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Starlike Functions Of order α With Respect To $2(j, k)$-Symmetric Conjugate Points

E. Amini [1]
[1] Department of Mathematics, Payme Noor University, P. O. Box 19395-4697 Tehran, IRAN.

Abstract. In this paper, we introduced and investigated starlike and convex functions of order α with respect to $2(j, k)$-symmetric conjugate points and coefficient inequality for function belonging to these classes are provided. Also, we obtain some convolution condition for functions belonging to this class.

Keywords: Univalent functions, $2(j, k)$-Symmetric conjugate, Coefficient bound, Convolution.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{S}^*(\alpha)$, $\mathcal{CV}$, $\mathcal{CV}(\alpha)$ and $\mathcal{C}(\alpha)$ denote the familiar subclass of $\mathcal{A}$ consisting of functions which are, respectively, univalent, starlike, α-starlike, convex, α-convex and close-to-convex functions of order α in $\mathcal{U}$ (See, for details, [2, 3, 5, 10]).

[1]Corresponding author: eb.amini.s@pnu.ac.ir
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Lemma 1.1. [4] *Let* $\alpha \in [0,1)$ *and* f *given by* (1.1) *be a holomorphic function on* $\mathcal{U}$. *If*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| < 1,$$

then $f \in \mathcal{S}^*(\alpha)$.

Let two function $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

H. Silverman, E. M. Silvia and D. Telage in [9] generated a great deal of intrinsic interest properties of convolution for α-starlike functions, α-convex functions and λ-spirallike functions.

The concept of dual set has proved to be very useful in the study of properties of analytic functions, (see for example [6, 7]).

Lemma 1.2. [6, 7] *If* $\phi \in \mathcal{CV}$ *and* $f \in \mathcal{ST}$, *then for any analytic function* F *in* D, *the image of* D *under* $\frac{\phi * F f(z)}{\phi * f(z)}$ *is a subset of the convex hull of* $F(D)$.

Sakaguchi [8] once introduce a class $\mathcal{S}_s^*$ of function starlike with respect to symmetric points, which consists of functions $f(z) \in \mathcal{S}$ satisfying the inequality

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

Al-Amiri, Coman and Mocano [1] once introduce and investigate a class of functions starlike with respect to $2k$-symmetric conjugate points which satisfy the inequality

$$Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\} > 0 \quad z \in \mathcal{U},$$

where $k \geq 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality

$$f_{2k}(z) = \frac{1}{2k} \sum_{v=0}^{k-1} \left(\epsilon^{-v} f(\epsilon^v z) + \epsilon^v \overline{f(\epsilon^v \overline{z})}\right) \quad (\epsilon = \exp(2\pi i/k);\ z \in \mathcal{U}). \tag{1.2}$$

But until now, a new subclass of $2(j,k)$-symmetric functions of order α is defined and some properties for this class are obtained such as coefficient bounds and convolution condition.

2. Definitions and coefficient bounds

In this chapter we introduce subclass of functions with $2k$-symmetric conjugate points and obtain some properties such as coefficient bounds.

Definition 2.1. Let $0 \le \alpha < 1$. The class of α-starlike functions with $2(j,k)$-symmetric conjugate points of the class of functions in $\mathcal{ST}$ denoted by $\mathcal{ST}_{sc}^{(j,k)}(\alpha)$ satisfying the condition

$$Re\Big\{\frac{zf'(z)}{f_{2(j,k)}(z)}\Big\} > \alpha, \tag{2.1}$$

for all z in $\mathcal{U}$, where $f_{2(j,k)}(z)$ is given by

$$f_{2(j,k)}(z) = \frac{1}{2k}\sum_{v=0}^{k-1}\Big(\epsilon^{-vj}f(\epsilon^{v}z) + \epsilon^{vj}\overline{f(\epsilon^{v}\bar{z})}\Big). \tag{2.2}$$

The function $f(z) \in \mathcal{A}$ is in the $\mathcal{CV}_{sc}^{(j,k)}(\alpha)$, α-convex functions with respect to $2(j,k)$-symmetric conjugate points, if and only if $zf'(z) \in \mathcal{ST}_{sc}^{(j,k)}(\alpha)$.

Remark 2.2. Since f is given by (1.1) and $f_{2(j,k)}$ is given by (2.2), we obtain

$$\begin{aligned} f_{2(j,k)}(z) &= \frac{1}{2k}\sum_{v=0}^{k-1}\Big(\epsilon^{-vj}f(\epsilon^{vj}z) + \epsilon^{vj}\overline{f(\epsilon^{v}\bar{z})}\Big) \\ &= \sum_{n=1}^{\infty}\Big(\frac{1}{2k}\sum_{v=0}^{k-1}\big(\epsilon^{(n-j)v}a_n + \overline{\epsilon^{(n-j)v}a_n}\big)\Big)z^n. \end{aligned}$$

Then

$$f_{2(j,k)}(z) = \sum_{n=1}^{\infty}\delta_{n,j}\frac{a_n+\bar{a}_n}{2}z^n,$$

where

$$\delta_{n,j} = \frac{1}{2k}\sum_{v=0}^{k-1}\Big(\epsilon^{(n-j)v}a_n + \overline{\epsilon^{(n-j)v}a_n}\Big) = \begin{cases} 1, & n = lk+j \\ 0, & n \ne nk+j. \end{cases} \tag{2.3}$$

Theorem 2.3. *Let f given by* (1.1) *and $f \in \mathcal{S}_{sc}^{2(j,k)}(\alpha)$, then $f_{2(j,k)} \in \mathcal{ST}(\alpha)$ and $\mathcal{ST}_{sc}^{2(j,k)}(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$.*

Proof. Since $f \in \mathcal{ST}_{sc}^{2(j,k)}(\alpha)$, Then we have

$$Re\Big\{\frac{zf'(z)}{f_{2(j,k)}(z)}\Big\} \geq \alpha$$

or equivalently

$$\mathfrak{T}\Big\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_{2(j,k)}(re^{i\theta})}\Big\} \geq \alpha. \tag{2.4}$$

In the inequality (2.4) substituting $re^{i\theta}$ by $\epsilon^v re^{i\theta}$, we obtain

$$\mathfrak{T}\Big\{\frac{\frac{\partial}{\partial\theta}f(\epsilon^v re^{i\theta})}{f_{2(j,k)}(\epsilon^v re^{i\theta})}\Big\} \geq \alpha. \tag{2.5}$$

Note that $f_{2(j,k)}(\epsilon^v z) = \epsilon^{vj} f_{2(j,k)}(z)$, the inequality (2.5) can be written as

$$\mathfrak{T}\Big\{\frac{\epsilon^{-vj}\frac{\partial}{\partial\theta}f(\epsilon^v re^{i\theta})}{f_{2(j,k)}(re^{i\theta})}\Big\} \geq \alpha. \tag{2.6}$$

In the inequality (2.4) substituting $re^{i\theta}$ by $\epsilon^v re^{-i\theta}$, we obtain

$$\mathfrak{T}\Big\{\frac{\frac{\partial}{\partial\theta}f(\epsilon^v re^{-i\theta})}{f_{2(j,k)}(\epsilon^v re^{-i\theta})}\Big\} \geq \alpha.$$

the above equation is equivalent to

$$\mathfrak{T}\Big\{\frac{\overline{\frac{\partial}{\partial\theta}f(\epsilon^v re^{-i\theta})}}{\overline{f_{2(j,k)}(\epsilon^v re^{-i\theta})}}\Big\} \geq \alpha. \tag{2.7}$$

Note that $f_{2(j,k)}(\epsilon^v \bar{z}) = \epsilon^{-vj} f_{2(j,k)}(z)$, the inequality (2.7) can be written as

$$\mathfrak{T}\Big\{\frac{\epsilon^{vj}\overline{\frac{\partial}{\partial\theta}f(\epsilon^v re^{i\theta})}}{f_{2(j,k)}(re^{i\theta})}\Big\} \geq \alpha. \tag{2.8}$$

Note that $\overline{\frac{\partial}{\partial\theta}f(\epsilon^v re^{-i\theta})} = \frac{\partial}{\partial\theta}\overline{f(\epsilon^v re^{-i\theta})}$, by applying the inequalities (2.8) and (2.6), we obtain

$$\mathfrak{T}\left\{\frac{\frac{1}{2k}\Big(\sum_{v=0}^{k-1}\big(\epsilon^{-vj}\frac{\partial}{\partial\theta}f(\epsilon^v re^{i\theta})) + \epsilon^{vj}\frac{\partial}{\partial\theta}\overline{f(\epsilon^v\bar{z})}\big)\Big)}{f_{2(j,k)}(z)}\right\} = \mathfrak{T}\Big\{\frac{\frac{\partial}{\partial\theta}f_{2(j,k)}(z)}{f_{2(j,k)}(z)}\Big\} \geq \alpha, \tag{2.9}$$

that is $f_{2(j,k)}(z) \in \mathcal{ST}(\alpha)$. This means that $\mathcal{ST}_{sc}^{2(j,k)}(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$ and the proof of theorem is complete. □

Theorem 2.4. *Let $f(z)$ given by* (1.1) *and $f_{2(j,k)}(z)$ given by* (2.2)*. Let*

$$\sum_{n=2}^{\infty}\frac{nk+j-\alpha+1}{1-\alpha\delta_{1j}}|a_{nk+j}| + \sum_{n=1,n\neq lk+j}^{\infty}\frac{n+1}{1-\alpha\delta_{1j}}|a_n| \leq 1, \tag{2.10}$$

where $a_1 = 1$, $0 \leq \alpha < 1$, $k = 1, 2, 3, ...,$ $j = 0, 1, ..., k-1$, $l \in \mathbb{N}$ *and* δ_{nj} *is defined by* (2.3). *Then* f *is starlike function of order* α *and* $f \in \mathcal{ST}_{sc}^{2(j,k)}(\alpha)$.

Proof. Since

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| \leq \sum_{n=2}^{\infty} \frac{n+1-\alpha}{1-\alpha}|a_n| \leq \sum_{n=2}^{\infty} \frac{n+(1-\alpha)\delta_{nj}}{1-\alpha\delta_{1j}}|a_n|$$
$$= \sum_{n=2}^{\infty} \frac{nk+j-\alpha+1}{1-\alpha\delta_{1j}}|a_{nk+j}| + \sum_{n=1, n\neq, lk+j}^{\infty} \frac{n+1}{1-\alpha\delta_{1j}}|a_n| \leq 1$$

where δ_{nj} defined in (2.3). Hence by applying Lemma 1.1, f is starlike function of order α.

To prove $f \in \mathcal{ST}_{sc}^{2(j,k)}(\alpha)$, we need to show that

$$\Im\Big\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_{2(j,k)}(re^{i\theta})}\Big\} = Re\Big\{\frac{zf'(z))}{f_{2(j,k)}(z))}\Big\} = Re\Big\{\frac{A(z)}{B(z)}\Big\} \geq \alpha,$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $0 \leq \alpha < 1$, $k = 1, 2, 3, ...,$ $j = 0, 1, ..., k-1$,

$$A(z) = zf'(z) = z + \sum_{n=1}^{\infty} na_n z^n,$$

and $B(z) = f_{2(j,k)}(z)$ is given by (2.2).

Using the fact that $Re\{w\} \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\Big|A(z) + (1-\alpha)B(z)\Big| - \Big|A(z) - (1+\alpha)B(z)\Big| \geq 0$$

Hence, we have

$$\begin{aligned}
&\Big|A(z)+(1-\alpha)B(z)\Big|-\Big|A(z)-(1+\alpha)B(z)\Big| \\
&=\Big|\big(1+(1-\alpha)\delta_{1j}\big)z+\sum_{n=2}^{\infty}\big(n+\frac{1-\alpha}{2}\delta_{nj}\big)a_nz^n+\sum_{n=2}^{\infty}\big(\frac{1-\alpha}{2}\delta_{nj}\big)\bar{a}_nz^n\Big| \\
&-\Big|\big(1-(1+\alpha)\delta_{1j}\big)z+\sum_{n=2}^{\infty}\big(n-\frac{1+\alpha}{2}\delta_{nj}\big)a_nz^n-\sum_{n=2}^{\infty}\big(\frac{1+\alpha}{2}\delta_{nj}\big)\bar{a}_nz^n\Big| \\
&\geq\big(1+(1-\alpha)\delta_{1j}\big)|z|-\sum_{n=2}^{\infty}\big(n+\frac{1-\alpha}{2}\delta_{nj}\big)|a_n||z|^n-\sum_{n=2}^{\infty}\big(\frac{1-\alpha}{2}\delta_{nj}\big)|\bar{a}_n||z|^n \\
&+\big(1-(1+\alpha)\delta_{1j}\big)|z|-\sum_{n=2}^{\infty}\big(n-\frac{1+\alpha}{2}\delta_{nj}\big)|a_n||z|^n-\sum_{n=2}^{\infty}\big(\frac{1+\alpha}{2}\delta_{nj}\big)|\bar{a}_n|z^n \\
&=2\big(1-\alpha\delta_{1j}\big)|z|-2\sum_{n=2}^{\infty}(n-\alpha\delta_{nj})|a_n||z|^n-2\sum_{n=2}^{\infty}|a_n||z|^n \\
&=2(1-\alpha\delta_{1j})|z|\Big\{1-\sum_{n=2}^{\infty}\frac{n-\alpha\delta_{nj}}{1-\alpha\delta_{1j}}|a_n||z|^{n-1}-\sum_{n=2}^{\infty}\frac{1}{1-\alpha\delta_{1j}}|a_n||z|^{n-1}\Big\}\geq 0 \\
&=2(1-\alpha\delta_{1j})|z|\Big\{1-\sum_{n=2}^{\infty}\frac{n+1-\alpha\delta_{nj}}{1-\alpha\delta_{1j}}|a_n||z|^{n-1}\Big\}\geq 0
\end{aligned}$$

From the definition of δ_{nj} in (2.3), we get

$$\begin{aligned}
&\Big|A(z)+(1-\alpha)B(z)\Big|-\Big|A(z)-(1-\alpha)B(z)\Big| \\
&=2(1-\alpha\delta_{1j})|z|\Big\{1-\sum_{n=2}^{\infty}\frac{nk+j+1-\alpha}{1-\alpha\delta_{1j}}|a_{nk+j}|-\sum_{n=2,n\neq lk+j}^{\infty}\frac{n+1}{1-\alpha\delta_{1j}}\Big|a_n\Big|\Big\}\geq 0
\end{aligned}$$

we note that in (2.10). This conclude the proof of the theorem. □

The Starlike function of order α with respect to $2(j,k)$-symmetric conjugate points

$$f(z)=z+\sum_{n=2}^{\infty}\frac{1-\alpha\delta_{1,j}}{nk+j-\alpha+1}x_{nk+j}z^{nk+j}+\sum_{n=1,n\neq lk+j}^{\infty}\frac{1-\alpha\delta_{1,j}}{n+1}x_nz^n$$

where $\sum_{m=2}^{\infty}|x_m|=1$, shows that the coefficient bounds in (2.10) is sharp.

3. Convolution Condition

Theorem 3.1. *Let $\phi\in\mathcal{CV}$ and $f(z)\in\mathcal{ST}_{sc}^{(j,k)}(\alpha)$. Then $(\phi*f)(z)\in\mathcal{ST}_{sc}^{(j,k)}(\alpha)$.*

Proof. Let Ω_α is a convex domain and $f_{2(j,k)}(z)$ is given by (2.2). Since $f(z) \in \mathcal{ST}_{sc}^{(j,k)}(\alpha)$ by Theorem 2.4, we conclude that $f_{2(j,k)}(z) \in \mathcal{ST}$. Hence by applying Lemma 1.2, we obtain

$$\frac{z(\phi * f)'(z)}{\phi * f_{2(j,k)}(z)} = \frac{(\phi * zf')(z)}{\phi * f_{2(j,k)}(z)} = \frac{\phi * \frac{zf'(z)}{f_{2(j,k)}(z)} f_{2(j,k)}(z)}{\phi * f_{2(j,k)}} \subseteq \overline{co}\Big(\frac{zf'(z)}{f_{2(j,k)}(z)}\Big) \subseteq \Omega_\alpha.$$

Since Ω_α is a convex domain and $f \in \mathcal{ST}_{sc}^{j,k}(\alpha)$. This prove that $\big(\phi * f\big)(z) \in \mathcal{ST}_{sc}^{(j,k)}(\alpha)$. □

Theorem 3.2. *Let $0 \le \alpha < 1$, $k = 1, 2, 3, ...,$ $j = 0, 1, ..., k-1$ and $f(z) \in \mathcal{A}$. Then the function $f(z) \in \mathcal{ST}_{sc}^{(j,k)}(\alpha)$ if and only if*

$$\frac{1}{z}\Big[f * \big(\frac{z}{(1-z)^2} - \frac{x+2\alpha-1}{2(x+1)}h(z)\big) - \frac{x+2\alpha-1}{2(x+1)}\overline{f * h(\overline{z})}\Big] \ne 0, \quad (3.1)$$

for all $z \in \mathcal{U}$ and $|x| = 1$ where $h(z)$ is given by

$$h(z) = \frac{1}{k}\sum_{v=0}^{k-1} \frac{\epsilon^{v(1-j)} z}{1-\epsilon^v z}. \qquad \big(\epsilon = exp(2\pi i/k)\big). \quad (3.2)$$

Proof. Let $f(z) \in \mathcal{ST}_{sc}^{(j,k)}(\alpha)$, from the definition 2.1, f satisfies the inequality (2.1). Since $\frac{zf'(z)}{f(z)} = 1$ at $z = 0$, the inequality (2.1) is equivalent to

$$\frac{\frac{zf'(z)}{f_{2(j,k)}(z)} - \alpha}{1-\alpha} \ne \frac{x-1}{x+1}$$

for all $|z| < R < 1$, $|x| = 1$ and $x \ne 1$. Which simplifies to

$$\frac{zf'(z) - \alpha f_{2(j,k)}(z)}{(1-\alpha) f_{2(j,k)}(z)} \ne \frac{x-1}{x+1}, \quad (3.3)$$

for all $|z| < R < 1$, $|x| = 1$ and $x \ne 1$. By condition on the inequality (3.3), we obtain

$$\frac{1}{z}\Big[zf'(z) - \frac{x+2\alpha-1}{x+1} f_{2(j,k)}(z)\Big] \ne 0. \quad (3.4)$$

On the other hand, it is well known that

$$zf'(z) = f(x) * \frac{z}{(1-z)^2}. \quad (3.5)$$

By the definition of $f_{2(j,k)}(z)$, we know

$$f_{2(j,k)}(z) = \frac{1}{2}\Big[f * h(z) + \overline{f * h(\overline{z})}\Big], \quad (3.6)$$

where $h(z)$ is given by (3.2). Substituting (3.5) and (3.6) in (3.4), we can get (3.1). This complete the proof of theorem. □

Corollary 3.3. *Let* $0 \leq \alpha < 1$ *and* $f(z) \in \mathcal{A}$. *Then the function* $f(z) \in \mathcal{C}_{sc}^{(j,k)}(\alpha)$ *if and only if*

$$\frac{1}{z}\Big[f*\Big(\frac{2z}{(1-z)^3}-\frac{x+2\alpha-1}{2(x+1)}\big(h(z)+zh'(z)\big)\Big)-\frac{x+2\alpha-1}{2(x+1)}\overline{f*h'(\overline{z})}\Big] \neq 0,$$

for all $z \in \mathcal{U}$ *and* $|x| = 1$ *where* $h(z)$ *is given by*

$$h(z) = \frac{1}{k}\sum_{v=0}^{k-1}\frac{\epsilon^{v(1-j)}z}{1-\epsilon^{v}z}. \qquad \big(\epsilon = exp(2\pi i/k)\big),$$

Proof. By definition 2.1 we have, $f \in \mathcal{A}$ is in the $\mathcal{CV}_{sc}^{(j,k)}(\alpha)$ if and only if $zf'(z) \in \mathcal{ST}_{sc}^{(j,k)}(\alpha)$. Now applying Theorem 3.2. This complete the proof of corollary. □

References

[1] H. AL-Amiri, D. Coman and P. T. Mocanu, Some properties of starlike functions with respect to symmetric conjugate points, *Internat. J. Math. Sci.* **18***(1995), 469-474.*

[2] P.L. Duren, Univalent Functions, *Grundelheren der Mathematischen Wissenchaften 259, Springer-Verlag, New York (1983).*

[3] I. Graham and G. Kohr, Geometric Function Theory in one and Higher Dimensions, *Marcel Dekker, Inc, NewYork, 2003.*

[4] E. P. Merkes, M. S. Robertson and W. T. Scott, On products of starlike functions,*Proc. Amer. Math. Soc.,* **13***(1962), 960-964.*

[5] S. Owa, M. Nunokawa, H. Satoh and H. M. Srivastava, Close-to-convity, starlikeness and convexity of certain analytic function, *Appl. Math. Lett.* **15***(2002), 63-69.*

[6] S. T. Ruscheweyh, Duality for hadmarad product with applications to extremal problems, *Trans. Am.* **210***, 63–74 (1975).*

[7] S. T. Ruscheweyh and T. Sheil-Small, Hadamard products of schlicht functions and the Polya-Schoenberg Con- jecture, *Comment. Math. Helv.,* **48** *(1973), 119–135.*

[8] K. Sakaguchi, On certain univalent mapping, *J, Math. Soc. Japan* **11***(1959)72-75.*

[9] H. Silverman, E. M. Silvia and D. Telage, Convolution conditions for convexity starlikeness and spiral-likenes, Math. Z, **162** (1978)125-130.

[10] H. M. Srivastava and S. Owa, Current topics in Analytic Function Theory, *World scientific, Singapore, (1992).*