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The Generalization of Helices

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Abstract. We defined new special curves in Euclidean 3-space which refer to clad helices and found the geometric invariants of clad helices [17]. These notions are generalizations of the notion of cylindrical helices and slant helices. Using the geometric invariants of clad helices in this article, we proposed approaches to construct examples of clad helices in $\mathbb{E}^3$ and on S^2. Moreover, we obtained the classification of special developable surfaces under the condition of the existence of clad helices as a geodesic and existence of slant helices as a line of curvature.

Keywords: clad helices, slant helices, gauss map, spherical involute, developable surface.
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1. Introduction

In Euclidean 3-space, helices are characterized by their constant curvature and torsion, resembling lines and circles. In other words, helices can be thought as a curve that generalizes a straight line or a circle. Curves that generalized helices are known as "Cylindrical helices","Bertrand curves" and "Mannheim curves". In this paper, we only focused on the cylindrical helices. Cylindrical helices have a geometric invariant that the ratio of curvature to torsion is constant and have long been studied in the field of differential geometry. Cylindrical helices are the subject of research, such as characterization of surfaces including cylindrical helices, classification of surfaces and classification

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of singularities using invariants of cylindrical helices [3, 7, 10, 12, 13, 14]. M.I.Munteanu and A.I.Nistor classified the surface called "Constant angle surfaces", where the normal vector of the surface always makes a constant angle with fixed line, that is, the surface whose Gauss map describes as a part of circle. They found that the tangent surface of cylindrical helices was one of constant angle surfaces [7].

Then, S.Izumiya and N.Takeuchi have defined "Slant helix" as the generalization of cylindrical helices. They found the geometric invariants of slant helices and obtained the classification of developable surfaces under the condition of the existence of slant helices as a geodesic [13]. Several studies have been conducted on slant helices such as studies on characterization of surfaces including slant helices, classification of developable surfaces and classification of singularities using invariants of slant helices [3, 9, 13]. L.Kula and Y.Yayli shows the examples on how to construct slant helices in $\mathbb{E}^3$ using the notion of involute and evolute of a space curve [5]. Many examples of slant helices on various surfaces have been found [1, 2, 4, 6, 8, 9].

We defined the notion of "Clad helices" which is a generalization of the notion of slant helices and found the geometric invariants of clad helices. We also classified singularities of special developable surfaces using the geometric invariants of clad helices [17]. Clad helices is the topic that recently gain interest from many researchers [11, 15, 16]. Furthermore, studies of curves further extending clad helices are also being conducted. S.Kaya and Y.Yayli defined "X-slant helix," which is an extension of clad helix, and performed a study of singularities of special developable surfaces [16]. S.Izumiya, K.Saji and N.Takeuchi defined a curve called "K-th order helices" which is an extension of clad helices, and studied a relation with a surface called "K-th order slope" which is an extension of constant angle surfaces [11].

The aims of this paper are to determine how to construct clad helices and to adapt the surface classifications studied in cylindrical helices and slant helices to clad helices. As resulted, we determined the way to construct clad helices in $\mathbb{E}^3$ and on S^2 (cf. Theorem 3.2 and 6.2). The theorems are extension method of making slant helices shown in [5]. Furthermore, we classified the ruled surfaces whose Gauss map describes as a part of cylindrical helix and slant helix (cf. Theorem 4.1). Theorem 4.1 is an extension of the results on the classification of constant angle surfaces shown in [7]. Then, we also obtained the classification of developable surfaces under the condition of the existence of clad helices as a geodesic and existence of slant helices as a line of curvature (cf. Theorem 5.5 and 5.7). The theorems are extension of the classification of developable surfaces under the condition of the existence of slant helices as geodesics

shown in [13].
In §2, we described basic notions and properties of space curves and define clad helices. In §3, using the notions of *evolute* and *involute*, we studied how to construct examples of clad helices in $\mathbb{E}^3$. In §4, we studied the classification of ruled surfaces whose Gauss maps describe as a part of slant helix. In §5, we studied the classification of developable surfaces under the condition of the existence of clad helices as a geodesic and existence of slant helices as a line of curvature. In §6, considering that a great circle on S^2 corresponds to a line in $\mathbb{E}^3$, we defined *spherical involute* and *spherical evolute*. Using these notions, we studied how to construct examples of cylindrical, slant, and clad helices on S^2. In §7, we give examples of clad helices in $\mathbb{E}^3$ and slant helices on S^2.
All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2. Basic Notions and Properties

In this section, we are reviewing some basic concepts on classical differential geometry of space curves and the definition of clad helices. Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a curve in $\mathbb{E}^3$ and parameterized by its *arc-length* s. Then, we called γ *unit speed curve*. Denote by $\{e_1(s), e_2(s), e_3(s)\}$ the Frenet frame along $\gamma(s)$, where $e_1(s)$ is the *unit tangent vector field*, $e_2(s)$ the *unit normal vector field* and $e_3(s)$ the *unit binormal vector field* of $\gamma(s)$. Then, given the Frenet-Serret formulas as:

$$\begin{cases} e_1'(s) = \kappa(s)e_2(s) \\ e_2'(s) = -\kappa(s)e_1(s) + \tau(s)e_3(s) \\ e_3'(s) = -\tau(s)e_2(s) \end{cases} \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of γ at s. Here, we used "dash" to denote the derivative with respect to s. It is possible in general, that $e_1'(s) = 0$ for some s, however, we assumed that this never happens. For any unit speed curve γ, we defined a vector field $D(s) = (\tau/\kappa)(s)e_1(s) + e_3(s)$ along γ and called it the *modified Darboux vector field* of γ. We also denoted the unit Darboux vector field by $\bar{D}(s) = ((\tau e_1 + \kappa e_3)/((\kappa^2+\tau^2)^{1/2}))(s)$, and defined $D^u(s) = ((-\kappa e_1 + \tau e_3)/(\kappa^2+\tau^2)^{1/2})(s)$ along γ. Then, an orthonormal frame can be defined as $\{e_2(s), D^u(s), \bar{D}(s)\}$. We called this frame as *D-frame*. The following is the Frenet-Serret type formulas :

$$\begin{cases} e_2'(s) = \sqrt{\kappa(s)^2 + \tau(s)^2} D^u(s) \\ (D^u)'(s) = \sqrt{\kappa(s)^2 + \tau(s)^2}(-e_2(s) + \sigma(s)\bar{D}(s)) \\ \bar{D}'(s) = -\sqrt{\kappa(s)^2 + \tau(s)^2}\sigma(s)D^u(s) \end{cases} \tag{2.2}$$

where $\sigma(s) = (\kappa^2(\tau/\kappa)'/(\kappa^2+\tau^2)^{3/2})(s)$.
Let M be a regular surface and $\gamma : I \subset \mathbb{R} \longrightarrow M$ be a unit speed curve on the surface. Then, the *Darboux frame* $\{T(s), B(s) = N(s) \times T(s), N(s)\}$ is well-defined along γ, where $T(s)$ is the unit tangent vector of $\gamma(s)$ and $N(s)$ is the *unit normal vector* of M. Here, "$\times$" is the cross product in $\mathbb{R}^3$. Darboux equations of this frame are given by

$$\begin{cases} T'(s) = \kappa_g(s)B(s) + \kappa_n(s)N(s) \\ B'(s) = -\kappa_g(s)T(s) + \tau_g(s)N(s) \\ N'(s) = -\kappa_n(s)T(s) - \tau_g(s)B(s) \end{cases} \tag{2.3}$$

where $\kappa_n(s)$, $\kappa_g(s)$ and $\tau_g(s)$ are the *normal curvature*, the *geodesic curvature* and the *geodesic torsion* of $\gamma(s)$, respectively. It has been known that if $\kappa_g = 0$, $\gamma(s)$ is a *geodesic* on surface M and if $\tau_g = 0$, $\gamma(s)$ is *a line of curvature* on surface M. With the above notations, we denoted $\phi(s)$ as the angle between the surface normal $N(s)$ and the principal normal of $\gamma(s)$. Then, given the following equations as:

$$\begin{cases} \kappa^2(s) = \kappa_n^2(s) + \kappa_g^2(s) \\ \kappa_g(s) = \kappa(s)\sin\phi(s) \\ \kappa_n(s) = \kappa(s)\cos\phi(s) \\ \tau_g(s) = \tau(s) + d\phi(s)/ds \end{cases} \tag{2.4}$$

and

$$\begin{cases} B(s) = \sin\phi(s)\cdot e_2(s) - \cos\phi(s)\cdot e_3(s) \\ N(s) = \cos\phi(s)\cdot e_2(s) + \sin\phi(s)\cdot e_3(s) \end{cases} \tag{2.5}$$

Let $\gamma(s)$ be a spherical curve on S^2. Then, Darboux frame along $\gamma(s)$ is given by $\{\gamma(s), T(s), B(s)\}$. This frame is called *Sabban frame*. Using equations (2.3) and (2.5), the equations can be derived as follow:

$$\begin{cases} \gamma'(s) = T(s) \\ T'(s) = -\gamma(s) + \kappa_g(s)B(s) \\ B'(s) = -\kappa_g(s)T(s) \end{cases} \tag{2.6}$$

and

$$\begin{cases} \gamma(s) = \cos\phi(s)e_2(s) + \sin\phi(s)e_3(s) \\ B(s) = \sin\phi(s)e_2(s) - \cos\phi(s)e_3(s) \end{cases} \tag{2.7}$$

Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a curve in $\mathbb{E}^3$ and parameterized by its arc-length s. Then, γ is called a *cylindrical helix* if $e_1(s)$ makes a constant angle with a fixed direction. This condition is equivalent to the condition that $e_2(s)$ is orthogonal to fixed direction. Furthermore, γ is called a *slant helix* if $e_2(s)$ makes a constant angle with a fixed direction. This condition is equivalent to the condition that $e_2(s)$ is a circle on S^2 [13]. Recently, we defined the notion of *clad helix* which is generalization of the notion of slant helix. γ is called a clad helix if $e_2(s)$ is a cylindrical helix on S^2

[17]. It has been known that γ is a cylindrical helix when the ratio of torsion, τ to curvature, κ is a constant. If both of κ and τ are constant, γ is a *circular helix*. S.Izumiya and N.Takeuchi showed that γ is a slant helix when

$$\sigma(s) = \left(\frac{\kappa^2}{\left(\kappa^2 + \tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s) \tag{2.8}$$

is a constant [13]. Therefore, $\sigma(s)$ is the invariant of slant helices. We found that γ is a clad helix when

$$\varphi(s) = \left(\frac{\sigma'}{\left(1 + \sigma^2\right)^{3/2} \left(\kappa^2 + \tau^2\right)^{1/2}} \right)(s) \tag{2.9}$$

is a constant [17]. Therefore, $\varphi(s)$ is the invariant of clad helices.

3. Characterization of Clad Helices

In this section, we investigated how to construct examples of clad helices in $\mathbb{E}^3$. Let $\gamma(s)$ be a space curve. We defined a space curve whose tangents are orthogonal to the family of tangents of $\gamma(s)$ as an *involute* of $\gamma(s)$ and defined a space curve which have an involute as $\gamma(s)$ as an *evolute* of $\gamma(s)$. L. Kula and Y. Yayli shows the examples on how to construct slant helices in $\mathbb{E}^3$ using the notion of involute and evolute of a space curve [5]. We obtained that a curve γ is a slant helix if and only if its evolute is a clad helix (cf. Theorem 3.2). Using this result, the examples of clad helices was constructed. It was shown in §7.

Lemma 3.1. [5] *Let $\gamma(s)$ be a space curve. Then,*

(*i*) *$\gamma(s)$ is a cylindrical helix if and only if its evolute is a slant helix.*

(*ii*) *$\gamma(s)$ is a slant helix if and only if its involute is a cylindrical helix.*

We generalized Lemma 3.1 and obtained Theorem 3.2.

Theorem 3.2. *Let $\gamma(s)$ be a space curve. Then,*

(*i*) *$\gamma(s)$ is a slant helix if and only if its evolute is a clad helix.*

(*ii*) *$\gamma(s)$ is a clad helix if and only if its involute is a slant helix.*

Proof. First, we investigated (*i*). We defined $\alpha(s)$ as an evolute of $\gamma(s)$. Then, $\alpha(s)$ was defined as the following equation: $\alpha(s) = \gamma(s) + ((e_2 + \tan(\phi + c)e_3)/\kappa)(s)$, where s is the arc-length parameter of $\gamma(s)$ and c is a constant function and $\phi = -\int \tau(s)ds$ and $\kappa(s)$, $\tau(s)$ are the curvature and the torsion of $\gamma(s)$ at s. Let $\bar{s}$ be the arc-length parameter of $\alpha(s)$. Then, we got $d\bar{s}/ds = \varepsilon_1 \left((1/\kappa)' - (\tau \tan(\phi + c)/\kappa)\right) / \cos(\phi + c)$, where $\varepsilon_1 = \pm 1$ and $\varepsilon_1 \left(\left((1/\kappa)' - (\tau \tan(\phi + c)/\kappa)\right) / \cos(\phi + c)\right) >$

0. We denoted $\kappa_\alpha(s)$ and $\tau_\alpha(s)$ the curvature and the torsion of $\alpha(s)$, respectively. Then,

$$\kappa_\alpha(s) = \varepsilon_2 \frac{\kappa \cos^2(\phi + c)}{\left(\dfrac{1}{\kappa}\right)' - \dfrac{\tau \tan(\phi + c)}{\kappa}}, \tau_\alpha(s) = \frac{\kappa \sin(\phi + c)\cos(\phi + c)}{\left(\dfrac{1}{\kappa}\right)' - \dfrac{\tau \tan(\phi + c)}{\kappa}} \quad (3.1)$$

where $\varepsilon_2 = \pm 1$ and $\varepsilon_2((1/\kappa)' - (\tau \tan(\phi + c)/\kappa)) > 0$. From that, we got

$$\sigma_\alpha(s) = \left(\frac{\kappa_\alpha^2 \dfrac{d}{d\bar{s}} \left(\dfrac{\tau_\alpha}{\kappa_\alpha} \right)}{\left(\kappa_\alpha^2 + \tau_\alpha^2\right)^{3/2}} \right)(s) = -\varepsilon_1 \varepsilon_2 \frac{\tau}{\kappa} \quad (3.2)$$

and

$$\varphi_\alpha(s) = \left(\frac{\dfrac{d}{d\bar{s}} \sigma_\alpha}{\left(1 + \sigma_\alpha^2\right)^{3/2} \left(\kappa_\alpha^2 + \tau_\alpha^2\right)^{1/2}} \right)(s) = -\varepsilon_1 \varepsilon_2 \sigma(s) \quad (3.3)$$

where σ_α and φ_α are invariants of slant helices and clad helices of $\alpha(s)$, respectively. This completes the proof of (i).
After that, we investigated (ii). Let $\beta(s)$ be involute of $\gamma(s)$. Then, $\beta(s)$ was defined as the following equation: $\beta(s) = \gamma(s) + (c - s)e_1(s)$, where c is a constant. We denoted $\sigma_\beta(s)$ the invariant of slant helices of $\beta(s)$. By the same method as in the proof of (i), we got $\sigma_\beta(s) = -\varepsilon\varphi(s)$, where $\varepsilon = \pm 1$ and $\varepsilon(c - s) > 0$. This completes the proof of (ii). $\square$

4. Classification of Ruled Surfaces with Gauss map

In this section, we give the classification of *ruled surfaces* in $\mathbb{E}^3$ with *Gauss map*. A ruled surface in $\mathbb{E}^3$ is the map $F_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{R}^3$ defined by $F_{(\gamma,\delta)}(s, v) = \gamma(s) + v\delta(s)$, where $\gamma : I \to \mathbb{R}^3$, $\delta : \to \mathbb{R}^3 \backslash \{0\}$ are smooth mappings and I is an open interval or a unit circle S^1. We called γ the *base curve* and δ the *director curve*. The straight lines $v \to \gamma(s) + v\delta(s)$ are called *rulings*. *Developable surfaces* are ruled surfaces and have the vanishing *Gaussian curvature* on the regular part. It has been known that developable surfaces are classified in *tangent developable*, *cylindrical surfaces* and *conical surfaces*.
Let $M \subset \mathbb{R}^3$ be an oriented surface and N be the unit normal vector of M. The Gauss map is the map $N : M \to S^2$. The Gauss map has been investigated in differential geometry and singularity theory. Surfaces whose unit normal vector makes a constant angle with a fixed direction at any point on the surfaces are defined [7]. These surfaces are called *constant angle surfaces*, the Gauss map of constant angle surfaces describes as a part of circle on S^2. Then, we remark that a cylindrical helix on S^2 is a generalization of the notion of a circle on S^2 and that a

slant helix is a generalization of the notion of a cylindrical helix on S^2. We studied the surfaces whose Gauss map describes as a part of a cylindrical helix and a slant helix on S^2. As resulted, we found the classification of ruled surfaces with Gauss map in Theorem 4.1.

Theorem 4.1. *Let M be a ruled surface. Suppose that the Gauss map of M is a slant helix on S^2. Then,*

(i) *M is a part of a cylindrical surface.*

(ii) *M is a part of the tangent developable of a clad helix.*

(iii) *M is a part of a conical surface whose director curve is a slant helix.*

Proof. If the Gauss map of M describes as a curve on S^2, the Gaussian curvature of M becomes zero. It follows that M is a developable surface. Therefore, we considered that M is classified in *tangent developable*, *cylindrical surfaces* and *conical surfaces*.
If M is a part of a cylindrical surface, the Gauss map of M describes as a part of a circle on S^2. This proved that (i) consists.
If M is a part of the tangent developable, we defines M: $F_{(\gamma,e_1)}(s,v) = \gamma(s) + ve_1(s)$, where $e_1(s)$ is the unit tangent vector of $\gamma(s)$. Then, we calculated the unit normal vector $N(s)$ of M: $N(s) = e_3(s)$. We denoted $\kappa_{e_3}(s)$ and $\tau_{e_3}(s)$ as the curvature and the torsion of the unit binormal vector $e_3(s)$ of $\gamma(s)$, respectively. Using D-frame, we calculated $\kappa_{e_3}(s) = \varepsilon((\kappa^2+\tau^2)^{1/2}/\tau)(s), \tau_{e_3}(s) = -(\sigma(\kappa^2+\tau^2)^{1/2}/\tau)(s)$, where $\varepsilon = \pm 1$ and $\varepsilon\tau(s) > 0$. It follows that $(\tau_{e_3}/\kappa_{e_3})(s) = \sigma(s)$. By straightforward computation, we got

$$\sigma_{e_3}(s) = \left(\frac{\kappa_{e_3}^2 \left(\dfrac{d}{d\bar{s}} \left(\dfrac{\tau_{e_3}}{\kappa_{e_3}} \right) \right)}{(\kappa_{e_3}^2 + \tau_{e_3}^2)^{3/2}} \right)(s) = -\varepsilon\varphi(s) \tag{4.1}$$

It implies that if the Gauss map of M describes as a part of a slant helix on S^2, the base curve $\gamma(s)$ is a part of a clad helix. Then, M is a part of the tangent developable of a clad helix. This proved that (ii) consists.
If M is a part of a conical surface, we defines M: $F_{(\boldsymbol{p},\alpha)}(s,v) = \boldsymbol{p}+v\alpha(s)$, where $\boldsymbol{p}$ is a constant vector, $\alpha(s)$ is a unit vector and s is arc-length parameter of $\alpha(s)$. Then, we defined Sabban frame as $\{\alpha(s), T(s), B(s)\}$ along $\alpha(s)$. Using equation (2.6), we calculated $N(s) = B(s)$. Then, we got $N'(s) = B'(s) = -\kappa_g(s)T(s)$, where $\kappa_g(s)$ is the geodesic curvature of M along $\alpha(s)$. If $\kappa_g(s) = 0$, the unit director curve is a part of a great circle. Then M is a right circular cone. From this point forward it is considered as $\kappa_g(s) \neq 0$. Let $\bar{s}$ be the arc-length parameter of the spherical image of the unit vector $B(s)$. Then we got $d\bar{s}/ds = \varepsilon\kappa_g(s)$, where $\varepsilon = \pm 1$ and $\varepsilon\kappa_g(s) > 0$. We denoted $\kappa_B(s)$ and $\tau_B(s)$

as the curvature and the torsion of $B(s)$, respectively. By straightforward computation, we calculated $\kappa_B(s) = (\varepsilon(1+\kappa_g^2)^{1/2}/\kappa_g)(s), \tau_B(s) = -(\kappa_g'/\kappa_g(1+\kappa_g^2))(s)$. The equations $\kappa_B(s)$ and $\tau_B(s)$ were written as $\kappa(s)$ and $\tau(s)$, where $\kappa(s)$ and $\tau(s)$ is the curvature and torsion of $\alpha(s)$. It consists that $\kappa_n(s) = -1$, $\tau_g(s) = 0$ on S^2. Then, using equation (2.4), we got $\kappa_g'(s) = \kappa^2(s)\tau(s)$ and $\kappa_g(s) = \varepsilon(\kappa^2(s)-1)^{1/2}$, where $\varepsilon = \pm 1$ and $\varepsilon(\kappa^2(s)-1)^{1/2} > 0$. Therefore, we calculated $\kappa_B(s) = (\kappa/(\kappa^2-1)^{1/2})(s), \tau_B(s) = -(\varepsilon\tau/(\kappa^2-1)^{1/2})(s)$. It follows that $(\tau_B/\kappa_B)(s) = -\varepsilon(\tau/\kappa)(s)$. By straightforward computation, we got $\sigma_B(s) = -\varepsilon\sigma(s)$, where σ_B is the invariant of slant helices of $B(s)$. It implies that if the Gauss map of M is a part of a slant helix on S^2, $\alpha(s)$ is a part of a slant helix. This proved that (iii) consists. Therefore, we completes the proof. $\square$

From the proof of Theorem 4.1, we have the following Corollary.

Corollary 4.2. *Let M be a ruled surface. Suppose that the Gauss map of M is a cylindrical helix on S^2. Then,*

(i) *M is a part of a cylindrical surface.*

(ii) *M is a part of the tangent developable of a slant helix.*

(iii) *M is a part of a conical surface whose director curve is a cylindrical helix.*

5. Curves on Developable Surfaces

In this section, we studied clad helices from the view point of the theory of curves on developable surfaces. Let γ be a unit speed space curve with $\kappa \neq 0$. A ruled surface $F_{(\gamma,D)}(s,v) = \gamma(s) + vD(s)$ is called the *rectifying developable of* γ. It has been classically known that γ was a geodesic of the rectifying developable of γ. S.Izumiya and N.Takeuchi shows that the converse was also true (cf. Proposition 5.1)[13]. Further, the result of Proposition 5.2 has been known for many years. As shown in §2, we called D as modified Darboux vector field. Using modified Darboux vector field, we proved Proposition 5.2 again.

Proposition 5.1. [13] *Let M be a ruled surface and $\gamma(s)$ a regular curve on M with nonvanishing curvature. Then, the following conditions are equivalent*:

(i) *M is the rectifying developable of $\gamma(s)$.*

(ii) *$\gamma(s)$ is a geodesic of M, which is transversal to rulings and M is a developable surface.*

Proposition 5.2. *Let $\gamma(s)$ be a regular curve with nonvanishing curvature. Then, rectifying developable M of $\gamma(s)$ is classified as follow*:

(*i*) *M is a part of a cylindrical surface if* $(\tau/\kappa)'(s) = 0$.
(*ii*) *M is a part of the tangent developable if* $(\tau/\kappa)'(s) \neq 0$ *and* $(\tau/\kappa)''(s) \neq 0$.
(*iii*) *M is a part of a conical surface if* $(\tau/\kappa)'(s) \neq 0$ *and* $(\tau/\kappa)''(s) = 0$.

Proof. We defined M: $F_{(\gamma,D)}(s,v) = \gamma(s) + vD(s)$ and got $D'(s) = (\tau/\kappa)'(s)e_1(s)$. If $(\tau/\kappa)'(s) = 0$, director curve is a constant vector. Therefore, M is a cylindrical surface. This proved that (i) consists. Moreover, we got $(\partial F/\partial s \times \partial F/\partial v) = -(1 + v\,(\tau/\kappa)'\,(s))e_2(s)$.
If $(\tau/\kappa)'\,(s) \neq 0$, we assumed that $v = -1/\,(\tau/\kappa)'\,(s)$. Then, have a space curve $\bar{\gamma}(s)$ on M:

$$\bar{\gamma}(s) = \gamma(s) - \frac{1}{(\tau/\kappa)'\,(s)}D(s) \tag{5.1}$$

where $\bar{\gamma}(s)$ is the locus of singular points of M. Then, the equation of the surface M was rewritten to $F_{(\bar{\gamma},D)}(s,v) = \bar{\gamma}(s) + (v + 1/(\tau/\kappa)')D(s)$. We have $d\bar{\gamma}/ds = \left((\tau/\kappa)''/(\tau/\kappa)'^2\right)D(s)$. Therefore, if $(\tau/\kappa)''\,(s) \neq 0$, we rewrote the equation of M to $F_{(\bar{\gamma},\bar{\gamma}')}(s,\bar{v}) = \bar{\gamma}(s) + \bar{v}\bar{\gamma}'(s)$. Therefore, M is the tangent developable of $\bar{\gamma}(s)$. This proved that (ii) consists.
If $(\tau/\kappa)''(s) = 0$, $\bar{\gamma}(s)$ is a constant vector. Therefore, we defined M: $F_{(\boldsymbol{p},D)}(s,\bar{v}) = \boldsymbol{p} + \bar{v}D(s)$, where $\boldsymbol{p}$ is a constant vector. This proved that (iii) consists. This completes the proof. □

S.Izumiya and N.Takeuchi have investigated the classification of special developable surfaces under the condition that there exists a slant helix as a geodesic on the surface(cf. Theorem 5.4)[13]. Now, we have the following classification of special developable surfaces under the condition that there exists a clad helix as a geodesic on the surface (cf. Theorem 5.5). We found that Theorem 5.5 was the generalization of Theorem 5.4. However, before we denote Theorem 5.5, we denote Lemma 5.3.

Lemma 5.3. *Let $\gamma(s)$ be a unit speed space curve and $\bar{D}(s)$ be a unit Darboux vector of $\gamma(s)$. Then,*

(*i*) *$\gamma(s)$ is a slant helix if and only if spherical images of $\bar{D}(s)$ is a part of a circle on S^2.*
(*ii*) *$\gamma(s)$ is a clad helix if and only if spherical images of $\bar{D}(s)$ is a part of a cylindrical helix on S^2.*

Proof. We denoted $\kappa_{\bar{D}}(s)$ and $\tau_{\bar{D}}(s)$ as the curvature and the torsion of the spherical image of the unit vector $\bar{D}(s)$, respectively. Using D-frame, we calculated $\kappa_{\bar{D}}(s) = \varepsilon((1+\sigma^2)^{1/2}/\sigma)(s)$, $\tau_{\bar{D}}(s) = -(\sigma'/\sigma(\kappa^2 + \tau^2)^{1/2}(1+\sigma^2))(s)$, where $\varepsilon = \pm 1$ and $\varepsilon\sigma(s) > 0$. Therefore, the image of $\bar{D}(s)$ is a part of a circle on S^2 when $\sigma(s)$ is a constant function. And it follows that $(\tau_{\bar{D}}/\kappa_{\bar{D}})(s) = -\varepsilon\varphi(s)$. Therefore, the image of $\bar{D}(s)$ is a part of a cylindrical helix on S^2 when $\varphi(s)$ is a constant function. This completes the proof. $\square$

Theorem 5.4. [13] *Let M be a developable surface and $\gamma(s)$ a regular curve on M with non-vanishing curvature. Suppose that $\gamma(s)$ is a slant helix of M and geodesic, which is transversal to rulings. Then,*

(*i*) *M is a part of a cylindrical surface if $(\tau/\kappa)'(s) = 0$.*

(*ii*) *M is a part of the tangent developable of a cylindrical helix if $\sigma'(s) = 0$, $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) \neq 0$.*

(*iii*) *M is a part of a circular cone if $\sigma'(s) = 0$, $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) = 0$.*

Using Proposition 5.2 and Lemma 5.3, we generalized Theorem 5.4.

Theorem 5.5. *Let M be a developable surface and $\gamma(s)$ a regular curve on M with non-vanishing curvature. Suppose that $\gamma(s)$ is a clad helix of M and geodesic, which is transversal to rulings. Then,*

(*i*) *M is a part of a cylindrical surface if $(\tau/\kappa)'(s) = 0$.*

(*ii*) *M is a part of the tangent developable of a cylindrical helix if $\sigma'(s) = 0$, $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) \neq 0$.*

(*iii*) *M is a part of the tangent developable of a slant helix if $\varphi'(s) = 0$, $\sigma'(s) \neq 0$, $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) \neq 0$.*

(*iv*) *M is a part of a circular cone if $\sigma'(s) = 0$, $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) = 0$.*

(*v*) *M is a part of a cone whose unit director curve is a cylindrical helix if $\varphi'(s) = 0$, $\sigma'(s) \neq 0$, $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) = 0$.*

Proof. Using Proposition 5.1, M is the rectifying developable of $\gamma(s)$. Therefore, we defined $M : F_{(\gamma,D)}(s,v) = \gamma(s) + vD(s)$.
If $(\tau/\kappa)'(s) = 0$, by Proposition 5.2, M was a cylindrical surface. This proved that (i) consists.
If $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) \neq 0$, by Proposition 5.2, M was a part of the tangent developable. To prove the Proposition 5.2, we considered the locus of singular points of rectifying developable, which is given by $\bar{\gamma}(s) = \gamma(s) - (1/(\tau/\kappa)'(s))D(s)$. $\bar{\gamma}(s)$ is a base curve of M. Let $\bar{s}$ be the arc-length parameter of $\bar{\gamma}$. Then, $d\bar{s}/ds = \|\bar{\gamma}'(s)\| = \varepsilon_1((\kappa^2+\tau^2)^{1/2}(\tau/\kappa)''/\kappa\,(\tau/\kappa)'^2)(s)$, where $\varepsilon_1 = \pm 1$ and $\varepsilon_1(\tau/\kappa)''(s) > 0$. We denoted $\kappa_{\bar{\gamma}}(s)$ and $\tau_{\bar{\gamma}}(s)$ as the curvature and the torsion of $\bar{\gamma}$, respectively. By straightforward computation, we got

$$\kappa_{\bar{\gamma}}(s) = \varepsilon_2 \left(\sigma \frac{\kappa\,(\tau/\kappa)'^2}{(\tau/\kappa)''}\right)(s), \tau_{\bar{\gamma}}(s) = \left(\frac{\kappa\,(\tau/\kappa)'^2}{(\tau/\kappa)''}\right)(s) \qquad (5.2)$$

where $\varepsilon_2 = \pm 1$ and $\varepsilon_2(\sigma(\tau/\kappa)'')(s) > 0$. From $(\tau/\kappa)'(s) \neq 0$,thus, $\sigma(s) \neq 0$. Then, $(\tau_{\bar{\gamma}}/\kappa_{\bar{\gamma}})(s) = (\varepsilon_2/\sigma)(s)$. This means that the base curve of M is a cylindrical helix when $\sigma(s)$ is constant. Therefore, this proved that (ii) consists. By straightforward computation, we got

$$\sigma_{\bar{\gamma}}(s) = \left(\frac{\kappa_{\bar{\gamma}}^2}{(\kappa_{\bar{\gamma}}^2 + \tau_{\bar{\gamma}}^2)^{\frac{3}{2}}} \left(\frac{d}{d\bar{s}} \left(\frac{\tau_{\bar{\gamma}}}{\kappa_{\bar{\gamma}}} \right) \right) \right)(s) = -\varepsilon_1\varepsilon_2\varepsilon_3\varphi(s) \qquad (5.3)$$

where $\varepsilon_3 = \pm 1$ and $\varepsilon_3 \left(\tau/\kappa\right)'' (s) > 0$. This means that the base curve of M is a slant helix when $\varphi(s)$ is a constant. This proved that (iii) consists.
If $(\tau/\kappa)'(s) \neq 0$ and $(\tau/\kappa)''(s) = 0$, by Proposition 5.2, M was a part of a conical surface. If $(\tau/\kappa)''(s) = 0$, $\bar{\gamma}(s)$ is a fixed vector. Then, we defined $M : F_{(\boldsymbol{p},\bar{D})}(s,\bar{v}) = \boldsymbol{p} + \bar{v}\bar{D}(s)$, where $\boldsymbol{p}$ is a fixed vector and $\bar{D}$ is the unit Darboux vector. By Lemma 5.3, this proved that (iv) and (v) consist. This completes the proof. □

In §5, we have considered about geodesics. Then, we focused on a line of curvature. The problem of classifying surfaces under the condition of the existence of a curve as a line of curvature has long been studied. For instance, the surface that all curvature lines is a circle was called " *Cycloid of Dupin*" and these surfaces have been investigated in the past. Similarly, the surface that all curvature lines is a plane curve with regard to minimal surfaces were "*Enneper surfaces*" and "*Shark surfaces*" and others. However, surfaces that have cylindrical helices or slant helices as a line of curvature was not well known. In this section, we studied the classifications of surfaces that have cylindrical helices or slant helices as a line of curvature. First, we denoted Lemma 5.6.

Lemma 5.6. *Let M be a surface and $\gamma(s)$ a regular curve on M. Suppose that $\gamma(s)$ is a slant helix and a line of curvature on M. The Gauss map of M along $\gamma(s)$ is a slant helix.*

Proof. If $\gamma(s)$ is a line of curvature on M, the following formulas consist along $\gamma(s)$.

$$\tau_g(s) = 0, \kappa_g(s) = \kappa(s)\sin\phi(s), \kappa_n(s) = \cos\phi(s), \tau(s) = -\phi'(s) \qquad (5.4)$$

We defined the unit normal of M along $\gamma(s)$ as $N(s)$. Then, using equation (2.3), we got $N'(s) = -\kappa_n(s)T(s)$. Let $\bar{s}$ the arc-length parameter of $N(s)$. Then, $d\bar{s}/ds = \varepsilon_1\kappa_n(s)$,where $\varepsilon_1 = \pm 1$ and $\varepsilon_1\kappa_n(s) > 0$. We denoted $\kappa_N(s)$ and $\tau_N(s)$ as the curvature and the torsion of $N(s)$, respectively. By straightforward computation, we got $\kappa_N(s) = \varepsilon_1((\kappa_g^2 + \kappa_n^2)^{1/2}/\kappa_n)(s), \tau_N(s) = \left((\kappa_g'\kappa_n - \kappa_g\kappa_n')/(\kappa_n(\kappa_g^2 + \kappa_n^2))\right)(s)$. Using equation (2.4) and (5.4), we rewrote equations of $\kappa_N(s)$ and $\tau_N(s)$ to $\kappa_N(s) = \varepsilon_1(\kappa/\kappa_n)(s)$ and $\tau_N(s) = -(\tau/\kappa_n)(s)$. Therefore, we got

$(\tau_N/\kappa_N)(s) = -\varepsilon_1(\tau/\kappa)(s)$ and $\sigma_N(s) = -\varepsilon_1\sigma(s)$, where $\sigma_N(s)$ is the invariant of slant helices of $N(s)$. Therefore, this completes the proof. $\square$

By Lemma 5.6 and Theorem 4.1, we have the following Theorem.

Theorem 5.7. *Let M be a developable surface and $\gamma(s)$ a regular curve on M with non-vanishing curvature. Suppose that $\gamma(s)$ is a slant helix and a line of curvature on M which is transversal to rulings. Then,*

(*i*) *M is a part of a cylindrical surface.*

(*ii*) *M is a part of the tangent developable of a clad helix.*

(*iii*) *M is a part of a conical surface whose director curve is a slant helix.*

Proof. The normal vector of developable surfaces M is constant along rulings. Therefore, the Gauss map of M is a slant helix by Lemma 5.6. By Theorem 4.1, this completes the proof. $\square$

From the proof of Lemma 5.6 and Corollary 4.2, we have Corollary 5.8.

Corollary 5.8. *Let M be a developable surface and $\gamma(s)$ a regular curve on M with non-vanishing curvature. Suppose that $\gamma(s)$ is a cylindrical helix and a line of curvature on M which is transversal to rulings. Then,*

(*i*) *M is a part of a cylindrical surface.*

(*ii*) *M is a part of the tangent developable of a slant helix.*

(*iii*) *M is a part of a conical surface whose director curve is a cylindrical helix.*

6. Clad Helices as Spherical Curves

In §3, we shows the examples on how to construct clad helices in $\mathbb{E}^3$ using the notion of involute and evolute of a space curve. In this section, we investigated how to construct cylindrical, slant, and clad helices on S^2. A great circle on S^2 corresponds to a line in $\mathbb{E}^3$ from the point of view of a geodesic line. Then, we defined *spherical involute* and *spherical evolute* of a spherical curve on S^2 and found how to construct cylindrical, slant, and clad helices on S^2.

Definition 6.1. *Let $\gamma(s)$ be a spherical curve on S^2. We can make great circles such as contact with a curve $\gamma(s)$ at each point. We can define a curve on S^2, which is perpendicular to each great circle as spherical involute of $\gamma(s)$.*

First, we investigated the equation of a *spherical involute*. Let $\gamma(s)$ be a spherical curve on S^2. We defined Sabban frame as $\{\gamma(s), T(s), B(s)\}$. Then, defined a spherical curve $\alpha(s)$ on S^2 as follow:

$$\alpha(s) = \cos\theta(s)\gamma(s) + \sin\theta(s)T(s) \tag{6.1}$$

Using equations (2.6), we got

$$\alpha'(s) = (\theta'(s)+1)(\cos\theta(s)T - \sin\theta(s)\gamma) + \kappa_g(s)\sin\theta(s)B \tag{6.2}$$

Then, defined the points on great circles as follow:

$$g(\theta) = \cos\theta\gamma + \sin\theta T \tag{6.3}$$

From this equation, we got the unit tangent vector of great circles : $(dg/d\theta)(\theta) = -\sin\theta\gamma + \cos\theta T$. If $\alpha(s)$ is a spherical involute of $\gamma(s)$, the tangent vector of great circles are perpendicular to $\alpha'(s)$. Therefore, it consists that $<\alpha'(s), (dg/d\theta)(\theta)>= 0$. By straightforward computation, we got $\theta(s) = -s + c$, where c is a constant function. Substituting this equation in $\alpha(s)$, the equation of a spherical involute of $\gamma(s)$ is

$$\alpha(s) = \cos(-s+c)\gamma(s) + \sin(-s+c)T(s) \tag{6.4}$$

Using this equation, we have Theorem 6.2.

Theorem 6.2. *Let $\gamma(s)$ be a spherical curve on S^2 and $\alpha(s)$ be a spherical involute of $\gamma(s)$. Then,*

(*i*) *If $\gamma(s)$ is a circle, $\alpha(s)$ is a cylindrical helix.*

(*ii*) *If $\gamma(s)$ is a cylindrical helix, $\alpha(s)$ is a slant helix.*

(*iii*) *If $\gamma(s)$ is a slant helix, $\alpha(s)$ is a clad helix.*

Proof. We denoted $\kappa_\alpha(s)$ and $\tau_\alpha(s)$ as the curvature and the torsion of $\alpha(s)$, respectively. By straightforward computation, we got $\kappa_\alpha(s) = \varepsilon_1/\sin(-s+c)$ and $\tau_\alpha(s) = 1/\kappa_g(s)\sin(-s+c)$, where $\varepsilon_1 = \pm 1$ and $\varepsilon_1 \sin(-s+c) > 0$. Therefore, $(\tau_\alpha/\kappa_\alpha)(s) = (\varepsilon_1/\kappa_g)(s)$. If $\gamma(s)$ is a circle, $\kappa_g(s)$ is a constant. It proved that (i) consists. For (ii), we got $\sigma_\alpha(s) = -\varepsilon_1(\kappa_g'/(1+\kappa_g^2)^{3/2})(s)$, where σ_α is the invariant of slant helices of $\alpha(s)$. Then, as seen in the proof of Theorem 4.1, it consists that $\kappa_g'(s) = \kappa^2(s)\tau(s)$ and $\kappa_g(s) = \varepsilon(\kappa^2(s)-1)^{1/2}$. Therefore, $\sigma_\alpha(s) = -\varepsilon_1(\tau/\kappa)(s)$. It proved that ($ii$) consists. For ($iii$), we got $\varphi_\alpha(s) = -\varepsilon_1\sigma(s)$, where φ_α is the invariant of clad helices of $\alpha(s)$. It proved that (iii) consists. This completes the proof. □

Then, we defined *spherical evolute* of a curve on S^2.

Definition 6.3. *Let $\gamma(s)$ be a spherical curve on S^2. We can make great circles such as contact with a curve $\gamma(s)$. We can define a curve on S^2, which is contact with such great circles as spherical evolute.*

First, we investigated the equation of a spherical involute. Let $\gamma(s)$ be a spherical curve on S^2 and s be arc-length parameter of $\gamma(s)$. We defined Sabban frame as $\{\gamma(s), T(s), B(s)\}$. Then, defined a curve $\beta(s)$ on S^2 as follow:

$$\beta(s) = \cos\theta(s)\gamma(s) + \sin\theta(s)B(s) \tag{6.5}$$

Using equation (2.6), we got

$$\beta' = \theta'(s)(\cos\theta(s)B - \sin\theta(s)\gamma) + (-\kappa_g(s)\sin\theta(s) + \cos\theta(s))T \quad (6.6)$$

Then, the points on great circles were represented by the following equation: $g(\theta) = \cos\theta\gamma + \sin\theta B$. The unit tangent vector of great circles at these points were represented by the following formula: $(dg/d\theta)(\theta) = -\sin\theta\gamma + \cos\theta B$. If $\beta(s)$ is a spherical evolute of $\gamma(s)$, by definition, we considered the condition that the tangent vector of great circles was the same direction as the tangent vector of $\beta(s)$. Therefore,

$$-\kappa_g(s)\sin\theta(s) + \cos\theta(s) = 0 \quad (6.7)$$

Differentiating this equation with respect to s, we got

$$\kappa_g'(s) = -(\theta'/\sin^2\theta)(s) \quad (6.8)$$

We denoted $\kappa(s)$ and $\tau(s)$ as the curvature and the torsion of $\gamma(s)$, respectively. Using equation (2.4) and (6.7), we got $\kappa_g'(s) = \kappa^2(s)\tau(s)$, $\tau(s) + \phi'(s) = 0$ and $\sin^2\theta(s) = 1/(1+\kappa_g^2(s)) = 1/\kappa^2(s)$. Then, substituting these equations in equation (6.8), $\theta'(s) = \phi'(s)$. Therefore, $\theta(s) = \phi(s) + c$, where c is a constant function and $\phi(s)$ denote the angle between $B(s)$ and $e_2(s)$. We rewrote the equation of $\beta(s)$ to $\beta(s) = \cos(\phi(s)+c)\gamma(s) + \sin(\phi(s)+c)B(s)$. Then, using equation (2.7), we got $\beta(s) = \cos c\, e_2(s) - \sin c\, e_3(s)$, $\beta'(s) = -\kappa(s)\cos c\, T(s) + \tau(s)\sin c\, e_2(s) + \tau(s)\cos c\, e_3(s)$ and $(dg/d\theta)(\theta) = -\sin c\, e_2 - \cos c\, e_3$. If $\beta(s)$ is a spherical evolute of $\gamma(s)$, the direction of $\beta'(s)$ and the tangent of great circles at each point should coincide. By straightforward computation, we got $c = \pi/2 + n\pi$. Therefore, the equation of a spherical evolute of $\gamma(s)$ is

$$\beta(s) = \pm e_3(s) \quad (6.9)$$

From this equation, we have Theorem 6.4.

Theorem 6.4. *Let $\gamma(s)$ be a spherical curve on S^2 and $\beta(s)$ be a spherical evolute of $\gamma(s)$. Then,*

(*i*) *If $\gamma(s)$ is a cylindrical helix, $\beta(s)$ is a circle.*

(*ii*) *If $\gamma(s)$ is a slant helix, $\beta(s)$ is a cylindrical helix.*

(*iii*) *If $\gamma(s)$ is a clad helix, $\beta(s)$ is a slant helix.*

Proof. Let $\beta(s)$ be a spherical evolute of $\gamma(s)$. Then, we defined $\beta(s)$ as $e_3(s)$. We denoted $\kappa_\beta(s)$ and $\tau_\beta(s)$ as the curvature and the torsion of $\beta(s)$, respectively. By straightforward computation, we got $\kappa_\beta(s) = (\varepsilon(\kappa^2+\tau^2)^{1/2}/\tau)(s), \tau_\beta(s) = -(\sigma(\kappa^2+\tau^2)^{1/2}/\tau)(s)$, where $\varepsilon = \pm 1$ and $\varepsilon\tau(s) > 0$. Therefore, $(\tau_\beta/\kappa_\beta)(s) = -\varepsilon\sigma(s)$ and $\sigma_\beta(s) = -\varepsilon\varphi(s)$, where $\sigma_\beta(s)$ is the invariant of slant helices of $\beta(s)$. This completes the proof of Theorem 6.4. ∎

Remark. By Theorem 3.2, Theorem 6.2 and Theorem 6.4, *spherical evolute* and *spherical involute* are inverse interpolation with evolute and involute in $\mathbb{E}^3$ from view point of the generalization of helices.

7. Examples

In this section, we shows the examples of clad helices in $\mathbb{E}^3$ and slant helices on S^2. By using **Mathematica**, we have drawn the examples.

Example 5.1. By Theorem 3.2, we constructed the example of a clad helix in $\mathbb{E}^3$. We considered a space curve called *constant precession*. This curve is known as slant helix on circular hyperboloid of one sheet [8]. We denoted $\gamma(s)$ as *constant precession*.

$$\begin{aligned}\gamma(s) =&\Big(-m\Big(\sqrt{1+m^2}\left(1+2m^2\right)\cos(us)\cos\left(\frac{\sqrt{1+m^2}us}{m}\right)\\ &+2\left(m+m^3\right)\sin(us)\sin\left(\frac{\sqrt{1+m^2}us}{m}\right)\Big)\Big/\left(1+m^2\right)u,\\ &m\Big(2m\left(1+m^2\right)\cos\left(\frac{\sqrt{1+m^2}us}{m}\right)\sin(us)\\ &-\sqrt{1+m^2}\left(1+2m^2\right)\cos(us)\sin\left(\frac{\sqrt{1+m^2}us}{m}\right)\Big)\Big/\left(1+m^2\right)u,-\frac{\cos(us)}{\sqrt{1+m^2}u}\Big)\end{aligned} \tag{7.1}$$

where s is arc-length parameter of $\gamma(s)$ and $m(\neq 0), u$ are constant functions. Let $\sigma(s)$ be the invariant of slant helix of $\gamma(s)$. We got $\sigma(s)=m$. Therefore, $\gamma(s)$ is a slant helix. Then, denoted $\alpha(s)$ as evolute of $\gamma(s)$. By straightforward computation, we got

$$\begin{aligned}\alpha(s) =&\Big(-m\Big((1+2m^2)\cos(us)\cos(\frac{\sqrt{1+m^2}us}{m})+\cos(\frac{\sqrt{1+m^2}us}{m})(-\sec(us)\\ &+m\tan(c+\frac{\cos(us)}{m}))+\sqrt{1+m^2}\sin(\frac{\sqrt{1+m^2}us}{m})(2m\sin(us)\\ &+\tan(us)\tan(c+\frac{\cos(us)}{m}))\Big)\Big/\sqrt{1+m^2}u,\\ &m\Big(-\sin(\frac{\sqrt{1+m^2}us}{m})((1+2m^2)\cos(us)-\sec(us)+m\tan(c+\frac{\cos(us)}{m}))\\ &+\sqrt{1+m^2}\cos(\frac{\sqrt{1+m^2}us}{m})(2m\sin(us)+\tan(us)\tan(c+\frac{\cos(us)}{m}))\Big)\Big/\sqrt{1+m^2}u,\\ &\Big(-\cos(us)+m(m\sec(us)+\tan(c+\frac{\cos(us)}{m}))\Big)\Big/\sqrt{1+m^2}u\Big)\end{aligned} \tag{7.2}$$

where c is a constant function. Let us denote φ_α as the invariant of clad helix of $\alpha(s)$. By straight computation, we found that $\varphi_\alpha = m$. Therefore, $\alpha(s)$ is an example of a clad helix. Then, we draw $\gamma(s)$ $(m=1, u=1)$ as shown in Figure 1 and $\alpha(s)$ $(m=1, u=1, c=0)$ in

Figure 2.

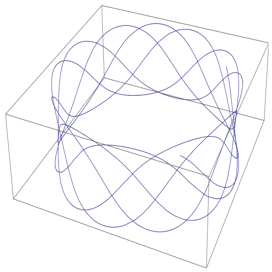

FIGURE 1. Slant helix in $\mathbb{E}^3$

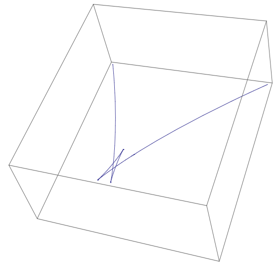

FIGURE 2. Clad helix in $\mathbb{E}^3$

Example 5.2. By Theorem 6.2, we constructed the examples of a cylindrical helix and a slant helix on S^2. We considered a circle on S^2 and defined $\gamma(s) = \left(\cos 2s, \sin 2s, \sqrt{3}\right)/2$. Then, denoted $\bar{\gamma}(s)$ as spherical involute of $\gamma(s)$. By straightforward computation, $\bar{\gamma}(s) = \left((3\cos s - \cos 3s)/4, \sin^3 s, (\sqrt{3}\cos s)/2\right)$, where integration constant $c = 0$. We denoted $\bar{\kappa}$ and $\bar{\tau}$ as the curvature and the torsion of $\bar{\gamma}(s)$, respectively. Then, we got $(\bar{\tau}/\bar{\kappa})(s) = -\sqrt{1/3}$. Therefore, $\bar{\gamma}(s)$ is a cylindrical helix on S^2. After that, we denoted t as arc-length parameter of $\bar{\gamma}(s)$. We got $t = -\sqrt{3}\cos s (0 < s < \pi), t = \sqrt{3}\cos s (\pi < s < 2\pi)$. Therefore, $\bar{\gamma}(s)$ is represented by following equations:

$$\bar{\gamma}_1(t) = \left(\frac{t\left(-9+2t^2\right)}{6\sqrt{3}}, \left(1-\frac{t^2}{3}\right)^{3/2}, -\frac{t}{2}\right), \bar{\gamma}_2(t) = \left(\frac{t\left(9-2t^2\right)}{6\sqrt{3}}, -\left(1-\frac{t^2}{3}\right)^{3/2}, \frac{t}{2}\right) \tag{7.3}$$

We denoted $\beta_1(t)$ and $\beta_2(t)$ as the spherical involute of $\bar{\gamma}_1(t)$ and $\bar{\gamma}_2(t)$. Then, we got

$$\beta_1(t) = \left(\frac{t\left(-9+2t^2\right)\cos t + 3\left(3-2t^2\right)\sin t}{6\sqrt{3}}, \left(1-\frac{t^2}{3}\right)^{3/2}\cos t + t\sqrt{1-\frac{t^2}{3}}\sin t, \right.$$
$$\left. \frac{1}{2}(-t\cos t + \sin t)\right) \tag{7.4}$$

$$\beta_2(t) = \left(\frac{t\left(-9+2t^2\right)\cos t + 3\left(3-2t^2\right)\sin t}{6\sqrt{3}}, -\left(1-\frac{t^2}{3}\right)^{3/2}\cos t - t\sqrt{1-\frac{t^2}{3}}\sin t, \right.$$
$$\left. \frac{1}{2}(-t\cos t + \sin t)\right) \tag{7.5}$$

where integration constant $c = 0$. By straight computation, we got $\sigma_{\beta_1}(t) = \sigma_{\beta_2}(t) = \sqrt{1/3}$, where $\sigma_{\beta_1}(t)$ and $\sigma_{\beta_2}(t)$ are invariants of slant helices of $\beta_1(t)$ and $\beta_2(t)$. Therefore, $\beta_1(t)$ and $\beta_2(t)$ are examples of slant helices on S^2. Then, we draw $\bar{\gamma}(s)$ as shown in Figure 3 and $\beta_1(t)$, $\beta_2(t)$ in Figure 4.

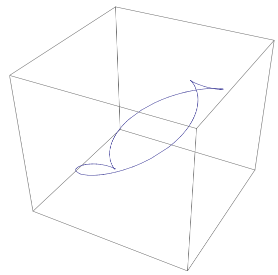

FIGURE 3. Cylindrical helix on S^2

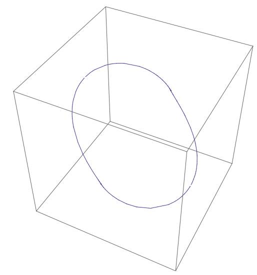

FIGURE 4. Slant helix on S^2

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