)}(x) = \{x\}$. For nonempty subsets $A_1, ..., A_n$ of H we define $f(A_1^n) = \bigcup_{x_i \in A_i} f(x_1^n)$ such that $1 \le i \le n$. An n-ary hyperoperation f is called *associative* if

$$f\left(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\right),$$

hold for every $1 \le i < j \le n$ and all $x_1^{2n-1} \in H$. An n-ary hypergroupoid with the associative n-ary hyperoperation is called an *n-ary semihypergroup*.

An n-ary semihypergroup (H, f) is called an *n-ary hypergroup*, if $f(x_1^{i-1}, H, x_{i+1}^n) = H$ for all $x_1^n \in H$ and $1 \le i \le n$. An n-ary hypergroupoid (H, f) is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$ we have $f(a_1^n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)})$. We denote $(a_{\sigma(1)}, ..., a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$. By ([19]) A non-empty subset B of an n-ary hypergroup (H, f) is called an *n-ary subhypergroup* of H, if $f(x_1^n) \subseteq B$ for all $x_1^n \in B$, and the equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution $x_i \in B$ for all $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \le i \le n$.

Definition 2.1. ([19]) An (m, n)-hyperring is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

(i) (R, f) is an m-ary hypergroup;

(ii) (R, g) is an n-ary semihypergroup;

(iii) the n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \le i \le n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \cdots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

An (m, n)-hyperring (R, f, g) is said to be Krasner ([14],[19]), if (R, f) is a canonical n-ary hypergroup, i.e.,

(1) f is commutative;

(2) there exists a unique $e \in H$, such that $f(x, \underbrace{e, \cdots, e}_{n-1}) = \{x\}$, for all $x \in H$;

(3) for all $x \in H$ there exists a unique $x^{-1} \in H$, such that $e \in f(x, x^{-1}, \underbrace{e, \cdots, e}_{n-2})$;

(4) if $x \in f(x_1^n)$, then for all $1 \le i \le n$, we have

$$x_i \in f(x, x^{-1}, \cdots, x_{i-1}^{-1}, x_{i+1}^{-1}, \cdots, x_n^{-1}).$$

and (R, g) is an n-ary semigroup such that 0 is a zero element (absorbing element) of the n-ary operation g, i.e. for all $x_2^n \in R$ we have

$$g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0).$$

Example 2.2. Consider the set of all integers, $\mathbb{Z}$, with the following hyperoperations defined for $x, y \in \mathbb{Z}$,

$$x \oplus y = \{x, y, x + y\}, \quad \text{and} \quad x \otimes y = \{x \cdot y\},$$

where "$+$" and "$\cdot$" are ordinary addition and multiplication on $\mathbb{Z}$. It is routine to check that $(\mathbb{Z}, \oplus, \otimes)$ is a hyperring. For $x_1^m, y_1^n \in \mathbb{Z}$, set $g(y_1^n) = \bigotimes_{i=1}^n y_i = \{\prod_{j=1}^n y_j\}$ and

$$f(x_1^m) = \bigoplus_{i=1}^m x_i = \left\{x_1^m, \ x_{i_1} + x_{i_2}, \ \dots, \ x_{i_1} + x_{i_2} + \cdots + x_{i_m}\right\}$$

such that $i_1, i_2, \dots, i_m$ are different natural numbers from 1 to m. Then, $(\mathbb{Z}, f, g)$ is an (m, n)-hyperring, by [4]. Note that $(\mathbb{Z}, f, g)$ is not a Krasner (m, n)-hyperring although $(\mathbb{Z}, g)$ is a trivial n-ary semihypergroup.

Example 2.3. ([19]) Suppose that $(L, \vee, \wedge)$ is a relatively complemented distributive lattice and "f" and "g" are defined on L as follows:

$$f(a_1, a_2) = \{c \in L \mid a_1 \wedge c = a_2 \wedge c = a_1 \wedge a_2\}, \quad \forall a_1, a_2 \in L,$$
$$g(a_1^n) = \vee_{i=1}^n a_i, \quad \forall a_1^n \in L.$$

It follows that (L, f, g) is a Krasner $(2, n)$-hyperring.

An equivalence relation ρ on an n-ary hypergroup (H, f) is called *regular*, if $a_1 \rho b_1, \dots, a_n \rho b_n$, for $a_1^n, b_1^n \in R$, then $f(a_1^n) \bar{\rho} f(b_1^n)$, that is,

$$\forall x \in f(a_1^n), \ \exists y \in f(b_1^n) ; \ x \rho y \quad \text{and} \quad \forall u \in f(b_1^n), \ \exists v \in f(a_1^n) ; \ u \rho v.$$

Also, ρ is called *strongly regular* if $a_i \rho b_i$ for all $1 \le i \le n$, implies that $x \rho y$ for all $x \in f(a_1^n)$ and for all $y \in f(b_1^n)$, that is shown by $f(a_1^n) \bar{\bar{\rho}} f(b_1^n)$.

Let (R, f, g) be an (m, n)-hyperring, then we say that ρ is (strongly) regular on R, if ρ is (strongly) regular with respect to both f and g.

Theorem 2.4. ([20]) *If (R, f, g) is an (m, n)-hyperring and the relation ρ is a strongly regular relation on both (R, f) and (R, g), then the quotient $R/\rho = \{\rho(x) \mid x \in R\}$ with the following m-ary and n-ary operations is an (m, n)-ring.*

$$f/\rho\big(\rho(x_1), \dots, \rho(x_m)\big) = \rho(z); \quad \forall z \in f(x_1^m),$$
$$g/\rho\big(\rho(y_1), \dots, \rho(y_n)\big) = \rho(d); \quad \forall d \in g(y_1^n).$$

Also, Mirvakili and Davvaz in [20] defined the relation Γ on (m, n)-hyperrings as follow:
Let (R, f, g) be an (m, n)-hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$ where

$s = k(m-1) + 1$, define a relation $\Gamma_{k;l_1^s}$, as follow:

$x\Gamma_{k;l_1^s}y$ if and only if there exist $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1) + 1$ and $i = 1, ..., s$ such that $\{x, y\} \subseteq f_{(k)}(u_1, ..., u_s)$ where for every $i = 1, ..., s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Now, set $\Gamma_k = \bigcup_{l_1^s \in \mathbb{N}} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbb{N}^*} \Gamma_k$. This definition is a natural generalization of the relation Γ on hyperrings $((2,2)$-hyperrings) defined by Vougiouklis in [23]. In [20], it is shown that the transitive closure of Γ, Γ^*, is a strongly regular relation on (m,n)-hyperrings such that $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m,n)-ring. Moreover, it was shown that Γ^* is the smallest equivalence relation such that $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m,n)-ring. Hence, $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is said to be fundamental (m,n)-ring obtained from Γ^*-relation.

3. α-RELATION ON (m,n)-HYPERRINGS

In this section, we introduce α-relation on (m,n)-hyperrings, as a generalization of α-relation on hyperrings, to construct a commutative fundamental (m,n)-ring.

Definition 3.1. Let (R, f, g) be an (m,n)-hyperring. For $k \in \mathbb{N}^*$, $l_1^r \in \mathbb{N}$ and $x, y \in R$, define the relation $\alpha_{k;l_1^r}$ as follow:

$$x\alpha_{k;l_1^r}y \iff \exists z_{11}^{1t_1}, ..., z_{r1}^{rt_r} \in R, \ \exists \sigma \in \mathbb{S}_r, \ \exists \sigma_i \in \mathbb{S}_{t_i};$$

$$x \in f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r})\right) \ \text{ and } \ y \in f_{(k)}\left(u_{\sigma(1)}, ..., u_{\sigma(r)}\right)$$

where $u_i = g_{(l_i)}(z_{i\sigma_i(1)}^{i\sigma_i(t_i)})$, $r = k(m-1) + 1$, $t_i = l_i(n-1) + 1$, and $1 \leq i \leq r$. Now, set $\alpha = \bigcup_{k \geq 0}\left(\bigcup_{l_1^r \in \mathbb{N}} \alpha_{k;l_1^r}\right)$.

It is easy to see that the relation α is reflexive and symmetric. Let α^* be transitive closure of α.

Theorem 3.2. *The relation α^* is a strongly regular equivalence relation on (R, f, g).*

Proof. Since α is reflexive and symmetric, and α^* is transitive closure of α, then α^* is an equivalence relation. Hence, we show that α^* is strongly regular. In order to let $a_1\alpha b_1, ..., a_m\alpha b_m$, for all $a_1^m, b_1^m \in R$, and let $x \in f(a_1^m)$ and $y \in f(b_1^m)$. Thus, for $i \in \{1, ..., m\}$ there exist $x_{1 1}^{1t_{ij}}, ..., x_{r1}^{rt_{ij}} \in R$, $\sigma_i \in \mathbb{S}_{r_i}$ and $\sigma_{ij} \in \mathbb{S}_{t_{ij}}$ such that $r_i = k_i(m-1) + 1$, $t_{ij} = l_{ij}(n-1) + 1$, $1 \leq j \leq r_i$ and

$$a_i \in f_{(k_i)}\left(g_{(l_{i1_i})}(x_{11}^{1_it_{i1}}), ..., g_{(l_{ir_i})}(x_{r_i1}^{r_it_{ir_i}})\right)$$
$$b_i \in f_{(k_i)}\left(u_{\sigma_i(i1)}, ..., u_{\sigma_i(ir_i)}\right),$$

where $u_{ij} = g_{(l_{ij})}(x_{1_i\sigma_{ij}(1)}^{1_i\sigma_{ij}(t_{ij})})$. Hence, we have

$$x \in f(a_1^m) \subseteq f\Big(f_{(k_1)}\big(g_{(l_{11})}(x_{111}^{1_1t_{11}}), ..., g_{(l_{1r_1})}(x_{r_11}^{r_1t_{1r_1}})\big),$$
$$..., f_{(k_m)}\big(g_{(l_{m1})}(x_{1m1}^{1_mt_{m1}}), ..., g_{(l_{mr_m})}(x_{rm1}^{r_mt_{mr_m}})\big)\Big)$$
$$= f_{(1+k_1+\cdots+k_m)}\Big(g_{(l_{11})}(x_{111}^{1_1t_{11}}), ..., g_{(l_{1r_1})}(x_{r_11}^{r_1t_{1r_1}}),$$
$$..., g_{(l_{m1})}(x_{1m1}^{1_mt_{m1}}), ..., g_{(l_{mr_m})}(x_{rm1}^{r_mt_{mr_m}})\Big),$$

similarly,

$$y \in f(b_1^m) \subseteq f_{(1+k_1+\cdots+k_m)}\big(u_{\sigma_1(11)}, ..., u_{\sigma_1(1r_1)}, ..., u_{\sigma_m(m1)}, ..., u_{\sigma_m(mr_m)}\big).$$

In this case, we have $r_1 + \cdots + r_m = (1 + k_1 + \cdots + k_m)(m-1) + 1$. Consider $\tau \in \mathbb{S}_{r_1+\cdots+r_m}$ such that $\tau(ij) = \sigma_i(ij)$, then we have

$$x \in f_{(1+k_1+\cdots+k_m)}\Big(g_{(l_{11})}(x_{111}^{1_1t_{11}}), ..., g_{(l_{1r_1})}(x_{r_11}^{r_1t_{1r_1}}),$$
$$..., g_{(l_{m1})}(x_{1m1}^{1_mt_{m1}}), ..., g_{(l_{mr_m})}(x_{rm1}^{r_mt_{mr_m}})\Big),$$

and $y \in f_{(1+k_1+\cdots+k_m)}\big(u_{\tau(11)}, ..., u_{\tau(1r_1)}, ..., u_{\tau(m1)}, ..., u_{\tau(mr_m)}\big)$. Therefore $(x,y) \in \alpha_{1+k_1+\cdots+k_m;l_{11}^{mr_m}}$ and so $x\alpha y$. Hence, $f(a_1^m)\ \overline{\overline{\alpha}}\ f(b_1^m)$. Now, if $x_1\alpha^*y_1, ..., x_m\alpha^*y_m$ for all $x_1^m, y_1^m \in R$, then

$$\exists\ x_1 = w_{01}, w_{11}, ..., w_{n1} = y_1 \in R\ ;\ x_1 = w_{01}\alpha w_{11}\alpha\cdots\alpha w_{n1} = y_1$$

$$\vdots$$

$$\exists\ x_m = w_{0m}, w_{1m}, ..., w_{nm} = y_m \in R\ ;\ x_m = w_{0m}\alpha w_{1m}\alpha\cdots\alpha w_{nm} = y_m.$$

By strongly regularity of α, we have

$$f(x_1^m) = f(w_{01}^{0m})\ \overline{\overline{\alpha}}\ f(w_{11}^{1m})\ \overline{\overline{\alpha}}\cdots\overline{\overline{\alpha}}\ f(w_{n1}^{nm}) = f(y_1^m).$$

Therefore, for all $u \in f(x_1^m)$ and $v \in f(y_1^m)$, there exist $z_1 \in f(w_{11}^{1m})$, ..., $z_{n-1} \in f(w_{(n-1)1}^{(n-1)m})$ such that $u\alpha z_1\alpha\cdots\alpha z_{n-1}\alpha v$. Thus $u\alpha^*v$, and so $f(x_1^m)\ \overline{\overline{\alpha^*}}\ f(y_1^m)$. On the other hand, for $1 \le i \le n$, $r_i = k_i(m-1) + 1$, $t_{ij} = l_{ij}(n-1) + 1$, $1 \le j \le r_i$ and $x_{1_i1}^{1_it_{ij}}, ..., x_{r_i1}^{r_it_{ij}}$ of R, there exists

$k' > 0$ such that

$$g\Big(f_{(k_1)}\big(g_{(l_{11})}(x_{111}^{11t_{11}}),...,g_{(l_{1r_1})}(x_{r_11}^{r_1t_{1r_1}})\big),...,f_{(k_n)}\big(g_{(l_{n1})}(x_{1n1}^{1nt_{n1}}),...,g_{(l_{nr_n})}(x_{rn1}^{r_nt_{nr_n}})\big)\Big)$$

$$= f_{(k')}\Big(g_{(1+l_{11}+\cdots+l_{n1})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{1n-11}^{1n-1t_{(n-1)1}},x_{1n1}^{1nt_{n1}}),$$

$$g_{(1+l_{11}+\cdots+l_{(n-1)1}+l_{n2})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{1n-11}^{1n-1t_{(n-1)1}},x_{2n1}^{2nt_{n2}}),$$

$$\cdots,g_{(1+l_{11}+\cdots+l_{(n-1)1}+l_{nr_n})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{1n-11}^{1n-1t_{(n-1)1}},x_{rn1}^{r_nt_{nr_n}}),$$

$$g_{(1+l_{11}+\cdots+l_{(n-1)2}+l_{n1})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{2n-11}^{2n-1t_{(n-1)2}},x_{1n1}^{1nt_{n1}}),$$

$$g_{(1+l_{11}+\cdots+l_{(n-1)2}+l_{n2})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{2n-11}^{2n-1t_{(n-1)2}},x_{2n1}^{2nt_{n2}}),$$

$$\cdots,g_{(1+l_{11}+\cdots+l_{(n-1)2}+l_{nr_n})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{2n-11}^{2n-1t_{(n-1)2}},x_{rn1}^{r_nt_{nr_n}}),$$

$$\vdots$$

$$g_{(1+l_{11}+\cdots+l_{(n-1)r_{n-1}}+l_{n1})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{r_{n-1}1}^{r_{n-1}t_{(n-1)r_{n-1}}},x_{1n1}^{1nt_{n1}}),$$

$$g_{(1+l_{11}+\cdots+l_{(n-1)r_{n-1}}+l_{n2})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{r_{n-1}1}^{r_{n-1}t_{(n-1)r_{n-1}}},x_{2n1}^{2nt_{n2}}),$$

$$\cdots,g_{(1+l_{11}+\cdots+l_{(n-1)r_{n-1}}+l_{nr_n})}(x_{111}^{11t_{11}},x_{121}^{12t_{21}},...,x_{r_{n-1}1}^{r_{n-1}t_{(n-1)r_{n-1}}},x_{rn1}^{r_nt_{nr_n}}),$$

$$\vdots$$

$$g_{(1+l_{1r_1}+l_{2r_2}+\cdots+l_{(n-1)r_{n-1}}+l_{nr_n})}(x_{r_11}^{r_1t_{1r_1}},x_{r_21}^{r_2t_{2r_2}},...,x_{r_{n-1}1}^{r_{n-1}t_{(n-1)r_{n-1}}},x_{rn1}^{r_nt_{nr_n}})\Big).$$

Thus by a similar manner, it can be shown that

$$\forall\, i \in \{1,...,n\}\ ;\ a_i\alpha^*b_i \Longrightarrow g(a_1^n)\ \overline{\overline{\alpha^*}}\ g(b_1^n).$$

Consequently, α^* is a strongly regular relation on (R, f, g). $\qquad\square$

Theorem 3.3. *If* (R, f, g) *is an* (m, n)*-hyperring, then* $(R/\alpha^*, f/\alpha^*, g/\alpha^*)$ *is an* (m, n)*-ring.*

Proof. Since α^* is strongly regular relation, by Theorem 2.4 , the quotients $(R/\alpha^*, f/\alpha^*)$ and $(R/\alpha^*, g/\alpha^*)$ are m-ary group and n-ary semigroup, respectively, under the following m-ary and n-ary operations:

$$f/\alpha^*\big(\alpha^*(x_1),...,\alpha^*(x_m)\big) = \alpha^*(z);\quad \forall z \in f(x_1^m),$$
$$g/\alpha^*\big(\alpha^*(y_1),...,\alpha^*(y_n)\big) = \alpha^*(d);\quad \forall d \in g(y_1^n),$$

such that $x_1^m, y_1^n \in R$. Therefore, it is enough to show that n-ary operation g/α^* is distributive respect to f/α^*. For all $x_1^m, y_1^n \in R$, $A_i \subseteq \alpha^*(x_i)$ and $B_j \subseteq \alpha^*(y_j)$ such that $1 \le i \le m$ and $1 \le j \le n$, we can write

$$f/\alpha^*\big(\alpha^*(x_1),...,\alpha^*(x_m)\big) = \alpha^*(f(x_1^m)) = \alpha^*(f(A_1^m))\ \text{ and }$$
$$g/\alpha^*\big(\alpha^*(y_1),...,\alpha^*(y_n)\big) = \alpha^*(g(y_1^n)) = \alpha^*(g(B_1^n))$$

where $\alpha^*(A) = \bigcup_{a \in A} \alpha^*(a)$. Hence, we have

$$g/\alpha^*\Big(\alpha^*|_{y_1}^{y_{i-1}}, f/\alpha^*(\alpha^*|_{x_1}^{x_m}), \alpha^*|_{y_{i+1}}^{y_n}\Big) = g/\alpha^*\Big(\alpha^*|_{y_1}^{y_{i-1}}, \alpha^*(f(x_1^m)), \alpha^*|_{y_{i+1}}^{y_n}\Big)$$
$$= \alpha^*\Big(g(y_1^{i-1}, f(x_1^m), y_{i+1}^n)\Big)$$

and

$$f/\alpha^*\Big(g/\alpha^*\big(\alpha^*|_{y_1}^{y_{i-1}}, \alpha^*(x_1), \alpha^*|_{y_{i+1}}^{y_n}\big), ..., g/\alpha^*\big(\alpha^*|_{y_1}^{y_{i-1}}, \alpha^*(x_m), \alpha^*|_{y_{i+1}}^{y_n}\big)\Big)$$
$$= f/\alpha^*\Big(\alpha^*\big(g(y_1^{i-1}, x_1, y_{i+1}^n)\big), ..., \alpha^*\big(g(y_1^{i-1}, x_m, y_{i+1}^n)\big)\Big)$$
$$= \alpha^*\Big(f\Big(g(y_1^{i-1}, x_1, y_{i+1}^n), ..., g(y_1^{i-1}, x_m, y_{i+1}^n)\Big)\Big)$$

(for abbreviation, $\alpha^*(x_1), ..., \alpha^*(x_i)$ denoted by $\alpha^*|_{x_1}^{x_i}$). Since g is distributive with respect to f in R, the distributivity law is valid in R/α^*. $\square$

Corollary 3.4. *The quotient $(R/\alpha^*, f/\alpha^*, g/\alpha^*)$ is a commutative (m, n)-ring.*

Proof. Let $g/\alpha^*\big(\alpha^*|_{x_1}^{x_n}\big) = \alpha^*(c)$ and $g/\alpha^*\big(\alpha^*|_{x_{\sigma(1)}}^{x_{\sigma(n)}}\big) = \alpha^*(d)$ such that $x_1^n \in R$ and $\sigma \in \mathbb{S}_n$. If $c \in g(x_1^n)$ and $d \in g(x_{\sigma(1)}^{\sigma(n)})$. Then, for $\tau \in \mathbb{S}_1$, we have $c \in f_{(0)}\big(g_{(1)}(x_1^n)\big)$ and $d \in f_{(0)}\big(g_{(\tau(1))}(x_{\sigma(1)}^{\sigma(n)})\big)$. Hence $c\alpha_{0;1}d$ and so $c\alpha^*d$. Then $\alpha^*(c) = \alpha^*(d)$ and so g/α^* is a commutative n-ary operation. Also, if $c \notin g(x_1^n)$ and $d \notin g(x_{\sigma(1)}^{\sigma(n)})$, then there exist $p \in g(x_1^n)$ and $q \in g(x_{\sigma(1)}^{\sigma(n)})$ such that $\alpha^*(c) = \alpha^*(p)$ and $\alpha^*(d) = \alpha^*(q)$, and also $\alpha^*(p) = \alpha^*(q)$. Similarly, for other cases, it can be seen that g/α^* is commutative. Moreover, we can show that f/α^* is a commutative m-ary operation. Therefore, $(R/\alpha^*, f/\alpha^*, g/\alpha^*)$ is a commutative (m, n)-ring. $\square$

Let R be an (m, n)-hyperring. The mapping $\varphi : R \longrightarrow R/\alpha^*$, defined by $\varphi(x) = \alpha^*(x)$ for all $x \in R$, is called *canonical projection* with respect to commutative fundamental relation α^*.

Theorem 3.5. *The relation α^* is the smallest strongly regular equivalence such that the quotient R/α^* is a commutative (m, n)-rings.*

Proof. By Theorem 3.2 and Corollary 4.6, we know that α^* is a strongly regular equivalence relation on R, and the quotient R/α^* is a commutative (m, n)-rings. Hence, we show that α^* is the smallest. Let ρ be a strongly regular relation on R such that R/ρ is a commutative (m, n)-rings and let $\phi : R \longrightarrow R/\rho$ be the canonical projection. If

$x\alpha y$ then there exist $z_{11}^{1t_1}, ..., z_{r1}^{rt_r} \in R$, $\sigma \in \mathbb{S}_r$ and $\sigma_i \in \mathbb{S}_{t_i}$ such that $r = k(m-1) + 1$, $t_i = l_i(n-1) + 1$, $1 \le i \le r$ and

$$x \in f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r})\right) \quad \text{and}$$

$$y \in f_{(k)}\left(g_{(l_{\sigma(1)})}(z_{\sigma(1)\sigma(1)(1)}^{\sigma(1)\sigma(1)(t_{\sigma(1)})}), ..., g_{(l_{\sigma(r)})}(z_{\sigma(r)\sigma(r)(1)}^{\sigma(r)\sigma(r)(t_{\sigma(r)})})\right).$$

Hence, by Theorem 4.5, we have

$$\rho(x) = f/\rho_{(k)}\left(g/\rho_{(l_1)}(\rho|_{z11}^{z_1t_1}), ..., g/\rho_{(l_r)}(\rho|_{z_{r1}}^{z_{rt_r}})\right) \quad \text{and}$$

$$\rho(y) = f/\rho_{(k)}\left(g/\rho_{(l_{\sigma(1)})}(\rho|_{z_{\sigma(1)\sigma(1)(1)}}^{z_{\sigma(1)\sigma(1)(t_{\sigma(1)})}}), ..., g/\rho_{(l_{\sigma(r)})}(\rho|_{z_{\sigma(r)\sigma(r)(1)}}^{z_{\sigma(r)\sigma(r)(t_{\sigma(r)})}})\right).$$

Since f/ρ and g/ρ are commutative, $\rho(x) = \rho(y)$ and so $\alpha \subseteq \rho$. Now, let $x\alpha^* y$, then by transitivity of ρ we have $x\rho y$. Therefore, $\alpha^* \subseteq \rho$ and thus α^* is the smallest strongly regular relation such that R/α^* is a commutative (m,n)-ring.. $\qquad \square$

By Corollary 4.6 and Theorem 3.5, we conclude that α^* is the smallest strongly regular relation such that R/α^* is a commutative (m,n)-ring. Hence, the relation α^* is called *commutative fundamental relation* on (m,n)-hyperring R, and the quotient R/α^* is said to be the *fundamental commutative (m,n)-ring.*

In certain case, for x_1^m, $y_1^n \in R$ if m-ary and n-ary hyperoperations f and g defined by

$$f(x_1^m) = \sum_{i=1}^{m} x_i \quad \text{and} \quad g(y_1^n) = \prod_{j=1}^{n} y_j \ ,$$

then for $r = k(m-1) + 1$, $t_i = l_i(n-1) + 1$ and $1 \le i \le r$,

$$f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r})\right)$$

means that $\sum_{i=1}^{r}\left(\prod_{j=1}^{t_i} x_{ij}\right)$, which is a finite sums of finite products of elements of R. Also, for $\sigma \in \mathbb{S}_r$ and $\sigma_i \in \mathbb{S}_{t_i}$,

$$f_{(k)}\left(g_{(l_{\sigma(1)})}(z_{\sigma(1)\sigma(1)(1)}^{\sigma(1)\sigma(1)(t_{\sigma(1)})}), ..., g_{(l_{\sigma(r)})}(z_{\sigma(r)\sigma(r)(1)}^{\sigma(r)\sigma(r)(t_{\sigma(r)})})\right)$$

is equal to $\sum_{i=1}^{r} A_{\sigma(i)}$ such that $A_i = \prod_{j=1}^{t_i} x_{i\sigma_i(j)}$. Therefore, by reduction of the relation α^* to general hyperrings $(R, +, \cdot)$ $((2,2)$-hyperrings), we will obtain the α^*-relation defined on hyperrings by Davvaz and Vougiouklis in [9].

Remark 3.6. Let $(m,n) - HRg$ and $(m,n) - CRg$ denote the categories of (m,n)-hyperrings and commutative (m,n)-rings, respectively. Also, let $h : R \longrightarrow R'$ be a homomorphism of (m,n)-hyperrings. Consider

the commutative fundamental relation α^* on R and R'. Then the map $h^* : R/\alpha^* \longrightarrow R'/\alpha^*$ given by $h^*\big(\alpha^*(r)\big) = \alpha^*\big(h(r)\big)$ for all $r \in R$, is a homomorphism of (m, n)-rings. Moreover, the following diagram is commutative

$$
\begin{array}{ccc}
R & \xrightarrow{\ h\ } & R' \\
\varphi \downarrow & & \downarrow \varphi' \\
R/\alpha^* & \xrightarrow{\ h^*\ } & R'/\alpha^*
\end{array}
$$

and hence we have the next result.

Theorem 3.7. *The mapping $\Phi : (m,\ n)-HRg \longrightarrow (m,\ n)-CRg$, defined by $R \longmapsto R/\alpha^*$, is a functor.*

Proof. The proof is similar to corresponding result in [2].   □

Consider the fundamental relation Γ^* on (m, n)-hyperrings was defined and studied by Mirvakili and Davvaz in [20]. Then, we have the following corollary:

Corollary 3.8. *If (R, f, g) is a commutative (m, n)-hyperring, then α^* coincide with Γ^*.*

Proof. It is an immediate consequence from definition of α^*.   □

Let R be an (m, n)-hyperring and ϕ be a relation on R such that $\alpha^* \subseteq \phi$. Set

$$
\phi/\alpha^* = \{(\alpha^*(a), \alpha^*(b)) \in R/\alpha^* \times R/\alpha^* \mid (a, b) \in \phi\}.
$$

Then we have the following result:

Theorem 3.9. *If ϕ is a strongly regular relation on (m, n)-hyperring (R, f, g) such that $\alpha^* \subseteq \phi$, then ϕ/α^* is a strongly regular relation on R/α^*.*

Proof. Let $(\alpha^*(a_i), \alpha^*(b_i)) \in \phi/\alpha^*$ for $1 \leq i \leq m$, then $(a_1, b_1) \in \phi$, ..., $(a_m, b_m) \in \phi$. Since ϕ is a strongly regular relation, we have $f(a_1^m) \ \overline{\overline{\phi}} \ f(b_1^m)$. It implies that $(x, y) \in \phi$ for every $x \in f(a_1^m)$ and $y \in f(b_1^m)$. Thus $(\alpha^*(x), \alpha^*(y)) \in \phi/\alpha^*$, for

$$
\alpha^*(x) = f/\alpha^*\big(\alpha^*|_{a_1}^{a_m}\big) \quad \text{and} \quad \alpha^*(y) = f/\alpha^*\big(\alpha^*|_{b_1}^{b_m}\big).
$$

Then $f/\alpha^*\big(\alpha^*(a_1), ..., \alpha^*(a_m)\big) \ \overline{\overline{\phi/\alpha^*}} \ f/\alpha^*\big(\alpha^*(b_1), ..., \alpha^*(b_m)\big)$. Similarly, we can show that ϕ/α^* is strongly regular with respect to g/α^*. This completes the proof.   □

4. TRANSITIVITY OF α-RELATION

Transitivity of α-relation on hyperrings was investigated in [18], by α-parts on hyperrings. The concept of complete parts of n-ary hypergroups was studied by Leoreanu-Fotea and Davvaz in [16] as a generalization of this concept in hypergroups. In this section, the notion of complete parts of (m,n)-hyperings is defined, and some necessary and sufficient conditions are determined such that the relation α is transitive.

Definition 4.1. A non-empty subset B of an (m,n)-hyperring (R,f,g) is called a $\alpha - part$ of R, whenever for $z_{i1}^{it_i} \in R$, $\sigma \in \mathbb{S}_r$, $\sigma_i \in \mathbb{S}_{t_i}$ and $1 \leq i \leq r$

$$f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}),...,g_{(l_r)}(z_{r1}^{rt_r})\right) \cap B \neq \emptyset \implies f_{(k)}(u_{\sigma(1)},...,u_{\sigma(r)}) \subseteq B$$

such that $u_i = g_{(l_i)}(z_{i\sigma_i(1)}^{i\sigma_i(t_i)})$, $r = k(m-1)+1$ and $t_i = l_i(n-1)+1$.

Lemma 4.2. *If B is a non-empty subset of (m,n)-hyperring (R,f,g), then the following conditions are equivalent:*

   (i) *B is a α-part of R;*
   (ii) *$x \in B$ and $x\alpha y$ implies that $y \in B$;*
   (iii) *$x \in B$ and $x\alpha^* y$ implies that $y \in B$.*

Proof. $(i) \Rightarrow (ii)$ Let B be a α-part of R and $x,y \in R$ such that $x \in B$ and $x\alpha y$. Hence, there exist $z_{i1}^{it_i} \in R$, $\sigma \in \mathbb{S}_r$, $\sigma_i \in \mathbb{S}_{t_i}$, and $1 \leq i \leq r$ such that

$$x \in f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}),...,g_{(l_r)}(z_{r1}^{rt_r})\right) = T \text{ and } y \in f_{(k)}(u_{\sigma(1)}^{\sigma(r)}),$$

for $u_i = g_{(l_i)}(z_{i\sigma_i(1)}^{i\sigma_i(t_i)})$, $r = k(m-1)+1$ and $t_i = l_i(n-1)+1$. Therefore, $x \in T \cap B$, and so $f_{(k)}(u_{\sigma(1)}^{\sigma(r)}) \subseteq B$, since B is α-part. Thus $y \in B$.
$(ii) \Rightarrow (iii)$ Assume that $x \in B$ and $x\alpha^* y$ for $x,y \in R$. Then there exist $x = w_0, w_1, ..., w_m = y$ of R such that

$$x = w_0\alpha w_1 \alpha \cdots \alpha w_m = y.$$

Since $x \in B$, by applying (ii) m times, we obtain that $y \in B$.
$(iii) \Rightarrow (i)$ Suppose that $T \cap B \neq \emptyset$ and $x \in T \cap B$. Let $y \in f_{(k)}(u_{\sigma(1)}^{\sigma(r)})$, then we have $x\alpha y$ and so $x\alpha^* y$. By (iii), it implies that $y \in B$ and thus $f_{(k)}(u_{\sigma(1)}^{\sigma(r)}) \subseteq B$. Therefore, B is a α-part of R. $\square$

In the following we introduce some notions which will be used in the next results:

For $x, z_{i1}^{it_i} \in R$, $\sigma \in \mathbb{S}_r$, $\sigma_i \in \mathbb{S}_{t_i}$, $r = k(m-1)+1$, $t_i = l_i(n-1)+1$,
$u_i = g_{(l_i)}(z_{i\sigma_i(1)}^{i\sigma_i(t_i)})$ and $1 \leq i \leq r$, define

$$\mathcal{C}_r(x) = \bigcup \left\{ f_{(k)}(u_{\sigma(1)}^{\sigma(r)}) \mid \sigma \in \mathbb{S}_r, \ \sigma_i \in S_{t_i}, \ x \in f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r}) \right) \right\}$$

$$\mathcal{J}_\sigma(x) = \bigcup_{r \geq 1} \mathcal{C}_r(x).$$

Then, we have:

Lemma 4.3. *For all $x, y \in R$, $y \in \mathcal{J}_\sigma(x)$ if and only if $x \alpha y$.*

Proof. For every $(x, y) \in R^2$ we have

$$x \alpha y \Longleftrightarrow \exists \ z_{11}^{1t_1}, ..., z_{r1}^{rt_r} \in R \ \exists \sigma \in \mathbb{S}_r \ \exists \sigma_i \in \mathbb{S}_{t_i};$$

$$x \in f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r}) \right) \ \text{and} \ y \in f_{(k)}(u_{\sigma(1)}^{\sigma(r)})$$

$$\Longleftrightarrow \exists \ r \ ; \ y \in \mathcal{C}_r(x)$$

$$\Longleftrightarrow y \in \mathcal{J}_\sigma(x).$$

$\square$

Theorem 4.4. *If α is transitive on (R, f, g), then $\alpha^*(x) = \mathcal{J}_\sigma(x)$, for all $x \in R$.*

Proof. Let α be transitive, then by Lemma 4.4, we have

$$y \in \alpha^*(x) \Longleftrightarrow x \alpha^* y \Longleftrightarrow x \alpha y \Longleftrightarrow y \in \mathcal{J}_\sigma(x).$$

$\square$

Theorem 4.5. *Let (R, f, g) be an (m, n)-hyperring. For every $x \in R$, if $\alpha^*(x) = \mathcal{J}_\sigma(x)$, then $\mathcal{J}_\sigma(x)$ is a α-part of R.*

Proof. Suppose that $f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r}) \right) \cap \mathcal{J}_\sigma(x) \neq \emptyset$ and $y \in f_{(k)}(u_{\sigma(1)}^{\sigma(r)})$. Hence, there exists $t \in R$ such that $t \in f_{(k)}\left(g_{(l_1)}(z_{11}^{1t_1}), ..., g_{(l_r)}(z_{r1}^{rt_r}) \right)$ and $t \in \mathcal{J}_\sigma(x)$. Then $t \alpha y$. Moreover, by Lemma 4.4, we have

$$y \in \mathcal{J}_\sigma(t) = \alpha^*(t) = \alpha^*(x) = \mathcal{J}_\sigma(x).$$

Then $f_{(k)}(u_{\sigma(1)}^{\sigma(r)}) \subseteq \mathcal{J}_\sigma(x)$ and so $\mathcal{J}_\sigma(x)$ is a α-part of R. $\square$

Theorem 4.6. *Let (R, f, g) be an (m, n)-hyperring. If for every $x \in R$, $\mathcal{J}_\sigma(x)$ is a α-part of R, then α is transitive on R.*

Proof. Let $x \alpha y$ and $y \alpha z$, then there exist $a_{i1}^{it_i}, b_{j1}^{jt_j} \in R$, $\sigma \in \mathbb{S}_{r_1}$, $\tau \in \mathbb{S}_{r_2}$, $\sigma_i \in S_{t_i}$, $\tau_j \in S_{t_j}$, $r_1 = k_1(m-1)+1$, $r_2 = k_2(m-1)+1$, $t_i = l_i(n-1)+1$,

$t_j = l_j(n-1) + 1$, $u_i = g_{(l_i)}(a_{i\sigma_i(1)}^{i\sigma_i(t_i)})$, $v_j = g_{(l_j)}(b_{j\tau_j(1)}^{j\tau_i(t_j)})$, and $1 \le i \le r_1$, $1 \le j \le r_2$ such that

$$x \in f_{(k_1)}\left(g_{(l_1)}(a_{11}^{1t_1}), ..., g_{(l_{r_1})}(a_{r_1 1}^{r_1 t_{r_1}})\right) \quad \text{and} \quad y \in f_{(k_1)}(u_{\sigma(1)}^{\sigma(r_1)}), \quad \text{and}$$

$$y \in f_{(k_2)}\left(g_{(l_1)}(b_{11}^{1t_1}), ..., g_{(l_{r_2})}(b_{r_2 1}^{r_2 t_{r_2}})\right) \quad \text{and} \quad z \in f_{(k_2)}(v_{\tau(1)}^{\tau(r_2)}).$$

Since $\mathcal{J}_\sigma(x) = \{y \in R \mid x\alpha y\}$ and α is reflexive, $x \in \mathcal{J}_\sigma(x)$ and so

$$x \in f_{(k_1)}\left(g_{(l_1)}(a_{11}^{1t_1}), ..., g_{(l_{r_1})}(a_{r_1 1}^{r_1 t_{r_1}})\right) \cap \mathcal{J}_\sigma(x).$$

Since $\mathcal{J}_\sigma(x)$ is α-part, we have

$$f_{(k_1)}(u_{\sigma(1)}^{\sigma(r_1)}) \subseteq \mathcal{J}_\sigma(x) \Longrightarrow y \in \mathcal{J}_\sigma(x)$$
$$\Longrightarrow y \in f_{(k_2)}\left(g_{(l_1)}(b_{11}^{1t_1}), ..., g_{(l_{r_2})}(a_{r_2 1}^{r_2 t_{r_2}})\right) \cap \mathcal{J}_\sigma(x)$$
$$\Longrightarrow f_{(k_2)}(v_{\tau(1)}^{\tau(r_2)}) \subseteq \mathcal{J}_\sigma(x)$$
$$\Longrightarrow z \in \mathcal{J}_\sigma(x)$$
$$\Longrightarrow x\alpha z \qquad\qquad \text{(by Lemma 4.4)}$$

Therefore, α is transitive. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 4.7. *Let (R, f, g) be an (m, n)-hyperring. Then α-relation is transitive on R if and only if for every $x \in R$, $\alpha^*(x) = \mathcal{J}_\sigma(x)$ if and only if for every $x \in R$, $\mathcal{J}_\sigma(x)$ is a α-part of R.*

Proof. It is proved by Theorem 4.4, Theorem 4.5 and Theorem 4.6. $\quad\square$

Example 4.8. Let $R = \{1, 2, 3, 4, 5, 6, 7\}$. For defining a noncommutative 3-ary hyperoperation f on R, we use the following code in Matlab

software:

"*clear*

clc

$nPop = 7;$

empty-action.addition = [];

$z = repmat$(empty-action,nPop,nPop);

$z(1, 1).addition = [1, 2]; z(1, 2).addition = [1, 2]; z(1, 3).addition = 3; z(1, 4).addition = 4;$

$z(1, 5).addition = 5; z(1, 6).addition = 6; z(1, 7).addition = 7;$

$z(2, 1).addition = [1, 2]; z(2, 2).addition = [1, 2]; z(2, 3).addition = 3; z(2, 4).addition = 4;$

$z(2, 5).addition = 5; z(2, 6).addition = 6; z(2, 7).addition = 7;$

$z(3, 1).addition = 3; z(3, 2).addition = 3; z(3, 3).addition = [1, 2]; z(3, 4).addition = 6;$

$z(3, 5).addition = 7; z(3, 6).addition = 4; z(3, 7).addition = 5;$

$z(4, 1).addition = 4; z(4, 2).addition = 4; z(4, 3).addition = 7; z(4, 4).addition = [1, 2];$

$z(4, 5).addition = 6; z(4, 6).addition = 5; z(4, 7).addition = 3;$

$z(5, 1).addition = 5; z(5, 2).addition = 5; z(5, 3).addition = 6; z(5, 4).addition = 7;$

$z(5, 5).addition = [1, 2]; z(5, 6).addition = 3; z(5, 7).addition = 4;$

$z(6, 1).addition = 6; z(6, 2).addition = 6; z(6, 3).addition = 5; z(6, 4).addition = 3;$

$z(6, 5).addition = 5; z(6, 6).addition = 7; z(6, 7).addition = [1, 2];$

$z(7, 1).addition = 7; z(7, 2).addition = 7; z(7, 3).addition = 4; z(7, 4).addition = 5;$

$z(7, 5).addition = 3; z(7, 6).addition = [1, 2]; z(7, 7).addition = 6;$

$disp('$Enter Your function q as a vector with three components $:')$

$q = input('');$

$s = z(q(1), q(2)).addition;$

$n = numel(s);$

$Sol = [];$

for $i = 1 : n$

$\quad Sol = [Sol, z(s(i), q(3)).addition];$

end

$unique(Sol)$"

In this algorithm [a b] means that $\{a, b\}$ for $a, b \in R$. Also, consider 3-ary hyperoperation g as $g(a, b, c) = \{1, 2\}$ for all $a, b, c \in R$. Then, (R, f, g) is an $(3, 3)$-hyperring. Using this algorithm, it can be seen that

$$f(7, 4, 6) = \{3\}, \quad f(4, 6, 7) = \{4\}, \quad f(6, 4, 7) = \{5\}$$
$$f(7, 5, 4) = \{6\}, \quad f(4, 7, 5) = \{7\}, \quad f(5, 7, 4) = \{1, 2\}.$$

Hence, we have $\alpha(1) = \alpha(2) = \alpha(6) = \alpha(7) = \{1, 2, 6, 7\}$ and $\alpha(3) = \alpha(4) = \alpha(5) = \{3, 4, 5\}$. Therefore, α is transitive on R and so $\alpha^* = \alpha$. Moreover, $\Gamma(1) = \Gamma(2) = \{1, 2\}$ and $\Gamma(x) = \{x\}$ for all $x \in \{3, 4, 5, 6, 7\}$. Thus, $\Gamma \neq \alpha$.

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