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Schwarz boundary value problem on a triangle

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Abstract. In this paper, the Schwarz boundary value problem (BVP) for the inhomogeneous Cauchy-Riemann equation in a triangle is investigated explicitly. Firstly, by the technique of parqueting-reflection and the Cauchy-Pompeiu representation formula a modified Cauchy-Schwarz representation formula is obtained. Then, the solution of the Schwarz BVP is explicitly solved. In particular, the boundary behaviors at the corner points are considered.

Keywords: Schwarz problem, Cauchy-Pompeiu formula, Cauchy-Schwarz representation, Triangle.

2000 Mathematics subject classification: 35F15, 35C15; Secondary 30E25.

1. Introduction

Numerous results have been achieved for BVPs in different particular domains. Those special domains include the unit disc [8], half plane [3, 9], ring [12, 13], half disc and half ring [5], lens and lune [6], lens and half lens [10] and some convex polygons, e.g. equilateral triangle [7] and half hexagon [11].

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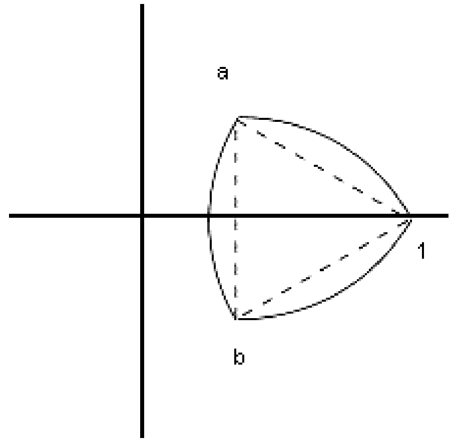

FIGURE 1. The triangle T

The Schwarz BVP is considered for an analytic function with given boundary values of its real part. Also, the Cauchy-Schwarz representation formula is obtained by the technique of parqueting–reflection and the Cauchy-Pompeiu representation formula, see e.g. [1, 2, 4, 14].

The main purpose of this paper is to solve Schwarz BVP on a triangle. Let T be the triangle formed in the complex plane $\mathbb{C}$. It is formed by the intersection of three circles $C_1 : |z-i| = \sqrt{2}$, $C_2 : |z+i| = \sqrt{2}$ and $C_3 : |z-\sqrt{3}| = \sqrt{2}$. The two circles C_1 and C_2 on the real axis meet with 1 and -1.

Obviously, the points $a = \frac{1}{2}\left(5-\sqrt{3}-\sqrt{6}) + i(1-\sqrt{3}+\sqrt{2}\right)$, $b = \frac{1}{2}\left(5-\sqrt{3}-\sqrt{6}) - i(1-\sqrt{3}+\sqrt{2}\right)$ and 1 are the corner points for T (Figure 1).

This paper is organized as follows. In section 2, the Cauchy-Schwarz representation formula on T is explicitly obtained by the technique of parqueting-reflection and the Cauchy-Pompeiu representation formula. In section 3, the Schwarz BVP in T for the inhomogeneous Cauchy-Riemann equation is studied and the expression of solution is explicitly obtained.

2. Cauchy-Schwarz representation formula on T

The point $z \in T$ is reflected at ∂C_1 onto $\frac{i\bar{z}+1}{\bar{z}+i}$, and both these points are reflected at ∂C_2 onto the points $\frac{-i\bar{z}+1}{\bar{z}-i}$ and $\frac{1}{z}$. Reflection of these four points at ∂C_3 are $\frac{\sqrt{3}\bar{z}-1}{\bar{z}-\sqrt{3}}$, $\frac{-(1+i\sqrt{3})z+(\sqrt{3}+i)}{-(\sqrt{3}+i)z+(1+i\sqrt{3})}$, $\frac{-\bar{z}+\sqrt{3}}{-\sqrt{3}\bar{z}+1}$ and $\frac{(-1+i\sqrt{3})z+(\sqrt{3}-i)}{(-\sqrt{3}+i)z+(1-i\sqrt{3})}$.

The Cauchy-Schwarz representation formula is derived by combining the Cauchy-Pompeiu representation formula applied to the points described above.

Theorem 2.1. *Any function $w \in C^1(T;\mathbb{C}) \cap C(\overline{T};\mathbb{C})$ for $T \subset \mathbb{C}$ can be represented as*

$$
\begin{aligned}
w(z) \;=\;& \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1} \mathrm{Re}\, w(\zeta)\Big[\frac{2(\zeta-i)}{\zeta-z} - 1 + \frac{2z(\zeta-i)}{\zeta z-1} - 1 \\
&+\frac{2\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} - 1 \\
&+\frac{2\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} - 1\Big] \\
&\times\frac{d\zeta}{\zeta-i} + \frac{1}{2\pi i}\int_{\partial T\cap\partial C_2} \mathrm{Re}\, w(\zeta)\Big[\frac{2(\zeta+i)}{\zeta-z} - 1 + \frac{2z(\zeta+i)}{\zeta z-1} - 1 \\
&+\frac{2\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} - 1 \\
&+\frac{2\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} - 1\Big] \\
&\times\frac{d\zeta}{\zeta+i} + \frac{1}{2\pi i}\int_{\partial T\cap\partial C_3} \mathrm{Re}\, w(\zeta)\Big[\frac{2(\zeta-\sqrt{3})}{\zeta-z} - 1 + \frac{2z(\zeta-\sqrt{3})}{\zeta z-1} \\
&-1+\frac{2\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
&-1 \\
&+\frac{2\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} - 1\Big] \\
&\times\frac{d\zeta}{\zeta-\sqrt{3}} \\
&+\frac{2}{\pi}\int_{\partial T\cap\partial C_1} \mathrm{Im}\, w(\zeta)\frac{d\zeta}{\zeta-i} + \frac{2}{\pi}\int_{\partial T\cap\partial C_2} \mathrm{Im}\, w(\zeta)\frac{d\zeta}{\zeta+i} \\
&+\frac{2}{\pi}\int_{\partial T\cap\partial C_3} \mathrm{Im}\, w(\zeta)\frac{d\zeta}{\zeta-\sqrt{3}} - \frac{1}{\pi}\int_T \Big\{ w_{\bar\zeta}(\zeta)\Big[\frac{1}{\zeta-z} + \frac{z}{\zeta z-1} \\
&+\frac{-(\sqrt{3}+i)z+(1+i\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
&+\frac{(-\sqrt{3}+i)z+(1-i\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\Big] \\
&+\overline{w_{\bar\zeta}(\zeta)}\Big[\frac{z-i}{-\bar\zeta(z-i)-iz+1} - \frac{z+i}{\bar\zeta(z+i)-iz-1} \\
&-\frac{z-\sqrt{3}}{\bar\zeta(z-\sqrt{3})-\sqrt{3}z+1} - \frac{-\sqrt{3}z+1}{\bar\zeta(-\sqrt{3}z+1)+z-\sqrt{3}}\Big]\Big\} d\xi d\eta,
\end{aligned}
\tag{2.1}
$$

where $\zeta = \xi + i\eta$.

Proof. The Cauchy-Pompeiu formula

$$\frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in T, \\ 0, & z \notin \overline{T}, \end{cases} \tag{2.2}$$

applied to $z \in T$ and $\frac{i\bar z+1}{\bar z+i}$, $\frac{1}{z}$, $\frac{-i\bar z+1}{\bar z-i}$, $\frac{-(1+i\sqrt{3})z+(\sqrt{3}+i)}{-(\sqrt{3}+i)z+(1+i\sqrt{3})}$, $\frac{\sqrt{3}\bar z-1}{\bar z-\sqrt{3}}$, $\frac{(-1+i\sqrt{3})z+(\sqrt{3}-i)}{(-\sqrt{3}+i)z+(1-i\sqrt{3})}$, $\frac{-\bar z+\sqrt{3}}{-\sqrt{3}\bar z+1} \notin \overline{T}$, respectively, gives the following eight equalities

$$w(z) = \frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{2.3}$$

$$0 = \frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{(\bar z + i)d\zeta}{\zeta(\bar z + i) - i\bar z - 1} - \frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{(\bar z + i)d\xi d\eta}{\zeta(\bar z + i) - i\bar z - 1}, \tag{2.4}$$

$$0 = \frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{z d\zeta}{\zeta z - 1} - \frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{z d\xi d\eta}{\zeta z - 1}, \tag{2.5}$$

$$0 = \frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{(\bar z - i)d\zeta}{\zeta(\bar z - i) + i\bar z - 1} - \frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{(\bar z - i)d\xi d\eta}{\zeta(\bar z - i) + i\bar z - 1}, \tag{2.6}$$

$$\begin{aligned} 0 = &\tfrac{1}{2\pi i}\textstyle\int_{\partial T} w(\zeta)\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)d\zeta}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\ &-\tfrac{1}{\pi}\textstyle\int_T w_{\bar\zeta}(\zeta)\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)d\xi d\eta}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 = &\frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{(\bar z - \sqrt{3})d\zeta}{\zeta(\bar z - \sqrt{3}) - \sqrt{3}\bar z + 1} \\ &-\frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{(\bar z - \sqrt{3})d\xi d\eta}{\zeta(\bar z - \sqrt{3}) - \sqrt{3}\bar z + 1}, \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
0 &= \tfrac{1}{2\pi i}\int_{\partial T} w(\zeta)\tfrac{\big((-\sqrt{3}+i)z+(1-i\sqrt{3})\big)d\zeta}{\zeta\big((-\sqrt{3}+i)z+(1-i\sqrt{3})\big)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
&\quad -\tfrac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\tfrac{\big((-\sqrt{3}+i)z+(1-i\sqrt{3})\big)d\xi d\eta}{\zeta\big((-\sqrt{3}+i)z+(1-i\sqrt{3})\big)+(1-i\sqrt{3})z+(-\sqrt{3}+i)},
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
0 &= \frac{1}{2\pi i}\int_{\partial T} w(\zeta)\frac{(-\sqrt{3}\bar z+1)d\zeta}{\zeta(-\sqrt{3}\bar z+1)+\bar z-\sqrt{3}} \\
&\quad -\frac{1}{\pi}\int_T w_{\bar\zeta}(\zeta)\frac{(-\sqrt{3}\bar z+1)d\xi d\eta}{\zeta(-\sqrt{3}\bar z+1)+\bar z-\sqrt{3}}.
\end{aligned}
\tag{2.10}
$$

Taking the complex conjugate of (2.4), (2.6), (2.8) and (2.10), where $\bar z$ appears, and adding the resulting eight relations, lead to the claimed representation formula. □

3. Schwarz BVP for T

The Cauchy-Schwarz representation formula (2.1), serves to solve the Schwarz BVP for the inhomogeneous Cauchy-Riemann equation in T.

Theorem 3.1. *The Schwarz BVP*

$$
\begin{gathered}
w_{\bar z} = f \ \text{in } T,\ \mathrm{Re}\, w = \gamma \ \text{on } \partial T, \\
\tfrac{2}{\pi i}\int_{\partial T\cap\partial C_1}\mathrm{Im}\, w(\zeta)\tfrac{d\zeta}{\zeta-i}+\tfrac{2}{\pi i}\int_{\partial T\cap\partial C_2}\mathrm{Im}\ w(\zeta)\tfrac{d\zeta}{\zeta+i} \\
+\tfrac{2}{\pi i}\int_{\partial T\cap\partial C_3}\mathrm{Im}\ w(\zeta)\tfrac{d\zeta}{\zeta-\sqrt{3}} = c,
\end{gathered}
$$

with given $f\in L_p(T;\mathbb{C}),\ 2<p,\ \gamma\in C(\partial T;\mathbb{C}),\ c\in\mathbb{R}$ *is uniquely solved by*

$$
\begin{aligned}
w(z) &= \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\gamma(\zeta)\Big[\frac{2(\zeta-i)}{\zeta-z}-1+\frac{2z(\zeta-i)}{\zeta z-1}-1 \\
&\quad +\frac{2\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
&\quad -1 \\
&\quad +\frac{2\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
&\quad -1\Big]\frac{d\zeta}{\zeta-i}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\gamma(\zeta)\Big[\frac{2(\zeta+i)}{\zeta-z}-1+\frac{2z(\zeta+i)}{\zeta z-1}-1\\
&+\frac{2\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1\\
&+\frac{2\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta+i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\gamma(\zeta)\Big[\frac{2(\zeta-\sqrt{3})}{\zeta-z}-1+\frac{2z(\zeta-\sqrt{3})}{\zeta z-1}-1\\
&+\frac{2\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\\
&+\frac{2\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta-\sqrt{3}}\\
&+ic\\
&-\frac{1}{\pi}\int_T\Big\{f(\zeta)\Big[\frac{1}{\zeta-z}+\frac{z}{\zeta z-1}\\
&+\frac{-(\sqrt{3}+i)z+(1+i\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{(-\sqrt{3}+i)z+(1-i\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\Big]\\
&+\overline{f(\zeta)}\Big[\frac{z-i}{-\bar{\zeta}(z-i)-iz+1}-\frac{z+i}{\bar{\zeta}(z+i)-iz-1}\\
&-\frac{z-\sqrt{3}}{\bar{\zeta}(z-\sqrt{3})-\sqrt{3}z+1}-\frac{-\sqrt{3}z+1}{\bar{\zeta}(-\sqrt{3}z+1)+z-\sqrt{3}}\Big]\Big\}d\xi d\eta.
\end{aligned}
\tag{3.1}
$$

Proof. The right-hand side of (3.1) up to the term

$$
Tf(z)=-\frac{1}{\pi}\int_T f(\zeta)\frac{d\xi d\eta}{\zeta-z},
$$

is an analytic function and $Tf(z)$ is a weak solution to the Cauchy-Riemann equation $w_{\bar z} = f$, so $w(z)$ is a weak solution of the inhomogeneous Cauchy-Riemann equation (see [14]).

Now, we consider the boundary behavior. Let

$$\begin{aligned}
w_0(z) \;=\; & -\frac{1}{\pi}\int_T \Big\{ f(\zeta)\Big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} \\
& + \frac{-(\sqrt{3}+i)z + (1+i\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z + (1+i\sqrt{3})\right) + (1+i\sqrt{3})z - (\sqrt{3}+i)} \\
& + \frac{(-\sqrt{3}+i)z + (1-i\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z + (1-i\sqrt{3})\right) + (1-i\sqrt{3})z + (-\sqrt{3}+i)}\Big] \\
& + \overline{f(\zeta)}\Big[\frac{z-i}{-\bar\zeta(z-i) - iz + 1} - \frac{z+i}{\bar\zeta(z+i) - iz - 1} \\
& - \frac{z-\sqrt{3}}{\bar\zeta(z-\sqrt{3}) - \sqrt{3}z + 1} - \frac{-\sqrt{3}z + 1}{\bar\zeta(-\sqrt{3}z + 1) + z - \sqrt{3}}\Big]\Big\} d\xi d\eta.
\end{aligned} \tag{3.2}$$

For $z \in \partial T \cap \partial C_1$,

$$\begin{aligned}
w_0(z) \;=\; & -\frac{1}{\pi}\int_T \Big\{ f(\zeta)\Big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} \\
& + \frac{-(\sqrt{3}+i)z + (1+i\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z + (1+i\sqrt{3})\right) + (1+i\sqrt{3})z - (\sqrt{3}+i)} \\
& + \frac{(-\sqrt{3}+i)z + (1-i\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z + (1-i\sqrt{3})\right) + (1-i\sqrt{3})z + (-\sqrt{3}+i)}\Big] \\
& - \overline{f(\zeta)}\Big[\frac{1}{\bar\zeta - \bar z} + \frac{\bar z}{\bar\zeta \bar z - 1} \\
& + \frac{(-\sqrt{3}+i)\bar z + (1-i\sqrt{3})}{\bar\zeta\left((-\sqrt{3}+i)\bar z + (1-i\sqrt{3})\right) + (1-i\sqrt{3})\bar z + (-\sqrt{3}+i)} \\
& + \frac{-(\sqrt{3}+i)\bar z + (1+i\sqrt{3})}{\bar\zeta\left(-(\sqrt{3}+i)\bar z + (1+i\sqrt{3})\right) + (1+i\sqrt{3})\bar z - (\sqrt{3}+i)}\Big]\Big\} \\
& d\xi d\eta.
\end{aligned}$$

So, $\mathrm{Re}w_0(z)=0$. Similarly, for $z\in\partial T\cap\partial C_2$ and $z\in\partial T\cap\partial C_3$, $\mathrm{Re}w_0(z)=0$. In fact

$$\begin{aligned}
\mathrm{Re}w(z) \quad = \quad & \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1+\frac{z(\zeta-i)}{\zeta z-1}\\
&+\frac{\bar z(\bar\zeta+i)}{\bar\zeta\bar z-1}-1\\
&+\frac{\left(-(\sqrt3+i)z+(1+i\sqrt3)\right)(\zeta-i)}{\zeta\left(-(\sqrt3+i)z+(1+i\sqrt3)\right)+(1+i\sqrt3)z-(\sqrt3+i)}\\
&+\frac{\left((-\sqrt3+i)\bar z+(1-i\sqrt3)\right)(\bar\zeta+i)}{\bar\zeta\left((-\sqrt3+i)\bar z+(1-i\sqrt3)\right)+(1-i\sqrt3)\bar z+(-\sqrt3+i)}\\
&-1\\
&+\frac{\left((-\sqrt3+i)z+(1-i\sqrt3)\right)(\zeta-i)}{\zeta\left((-\sqrt3+i)z+(1-i\sqrt3)\right)+(1-i\sqrt3)z+(-\sqrt3+i)}\\
&+\frac{\left(-(\sqrt3+i)\bar z+(1+i\sqrt3)\right)(\bar\zeta+i)}{\bar\zeta\left(-(\sqrt3+i)\bar z+(1+i\sqrt3)\right)+(1+i\sqrt3)\bar z-(\sqrt3+i)}\\
&-1\Big]\frac{d\zeta}{\zeta-i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1+\frac{z(\zeta+i)}{\zeta z-1}\\
&+\frac{\bar z(\bar\zeta-i)}{\bar\zeta\bar z-1}-1\\
&+\frac{\left((-\sqrt3+i)z+(1-i\sqrt3)\right)(\zeta+i)}{\zeta\left((-\sqrt3+i)z+(1-i\sqrt3)\right)+(1-i\sqrt3)z+(-\sqrt3+i)}\\
&+\frac{\left(-(\sqrt3+i)\bar z+(1+i\sqrt3)\right)(\bar\zeta-i)}{\bar\zeta\left(-(\sqrt3+i)\bar z+(1+i\sqrt3)\right)+(1+i\sqrt3)\bar z-(\sqrt3+i)}\\
&-1\\
&+\frac{\left(-(\sqrt3+i)z+(1+i\sqrt3)\right)(\zeta+i)}{\zeta\left(-(\sqrt3+i)z+(1+i\sqrt3)\right)+(1+i\sqrt3)z-(\sqrt3+i)}\\
&+\frac{\left((-\sqrt3+i)\bar z+(1-i\sqrt3)\right)(\bar\zeta-i)}{\bar\zeta\left((-\sqrt3+i)\bar z+(1-i\sqrt3)\right)+(1-i\sqrt3)\bar z+(-\sqrt3+i)}\\
&-1\Big]\frac{d\zeta}{\zeta+i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\gamma(\zeta)\Big[\frac{\zeta-\sqrt3}{\zeta-z}+\frac{\bar\zeta-\sqrt3}{\bar\zeta-\bar z}-1+\frac{z(\zeta-\sqrt3)}{\zeta z-1}\\
&+\frac{\bar z(\bar\zeta-\sqrt3)}{\bar\zeta\bar z-1}-1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{\left((-\sqrt{3}+i)\bar{z}+(1-i\sqrt{3})\right)(\bar{\zeta}-\sqrt{3})}{\bar{\zeta}\left((-\sqrt{3}+i)\bar{z}+(1-i\sqrt{3})\right)+(1-i\sqrt{3})\bar{z}+(-\sqrt{3}+i)}\\
&-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{\left(-(\sqrt{3}+i)\bar{z}+(1+i\sqrt{3})\right)(\bar{\zeta}-\sqrt{3})}{\bar{\zeta}\left(-(\sqrt{3}+i)\bar{z}+(1+i\sqrt{3})\right)+(1+i\sqrt{3})\bar{z}-(\sqrt{3}+i)}-1\Big]\\
&\times\frac{d\zeta}{\zeta-\sqrt{3}}\\
&+\mathrm{Re}\,w_0(z),
\end{aligned}
$$

where $w_0(z)$ is defined in (3.2). Therefore, on ∂C_1

$$
\begin{aligned}
\mathrm{Re}\,w(z) \;=\; &\frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z}+\frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{iz-1}{\zeta z-1}\\
&-1\\
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{2(z-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{2(-\sqrt{3}z+1)i}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})+(-\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta-i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z}+\frac{-(iz+1)}{\zeta z-1}-1+\frac{z(\zeta+i)}{\zeta z-1}\\
&+\frac{-(z+i)}{\zeta-z}-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(\sqrt{3}z-1)i}{\zeta\left(-(\sqrt{3}+i)+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\\
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{2(z-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta+i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\gamma(\zeta)\Big[\frac{\zeta-\sqrt{3}}{\zeta-z}\\
&+\frac{2(-z+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1+\frac{z(\zeta-\sqrt{3})}{\zeta z-1}\\
&+\frac{-2(z+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1\\
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{-z+\sqrt{3}}{\zeta-z}\\
&-1+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{\sqrt{3}z-1}{\zeta z-1}-1\Big]\frac{d\zeta}{\zeta-\sqrt{3}}\\
&+\mathrm{Re}\,w_0(z).
\end{aligned}
$$

Also, $\mathrm{Re}\,w(z)$ on ∂C_1 could be written as

$$
\begin{aligned}
\mathrm{Re}\,w(z) &= \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta-i}\\
&= \frac{1}{2\pi i}\int_{\partial C_1}\Gamma_1(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta-i},
\end{aligned}
$$

where

$$\Gamma_1(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial T\cap\partial C_1,\\ 0, & \zeta\in\partial C_1\setminus(\partial T\cap\partial C_1).\end{cases}$$

From the properties of the Poisson kernel for C_1, the equality

$$\lim_{z\to\zeta}\mathrm{Re}\,w(z)=\gamma(\zeta)$$

follows for $\zeta\in\partial T\cap\partial C_1$ up to the tips $b=\frac{1}{2}(5-\sqrt{3}-\sqrt{6})-i(1-\sqrt{3}+\sqrt{2})$ and 1 of T, because Γ_1 fails to be continuous there if γ does not vanish at these points. Also on ∂C_2, we have

$$\begin{aligned}
\mathrm{Re}\,w(z) = {} & \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z}+\frac{iz-1}{\zeta z-1}-1+\frac{z(\zeta-i)}{\zeta z-1}\\
&+\frac{-z+i}{\zeta-z}-1\\
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{2(-\sqrt{3}z+1)i}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{2(z-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta-i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z}+\frac{\bar{\zeta}-i}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta+i)}{\zeta z-1}\\
&+\frac{-(iz+1)}{\zeta z-1}-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{2(z-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{2(\sqrt{3}z-1)i}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta+i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\gamma(\zeta)\Big[\frac{\zeta-\sqrt{3}}{\zeta-z}\\
&+\frac{-2(z+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&-1+\frac{z(\zeta-\sqrt{3})}{\zeta z-1}\\
&+\frac{2(-z+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&-1\\
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{\sqrt{3}z-1}{\zeta z-1}-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{-z+\sqrt{3}}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta-\sqrt{3}}\\
&+\mathrm{Re}\, w_0(z).
\end{aligned}
$$

Also, $\mathrm{Re}\, w(z)$ on ∂C_2 could be written as

$$
\begin{aligned}
\mathrm{Re}\, w(z) &= \frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar{\zeta}-i}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+i}\\
&= \frac{1}{2\pi i}\int_{\partial C_2}\Gamma_2(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar{\zeta}-i}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+i},
\end{aligned}
$$

where

$$
\Gamma_2(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial T\cap\partial C_2,\\ 0, & \zeta\in\partial C_2\setminus(\partial T\cap\partial C_2).\end{cases}
$$

From the properties of the Poisson kernel for C_2 as above for $\zeta \in \partial T \cap \partial C_2$ the equality

$$\lim_{z\to\zeta} \mathrm{Re}\, w(z) = \gamma(\zeta)$$

holds with the possible exception of the tips 1 and $a = \frac{1}{2}(5-\sqrt{3}-\sqrt{6}) + i(1-\sqrt{3}+\sqrt{2})$. Similarly, on ∂C_3

$$\begin{aligned}
\mathrm{Re}\, w(z) \quad = \quad & \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1} \gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z} \\
& + \frac{2(z-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
& -1+\frac{z(\zeta-i)}{\zeta z-1} \\
& + \frac{-2(\sqrt{3}z-1)i}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
& -1 \\
& + \frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
& + \frac{-z+i}{\zeta-z}-1 \\
& + \frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
& + \frac{iz-1}{\zeta z-1}-1\Big]\frac{d\zeta}{\zeta-i} \\
& + \frac{1}{2\pi i}\int_{\partial T\cap\partial C_2} \gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z} \\
& + \frac{2(z-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
& -1+\frac{z(\zeta+i)}{\zeta z-1} \\
& + \frac{2(\sqrt{3}z-1)i}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
& -1 \\
& + \frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}
\end{aligned}$$

$$+\frac{-(z+i)}{\zeta-z}-1$$
$$+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}$$
$$+\frac{-(iz+1)}{\zeta z-1}-1\Big]\frac{d\zeta}{\zeta+i}$$
$$+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\gamma(\zeta)\Big[\frac{\zeta-\sqrt{3}}{\zeta-z}+\frac{\bar\zeta-\sqrt{3}}{\bar\zeta-\bar z}-1+\frac{z(\zeta-\sqrt{3})}{\zeta z-1}$$
$$+\frac{\sqrt{3}z-1}{\zeta z-1}-1$$
$$+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}$$
$$+\frac{2(-z+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}-1$$
$$+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}$$
$$+\frac{-2(z+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}-1\Big]$$
$$\times\frac{d\zeta}{\zeta-\sqrt{3}}$$
$$+\mathrm{Re}\,w_0(z).$$

Also, $\mathrm{Re}\,w(z)$ on ∂C_3 could be written as

$$\begin{aligned}\mathrm{Re}\,w(z) &= \frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\gamma(\zeta)\left[\frac{\zeta-\sqrt{3}}{\zeta-z}+\frac{\bar\zeta-\sqrt{3}}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-\sqrt{3}}\\ &= \frac{1}{2\pi i}\int_{\partial C_3}\Gamma_3(\zeta)\left[\frac{\zeta-\sqrt{3}}{\zeta-z}+\frac{\bar\zeta-\sqrt{3}}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-\sqrt{3}},\end{aligned}$$

where

$$\Gamma_3(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial T\cap\partial C_3,\\ 0, & \zeta\in\partial C_3\setminus(\partial T\cap\partial C_3).\end{cases}$$

From the properties of the Poisson kernel for C_3 as above for $\zeta\in\partial T\cap\partial C_3$ the equality

$$\lim_{z\to\zeta}\mathrm{Re}\,w(z)=\gamma(\zeta)$$

holds with the possible exception of the tips $a = \frac{1}{2}(5-\sqrt{3}-\sqrt{6})$ $+i(1-\sqrt{3}+\sqrt{2})$ and $b = \frac{1}{2}\left(5-\sqrt{3}-\sqrt{6}\right) - i(1-\sqrt{3}+\sqrt{2})$.

Now, we consider the boundary behavior at the corner points $1,\ a,\ b$. In fact, we show that

$$\lim_{z\to 1}\mathrm{Re}\,w(z)=\gamma(1),\ \lim_{z\to a}\mathrm{Re}\,w(z)=\gamma(a),\ \lim_{z\to b}\mathrm{Re}\,w(z)=\gamma(b).$$

We have

$$\begin{aligned}
1 \;=\; & \frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\Big[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{\bar z(\bar\zeta+i)}{\bar\zeta\bar z-1} \\
& -1 \\
& +\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
& +\frac{\left((-\sqrt{3}+i)\bar z+(1-i\sqrt{3})\right)(\bar\zeta+i)}{\bar\zeta\left((-\sqrt{3}+i)\bar z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})\bar z+(-\sqrt{3}+i)} \\
& -1 \\
& +\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
& +\frac{\left(-(\sqrt{3}+i)\bar z+(1+i\sqrt{3})\right)(\bar\zeta+i)}{\bar\zeta\left(-(\sqrt{3}+i)\bar z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})\bar z-(\sqrt{3}+i)} \\
& -1\Big]\frac{d\zeta}{\zeta-i} \\
& +\frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\Big[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1+\frac{z(\zeta+i)}{\zeta z-1}+\frac{\bar z(\bar\zeta-i)}{\bar\zeta\bar z-1} \\
& -1 \\
& +\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta+i)}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)} \\
& +\frac{\left(-(\sqrt{3}+i)\bar z+(1+i\sqrt{3})\right)(\bar\zeta-i)}{\bar\zeta\left(-(\sqrt{3}+i)\bar z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})\bar z-(\sqrt{3}+i)} \\
& -1 \\
& +\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta+i)}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)} \\
& +\frac{\left((-\sqrt{3}+i)\bar z+(1-i\sqrt{3})\right)(\bar\zeta-i)}{\bar\zeta\left((-\sqrt{3}+i)\bar z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})\bar z+(-\sqrt{3}+i)}
\end{aligned}$$

$$
\begin{aligned}
&-1\Big]\frac{d\zeta}{\zeta+i}\\
&+\frac{1}{2\pi i}\int_{\partial T\cap\partial C_3}\Big[\frac{\zeta-\sqrt{3}}{\zeta-z}+\frac{\bar\zeta-\sqrt{3}}{\bar\zeta-\bar z}-1+\frac{z(\zeta-\sqrt{3})}{\zeta z-1}\\
&+\frac{\bar z(\bar\zeta-\sqrt{3})}{\bar\zeta\bar z-1}-1\\
&+\frac{\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left(-(\sqrt{3}+i)z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})z-(\sqrt{3}+i)}\\
&+\frac{\left((-\sqrt{3}+i)\bar z+(1-i\sqrt{3})\right)(\bar\zeta-\sqrt{3})}{\bar\zeta\left((-\sqrt{3}+i)\bar z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})\bar z+(-\sqrt{3}+i)}\\
&-1\\
&+\frac{\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)(\zeta-\sqrt{3})}{\zeta\left((-\sqrt{3}+i)z+(1-i\sqrt{3})\right)+(1-i\sqrt{3})z+(-\sqrt{3}+i)}\\
&+\frac{\left(-(\sqrt{3}+i)\bar z+(1+i\sqrt{3})\right)(\bar\zeta-\sqrt{3})}{\bar\zeta\left(-(\sqrt{3}+i)\bar z+(1+i\sqrt{3})\right)+(1+i\sqrt{3})\bar z-(\sqrt{3}+i)}\\
&-1\Big]\frac{d\zeta}{\zeta-\sqrt{3}}.
\end{aligned}
\tag{3.3}
$$

Multiplying the relation (3.3) by $\gamma(1)$ and subtracting the resulting quantity from $\mathrm{Re}\, w(z)$, for $z\in\partial T\cap\partial C_1$, we get

$$\mathrm{Re}\, w(z)-\gamma(1)=\frac{1}{2\pi i}\int_{\partial T\cap\partial C_1}\tilde\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-i},$$

where $\tilde\gamma(\zeta)=\gamma(\zeta)-\gamma(1)$ and $\tilde\gamma(1)=0$. So

$$\lim_{z\to 1}\mathrm{Re}\, w(z)=\gamma(1).$$

Similarly, this relation can be shown to hold for $z\in\partial T\cap\partial C_2$. Also, multiplying (3.3) by $\gamma(a)$ and subtracting the resulting quantity from $\mathrm{Re}\, w(z)$, for $z\in\partial T\cap\partial C_2$, we get

$$\mathrm{Re}\, w(z)-\gamma(a)=\frac{1}{2\pi i}\int_{\partial T\cap\partial C_2}\hat\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta+i},$$

where $\hat\gamma(\zeta)=\gamma(\zeta)-\gamma(a)$ and $\hat\gamma(a)=0$. So

$$\lim_{z\to a}\mathrm{Re}\, w(z)=\gamma(a).$$

Similarly, this relation can be shown to hold for $z\in\partial T\cap\partial C_3$. Also, multiplying (3.3) by $\gamma(b)$ and subtracting the resulting quantity from

$\mathrm{Re}\, w(z)$, for $z \in \partial T \cap \partial C_3$, we get

$$\mathrm{Re}\, w(z) - \gamma(b) = \frac{1}{2\pi i} \int_{\partial T \cap \partial C_3} \dot{\gamma}(\zeta) \left[\frac{\zeta - \sqrt{3}}{\zeta - z} + \frac{\bar{\zeta} - \sqrt{3}}{\bar{\zeta} - \bar{z}} - 1 \right] \frac{d\zeta}{\zeta - \sqrt{3}},$$

where $\dot{\gamma}(\zeta) = \gamma(\zeta) - \gamma(b)$ and $\dot{\gamma}(b) = 0$. So

$$\lim_{z \to b} \mathrm{Re}\, w(z) = \gamma(b).$$

Similarly, this relation can be shown to hold for $z \in \partial T \cap \partial C_1$. □

References

[1] H. Begehr, Boundary value problems in complex analysis I, *Bol. Asoc. Math. Venezolana* **12***(2005), 65-85 (II. Bol. Asoc. Math. Venezolana* **12***(2005), 217-250).*

[2] H. Begehr, Complex analytic methods for partial differential equations: an introductory text, World Scientific, Singapore, 1994.

[3] H. Begehr, E. A. Gaertner, Dirichlet problem for the inhomogeneous polyharmonic equation in the upper half plane, *Georg. Math. J.* **14***(1)(2007), 33-52.*

[4] H. Begehr, D. Schmersau, The Schwarz problem for polyanalytic functions, *ZAA.* **24***(2)(2005), 341-351.*

[5] H. Begehr, T. Vaitekhovich, Harmonic boundary value problems in the half disc and half ring, *Funct. Approx.* **40***(2)(2009), 251-282.*

[6] H. Begehr, T. Vaitekhovich, Schwarz problem in lens and lune, *Complex Var. Elliptic Equ.* **59***(1)(2014), 76-84.*

[7] H. Begehr, T. Vaitekhovich, Harmonic Dirichlet problem for some equilateral triangle, *Complex Var. Ell. Eqs.* **57***(2012), 185-196.*

[8] S. Burgumbayeva, Boundary value problems for Tri-harmonic functions in the unit disc, Ph.D. thesis, FU Berlin, 2009. http://www.diss.fu-berlin.de/diss/receive/FUDISS-thesis-000000012636

[9] E. A. Gaertner, Basic complex boundary value problems in the upper half plane, Ph.D. thesis, FU Berlin, 2006. http://www.diss.fu-berlin.de/diss/receive/FUDISS-thesis-000000002129

[10] F. Joveini, M. Akbari, Schwarz problem in lens and half lens, *Eurasian Math. J.* **10***(2)(2019), 49-64.*

[11] B. Shupeyeva, Some basic boundary value problems complex partial differential equation in quarter ring and half hexagon, Ph.D. thesis, FU Berlin (2013). http://www.diss.fu-berlin.de/diss/receive/FUDISS-thesis-000000094596

[12] T. Vaitekhovich, Boundary value problems to second order complex partial differential equations in a ring domain, *Sauliai Math. Semin.* **2***(2007), 117-146.*

[13] T. Vaitekhovich, Boundary value problems to first order complex partial differential equations in a ring domain, *Integr. Transf. Special Funct.* **19***(2008), 211-233.*

[14] I. N. Vekua, Generalized analytic functions, Oxford, Pergamon Press, 1962.