Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260

CJMS. **9** (2)(2020), 321-339

Application of Legendre operational matrix to solution of two dimensional non-linear Volterra integro-differential equation

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Abstract. This article applies the operational matrix to find the numerical solution of two-dimensional nonlinear Volterra integro-differential equation (2DNVIDE). Form this prospect, two-dimensional shifted Legendre functions (2DSLFs) have been presented for integration, product, and differentiation. This method converts 2DNVIDE to an algebraic system of equations, so the numerical solution of 2DNVIDE is computable. The effectiveness and accuracy of the method were examined with some examples as well. The results and comparison with other methods have shown remarkable performance.

Keywords: Volterra integro-differential equation, Two-dimensional Legendre polynomials, Operational matrix, Error estimation.

2000 Mathematics subject classification: 45G10; Secondary 65R20.

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Received: 23 September 2019
Revised: 16 November 2019
Accepted: 15 November 2019

1. Introduction

The theory and applications of partial integral and integro-differential equations are used in many branches of scientific research from engineering, mechanics, and physics to economics, etc [1–7].

Recently, some numerical techniques and approaches have been used to evaluate the approximate solution of the nonlinear phenomena, whereas using the operational matrix could be the essential part of these numerical solutions [11,12,19]. In particular, Legendre polynomials as powerful tools have been employed to convert some nonlinear equations [13, 27]. There have been various numerical solutions for PDEs that benefit from the operational matrix to approximate the exact solutions in many fields of science; for example, in theory of anomalous relaxation processes in the vicinity of singular, IDE has been solved successfully by using operational matrix [14].

Some numerical methods, such as the homotopy perturbation method [24] and variational iteration method [26] have been represented to obtain the approximate solution of the mixed Volterra-Fredholm. The TFs methods have been used to approximate the numerical solution of Fredholm and Volterra integral equations [15]. Maleknejad in [18] and Khajehnasiri [17] have applied a TFs operational matrix to approximate the solution of nonlinear kind of Volterra-Fredholm integral equations and 2D nonlinear Volterra-Fredholm integro differential equations, respectively. The two-dimensional Block-Pulse functions (2D-BPFs) have been applied by Maleknejad and Mahdiani to find the solution of nonlinear mixed Volterra-Fredholm integral equations [16]. Imran Aziz has extended the Haar wavelet method to evaluate the numerical solution of 2D nonlinear integral equations [20], a class of 2D nonlinear Volterra integral equations in [25] have solved by Legendre polynomials, in [28] a 2DTFs have applied to find the nonlinear class of mixed Volterra-Fredholm integral equations. Aghazadeh in [10] has ameliorated the Block-pulse operational matrix to evaluate the approximate solution of the nonlinear 2D Volterra integro-differential equation.

Although the 2DNVIDEs have exciting applications in Physics, Mechanics, and applied sciences, there have been a few simple numerical methods for solving these equations with high accuracy. For this reason, in this paper, we formulate a Legendre operational matrix for 2D nonlinear Volterra integro-differential equations with given supplementary conditions. The most important part of our concept was extending this polynomial to convert the nonlinear kernels of FDVIDEs, which is unsolvable in a real evaluation. The two-dimensional nonlinear Volterra integro-differential equation in the general form of can be written as:

$$\frac{\partial^n u(x,t)}{\partial x^n} + \frac{\partial^m u(x,t)}{\partial t^m} + \frac{\partial^{n+m} u(x,t)}{\partial x^n \partial t^m} + u(x,t) = f(x,t) + \int_0^t \int_0^x G(x,t,y,z) R(y,z,U(y,z))) dydz, \quad (x,t) \in [0,l] \times [0,T], \tag{1.1}$$

with given supplementary initial conditions, where $u(x,t)$ is an unknown function in ($\Omega = [0,l] \times [0,T]$), the function $R(y,z,U(y,z))$ is given continuous in $\Omega \times (-\infty,+\infty)$, nonlinear in U, and the functions $f(x,t)$ and $G(x,t,y,z,u)$ are given smooth functions. Here we assume that R satisfy the following conditions:

$$\mid R(y,z,U(y,z)) - R(y,z,U^{'}(y,z)) \mid \leq \lambda \mid U(y,z) - U^{'}(y,z) \mid,$$

way in [21].
In the next section, we will define some basic definitions and properties of 2D shifted Legendre functions. In section three we will introduce the operational matrix for integration, product properties, as well as the operational matrix of differentiation. After which and in section 4, we are solving and obtaining the solution of two-dimensional nonlinear Volterra integro-differential equation by using 2D shifted Legendre functions. In section 5, we estimate the norm of error for the approximation of two variables smooth function on a specific domain Ω. Finally, we apply this the proposed method for some examples of 2DNVIDEs.

2. 2D shifted Legendre functions (Basic definitions and properties)

where λ is positive constants, and one can prove the uniqueness and existence of the solution to Eq. (1.1) in the same In the progress of this section, we define and represent some basic definition and properties of the two-dimensional shifted Legendre functions, which are used further in the following section.

2.1. **Definition and approximation of the function.**

The 2D shifted Legendre functions on Ω are defined as

$$\Psi_{m,n}(x,t) = L_m(\frac{2}{l}x - 1) L_n(\frac{2}{T}t - 1), \quad m,n = 0,1,2,..., \tag{2.1}$$

where L_m and L_n are f order m and n and they are the well-known Legendre functions, which are defined on the interval $[-1,1]$ and they can be obtained with the following formula

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= x, \\ L_{m+1}(x) &= \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), \quad m = 1, 2, 3, ..., -1 \leq x \leq 1. \end{aligned}$$

All of the 2D shifted Legendre functions pairs are orthogonal such as:

$$\int_0^T \int_0^l \Psi_{i,j}(x,t)\Psi_{m,n}(x,t)dxdt = \begin{cases} \frac{lT}{(2m+1)(2n+1)}, & i = m \quad and \quad j = n, \\ 0, & otherwise. \end{cases} \tag{2.2}$$

Let $X = L^2(\Omega)$, definition of the inner product in this space is as

$$\langle u(x,t), w(x,t)\rangle = \int_0^T \int_0^l u(x,t)w(x,t)dxdt, \tag{2.3}$$

where the norm is defined as:

$$\|u(x,t)\|_2 = \langle u(x,t), u(x,t)\rangle^{\frac{1}{2}} = \left(\int_0^T \int_0^l \mid u(x,t) \mid^2 dxdt\right)^{\frac{1}{2}}. \tag{2.4}$$

Suppose that

$$\Psi_{00}(x,t), \Psi_{01}(x,t), \cdots, \Psi_{0N}(x,t), \cdots, \Psi_{M0}(x,t), \Psi_{M1}(x,t), \cdots, \Psi_{MN}(x,t) \subset X \tag{2.5}$$

are the components of the 2D shifted Legendre functions and

$$X_{M,N} = span\{\Psi_{00}(x,t), \Psi_{01}(x,t), \cdots, \Psi_{0N}(x,t), \cdots, \Psi_{M0}(x,t), \Psi_{M1}(x,t), \cdots, \Psi_{MN}(x,t)\}$$

and $u(x,t)$ represent an arbitrary function in X. And $X_{M,N}$ be a finite dimensional vector space, so u is unique and it has a best approximation $u_{M,N} \in X_{M,N}$ [8], such that

$$\forall w \in X_{M,N}, \quad \|u - u_{M,N}\| \leq \|u - w\|_2. \tag{2.6}$$

In addition, since $u_{M,N} \in X_{M,N}$, unique coefficients $u_{00}, u_{01}, \cdots, u_{MN}$ are exist as follows:

$$u(x,t) \simeq u_{M,N}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} u_{ij}\Psi_{ij}(x,t) = U^T\Psi(x,t) = \Psi^T(x,t)U, \tag{2.7}$$

where the vectors U and $\Psi(x,t)$ of order $(M+1)(N+1) \times 1$ and given by

$$U = [u_{00}, \cdots, u_{0N}, u_{10}, \cdots, u_{1N}, \cdots, u_{M0}, \cdots, u_{MN}]^T, \tag{2.8}$$

$$\Psi(x,t) = [\Psi_{00}(x,t), \cdots, \Psi_{0N}(x,t), \Psi_{10}(x,t), \cdots \Psi_{1N}(x,t), \cdots \Psi_{M0}(x,t), \cdots \Psi_{MN}(x,t)]. \tag{2.9}$$

The coefficients u_{mn} of the 2D shifted Legendre function are defined by

$$u_{mn} = \frac{\langle u(x,t), \Psi_{mn}(x,t)\rangle}{\|\Psi_{mn}(x,t)\|_2^2}. \tag{2.10}$$

We could similarly expanding any functions g in $L^2(\Omega \times \Omega)$ respectively in terms of the 2D shifted Legendre functions as

$$g(x,t,y,z) \simeq \Psi^T(x,t) G \Psi(y,z), \tag{2.11}$$

here G is block matrices like as

$$G = [G^{(i,m)}]_{i,m=0}^{M}, \tag{2.12}$$

in which

$$G^{(i,m)} = [g_{ijmn}]_{j,n=0}^{N}, \quad i,m = 0,1,\cdots,M,$$

and the coefficients of the 2D shifted Legendre $k_{ijmn}, q = 1,2,3$ are defined by

$$g_{ijmn} = \frac{\langle\langle g(x,t,y,z), \Psi_{mn}(y,z)\rangle, \Psi_{ij}(x,t)\rangle}{\|\Psi_{ij}(x.t)\|_2^2 \|\Psi_{mn}(y,z)\|_2^2} \quad i,m = 0,1,\cdots,M, \quad j,n = 0,1,\cdots,N.$$

3. Construction of operational matrix of integration

We can approximate the integration defined by (2.9) for the vector $\Psi(x,t)$ as following role.

$$\int_0^t \int_0^x \Psi(x,t)\,dx\,dt \simeq \Upsilon_1 \Psi(x,t) = (E_1 \otimes E_2)\Psi(x,t), \tag{3.1}$$

where $x \in [0,l], t \in [0,T]$ and Υ_1 is the $(M+1)(N+1) \times (M+1)(N+1)$ operational matrix of integration, with setting out the E_1 and E_2 as the operational matrices of 1D shifted Legendre polynomials, respectively and they are defined on $[0,l]$ and $[0,T]$ as (See [22]):

$$E_1 = \frac{l}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2M-1} & 0 & \frac{1}{2M-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M+1} & 0 \end{bmatrix},$$

$$E_2 = \frac{T}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2M-1} & 0 & \frac{1}{2M-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M+1} & 0 \end{bmatrix},$$

and $\otimes$ denotes the Kronecker product, which defined for two arbitrary matrices A and B as [30]

$$A \otimes B = (a_{ij}B). \tag{3.2}$$

Analogously, we write

$$\int_0^x \Psi(x,t)dx \simeq \Upsilon_2 \Psi(x,t), \tag{3.3}$$

$$\int_0^t \Psi(x,t)dt \simeq \Upsilon_3 \Psi(x,t), \tag{3.4}$$

where Υ_2 and Υ_3 are matrices of order $(M+1)(N+1) \times (M+1)(N+1)$ of form

$$\Upsilon_2 = \frac{l}{2}\begin{bmatrix} I & I & O & \cdots & O & O & O \\ \frac{-I}{3} & O & \frac{I}{3} & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & \frac{-I}{2M-1} & O & \frac{I}{2M-1} \\ O & O & O & \cdots & O & \frac{-I}{2M+1} & O \end{bmatrix},$$

$$\Upsilon_3 = \begin{bmatrix} \Upsilon_2 & O & O & \cdots & O \\ O & \Upsilon_2 & O & \cdots & O \\ O & O & \Upsilon_2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & \Upsilon_2 \end{bmatrix},$$

where, O and I are zero and the identity matrix of order $N+1$, respectively.

3.0.1. *Product properties.* To following our approach to approximate the initial equations, we need to evaluate the product of two vectors $\Psi(x,t)$ and $\Psi^T(x,t)$, that is called the product matrix of Legendre functions. For this purpose, let

$$\Psi(x,t)\Psi^T(x,t)U_p \simeq \tilde{U_p}\Psi(x,t), \tag{3.5}$$

where U_p is defined by (2.8) and $\tilde{U_p}$ is the product operational matrix of order $(M+1)(N+1) \times (M+1)(N+1)$. We put

$$\Psi_{ij}(x,t)\Psi_{kh}(x,t) = \sum_{r=0}^{i+k}\sum_{s=0}^{j+h} b_{rs}\Psi_{rs}(x,t), \tag{3.6}$$

after which we can derive the b_{rs} in the following form. Multiplying both sides of Eq.(3.6) by $\Psi_{mn}(x,t), m = 0,1,\cdots,i+k, n = 0,1,\cdots,j+h,$

and integrating the prior result we have:

$$\begin{aligned}\int_0^T\int_0^1 \Psi_{ij}(x,t)\Psi_{kh}(x,t)\Psi_{mn}(x,t)dxdt &= \sum_{r=0}^{i+k}\sum_{s=0}^{j+k} b_{rs}\int_0^T\int_0^l \Psi_{rs}(x,t)\Psi_{mn}(x,t)dxdt,\\ &= \frac{lTb_{mn}}{(2m+1)(2n+1)},\end{aligned}$$

then we have

$$\begin{aligned} b_{mn} &= \frac{(2m+1)(2n+1)}{lT}\int_0^T\int_0^l \Psi_{ij}(x,t)\Psi_{kh}(x,t)\Psi_{mn}(x,t)dxdt\\ &= \frac{(2m+1)(2n+1)}{lT}\upsilon_{i,k,m}\upsilon'_{j,h,n}, \end{aligned} \quad (3.7)$$

where $\upsilon_{i,k,m}$ and $\upsilon'_{j,h,n}$ are represented as follows.

$$\begin{aligned}\upsilon_{i,k,m} &= \int_0^l L_i(\frac{2}{l}x-1)L_k(\frac{2}{l}x-1)L_m(\frac{2}{l}x-1)dx,\\ \upsilon'_{j,h,n} &= \int_0^T L_i(\frac{2}{T}t-1)L_h(\frac{2}{T}t-1)L_n(\frac{2}{T}t-1)dt,\end{aligned}$$

and it can be easily computed (See [23]). By substituting b_{mn} into Eq.(3.6) we have

$$\Psi_{ij}(x,t)\Psi_{kh}(x,t) = \sum_{m=0}^{i+k}\sum_{n=0}^{j+h}\frac{(2m+1)(2n+1)}{lT}\upsilon_{i,k,m}\upsilon'_{j,h,n}\Psi_{mn}(x,t). \quad (3.8)$$

The matrix $\tilde{U}_p$ in (3.5) is obtained. If we retain just the elements of $\Psi(x,t)$ in (2.9), as:

$$\tilde{U}_p = [U_p^{(i,j)}]_{i,j=0,1,\cdots,M} \quad (3.9)$$

moreover in Eq. (3.9), $U_p^{(i,j)}, i,j = 0,1,\cdots,M$ are defined as follows:

$$U_p^{(i,j)} = \frac{2j+1}{l}\sum_{m=0}^{M}\upsilon_{i,j,m}B_m,$$

and B_m, $m = 0,1,\cdots,M$ are matrices of order $(N+1)\times(N+1)$ which are given by

$$[B_m]_{kh} = \frac{2h+1}{T}\sum_{n=0}^{N}\upsilon'_{k,h,n}u_{mn}, \quad k,h = 0,1,\cdots,N. \quad (3.10)$$

Finally, for matrix $G = [G(i,j)], i,j = 0,1,\cdots,M$, with $(M+1)(N+1)\times(M+1)(N+1)$ such that

$$G^{(i,j)} = [g_{imjn}]_{m,n=0}^{N}, \quad i,j = 0,1,\cdots,M, \quad (3.11)$$

we have

$$\Psi^T(x,t)G\Psi(x,t) \simeq \tilde{G}\Psi(x,t), \tag{3.12}$$

where $\hat{G}$ is a vector of order $1 \times (M+1)(N+1)$ which is defined by

$$\hat{G} = [G_{00}, \cdots, G_{0N}, G_{10}, \cdots, G_{1N}, \cdots, G_{M0}, \cdots, G_{MN}], \tag{3.13}$$

and

$$G_{mn} = \frac{(2m+1)(2n+1)}{IT} \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{r=0}^{M} \sum_{s=0}^{N} \upsilon_{i,r,m} \upsilon'_{j,s,n} g_{ijrs},$$

where $m = 0, 1, \cdots, M, \quad n = 0, 1, \cdots, N.$

3.1. **Operational matrix of differentiation.** In this section, we evaluate the operational matrix of differentiation. For this purpose, we let

$$\begin{array}{l} u(x,t) = U^T\Psi(x,t), \\ u(x,0) = U_{x0}^T\Psi(x,t), \\ u(0,t) = U_{0t}^T\Psi(x,t), \\ u_t(x,t) = U_t^T\Psi(x,t), \\ u_x(x,t) = U_x^T\Psi(x,t), \\ u_t(x,0) = U_{tx0}^T\Psi(x,t), \\ u_{tt}(x,t) = U_{tt}^T\Psi(x,t), \\ u_x(0,t) = U_{x0t}^T\Psi(x,t), \\ u_{xx}(x,t) = U_{xx}^T\Psi(x,t), \\ u_{xt}(x,t) = U_{xt}^T\Psi(x,t). \end{array} \tag{3.14}$$

Now, we can write:

$$u(x,t) - u(x,0) = \int_0^t u_t(x,\tau)d\tau, \tag{3.15}$$

from (3.14), we obtain

$$\begin{aligned} U^T\Psi(x,t) - U_{x0}^T\Psi(x,t) &= \int_0^t U_t^T\Psi(x,\tau)d\tau \\ &= U_t^T \int_0^t \Psi(x,\tau)d\tau \\ &= U_t^T \Upsilon_3 \Psi(x,t). \end{aligned} \tag{3.16}$$

So we get

$$U^T - U_{x0}^T = U_t^T \Upsilon_3, \tag{3.17}$$

then

$$U_t^T = (U^T - U_{x0}^T)\Upsilon_3^{-1}. \tag{3.18}$$

Similarly, one can write:

$$u(x,t) - u(0,t) = \int_0^x u_x(\tau,t)d\tau, \tag{3.19}$$

then from (3.14), we have

$$\begin{aligned} U^T\Psi(x,t) - U_{0t}^T\Psi(x,t) &= \int_0^x U_x^T\Psi(\tau,t)d\tau, \\ &= U_x^T \int_0^x \Psi(\tau,t)d\tau, \\ &= U_x^T \Upsilon_2 \Psi(x,t), \end{aligned} \tag{3.20}$$

so we get

$$U^T - U_{0t}^T = U_x^T \Upsilon_2, \tag{3.21}$$

hence

$$U_x^T = (U^T - U_{0t}^T)\Upsilon_2^{-1}. \tag{3.22}$$

We can use such a way for the partial differential equations of the second-order. So one can write:

$$u_t(x,t) - u_t(x,0) = \int_0^t u_{tt}(x,\tau)d\tau, \tag{3.23}$$

by using (3.14), we have

$$\begin{aligned} U_t^T\Psi(x,t) - U_{tx0}^T\Psi(x,t) &= \int_0^t U_{tt}^T\Psi(x,\tau)d\tau, \\ &= U_{tt}^T \int_0^t \Psi(x,\tau)d\tau, \\ &= U_{tt}^T \Upsilon_3 \Psi(x,t), \end{aligned} \tag{3.24}$$

so we get

$$U_t^T - U_{tx0}^T = U_{tt}^T \Upsilon_3, \tag{3.25}$$

then

$$U_{tt}^T = (U_t^T - U_{tx0}^T)\Upsilon_3^{-1}. \tag{3.26}$$

In the same way, We get the following equation, which approximates $u_{xx}(x,t)$,

$$U_{xx}^T = (U_x^T - U_{x0t}^T)\Upsilon_2^{-1}. \tag{3.27}$$

And finally, to obtain the approximate $u_{xt}(x,t)$, the following procedure could be used as fallow.

$$u_t(x,t) - u_t(t,0) = \int_0^t u_{xt}(t,\tau)d\tau, \tag{3.28}$$

hence

$$\begin{aligned} U_x^T\Psi(x,t) - U_{x0t}^T\Psi(x,t) &= \int_0^t U_{xt}^T\Psi(x,\tau)d\tau, \\ &= U_{xt}^T\int_0^t \Psi(x,\tau)d\tau, \\ &= U_{xt}^T\Upsilon_3\Psi(x,t), \end{aligned} \tag{3.29}$$

so we get

$$U_x^T - U_{x0t}^T = U_{xt}^T\Upsilon_3, \tag{3.30}$$

then we have

$$U_{xt}^T = (U_x^T - U_{x0t}^T)\Upsilon_3^{-1}. \tag{3.31}$$

4. Applying the method

In this section, by using 2D shifted Legendre functions, we find the approximate solution of two-dimensional nonlinear Volterra integro-differential equation. As we evaluated in the previous section, we can write

$$\begin{aligned} &u(x,t) = U^T\Psi(x,t), \\ &f(x,t) = F^T\Psi(x,t), \\ &u_t(x,t) = U_x^T\Psi(x,t), \\ &u_t(x,t) = U_t^T\Psi(x,t), \\ &u_{xx}(x,t) = U_{xx}^T\Psi(x,t), \\ &u_{tt}(x,t) = U_{tt}^T\Psi(x,t), \\ &u_{tx}(x,t) = U_{tx}^T\Psi(x,t), \\ &g(x,t,y,z) = \Psi^T(x,t)\cdot G\cdot\Psi(y,z), \\ &R(y,z,U(y,z)) = R^T\Psi(y,z) = \Psi^T(y,z)R, \end{aligned} \tag{4.1}$$

where the m_1m_2-vectors $U, F, U_x, U_t, U_{xx}, U_{tt}, U_{tx}$ and matrix K and R are the 2D shifted Legendre coefficients of $u(x,t), f(x,t), u_x(x,t), u_t(x,t), u_{xx}(x,t), u_{tt}(x,t), u_{x,t}(x,t)$, $G(x,t,y,z)$ and $R(y,z,U(y,z)))$ respectively, now, consider the following equation,

$$u_{xx} + u_{tx} + u_{tt} + u(x,t) = f(x,t) + \int_0^t\int_0^x G(x,t,y,z)R(y,z,U(y,z)))dydz, \tag{4.2}$$

By using the proposed equations, we have

$$\begin{aligned}
\int_0^t\int_0^x G(x,t,y,z)R(y,z,U(y,z)))dydz &\simeq \int_0^t\int_0^x \Psi^T(x,t)\cdot G\cdot\Psi(y,z)\cdot\Psi(y,z)Rdydz \\
&= \Psi^T(x,t)\cdot G\cdot\int_0^t\int_0^x \Psi(y,z)\cdot\Psi(x,y)^T\cdot Rdydx \\
&\simeq \Psi^T(x,t)\cdot G\cdot\tilde{R}_p\int_0^t\int_0^x \Psi(x,y)dydx, \\
&= \Psi^T(x,t)\cdot G\cdot\tilde{R}_p\cdot\Upsilon_1\cdot\Psi(x,t) \qquad (4.3)
\end{aligned}$$

So we can rewrite the right part of Eq.(4.2) as:

$$\begin{aligned}
f(x,t)+\int_0^t\int_0^x G(x,t,y,z)R(y,z,U(y,z)))dydz &\simeq F^T\Psi(x,t)+\Psi^T(x,t)\cdot\Big(G\cdot\tilde{R}_p\cdot\Upsilon_1\Big)\cdot\Psi(x,t) \\
&\simeq F^T\Psi(x,t)+\big(\widehat{G\cdot\tilde{R}_p\cdot\Upsilon_1}\big)\cdot\Psi(x,t),
\end{aligned}$$

where $(\widehat{G\tilde{R}_p\Upsilon})$ is a $4m_1m_2$-vector with components equal to the diagonal components of the matrix $G\tilde{R}_p\Upsilon$. Since $\tilde{R}_p$ is a diagonal matrix, we get

$$(\widehat{G\tilde{R}_p\Upsilon_1})=\Pi\cdot R_p, \qquad (4.4)$$

in which Π is a $(4m_1m_2\times 4m_1m_2)$-matrix with components

$$\Pi_{i,j}=(KR)_{i,j}\cdot(\Upsilon_1)_{j,i}, \quad i,j=1,2,\cdots,4m_1m_2. \qquad (4.5)$$

So

$$\begin{aligned}
\Psi(x,t)^T(U_{xx}+U_{tx}+U_{tt}+U) &= F^T\cdot\Psi(x,t)+(\Pi\cdot R_p)^T\cdot\Psi(x,t) \\
&= \Psi^T(x,t)(F+\Pi\cdot R_p)
\end{aligned}$$

such that

$$U_{xx}+U_{tx}+U_{tt}+U=F^T+\Pi\cdot R_p \qquad (4.6)$$

Now, by using the equations (3.18), (3.22), (3.26), (3.27) and (3.31) we can obtain

$$AU=F \qquad (4.7)$$

in system of (4.7) A and F are the combination of 2DSLFs coefficient matrix and U which can be solved by Newton-Raphson method.

5. Error estimation for two variables functions

This section estimates the norm of error for approximating two variables smooth function on a specific domain Ω. First, we suppose that $u(x,t)$ is a two variables smooth function on a specific domain Ω and $P_{M,N}(x,t)$ is the approximation of u in point (x_i,t_i), where $x_i, i=$

$0, 1, \cdots, M$ belong to interval $[0, l]$ which are the shifted Chebyshev polynomials' roots and $t_j, j = 0, 1, \cdots, N$ belong to interval $[0, T]$ which are the the shifted Chebyshev polynomials' roots, we can evaluate upper band for derivatives as fallow.

$$u(x,t) - P_{M,N}(x,t) = \frac{\partial^{M+1}u(\xi,t)}{\partial x^{M+1}(M+1)!}\prod_{i=0}^{M}(x-x_i) + \frac{\partial^{N+1}u(x,\eta)}{\partial t^{N+1}(N+1)!}\prod_{j=0}^{N}(t-t_j) - \frac{\partial^{M+N+2}u(\xi',\eta')}{\partial x^{M+1}\partial t^{N+1}(M+1)!(N+1)!}\prod_{i=0}^{M}(x-x_i)\prod_{j=0}^{N}(t-t_j),$$

where $(\xi, \xi') \in [0, l]$, and $(\eta, \eta') \in [0, T]$. Then we have

$$\left|u(x,t) - P_{M,N}(x,t)\right| \leq \max_{(x,t)\in\Omega}\left|\frac{\partial^{M+1}u(x,t)}{\partial x^{M+1}}\right|\frac{\prod_{i=0}^{M}|x-x_i|}{(M+1)!} + \max_{(x,t)\in\Omega}\left|\frac{\partial^{N+1}u(x,t)}{\partial t^{N+1}}\right|\frac{\prod_{j=0}^{N}|t-t_j|}{(N+1)!} + \max_{(x,t)\in\Omega}\left|\frac{\partial^{M+N+2}u(x,t)}{\partial x^{M+1}\partial t^{N+1}}\right|\frac{\prod_{i=0}^{M}|x-x_i|\prod_{j=0}^{N}|t-t_j|}{(M+1)!(N+1)!}, \tag{5.1}$$

we set M_1, M_2 and M_3 as follows.

$$\max_{(x,t)\in\Omega}\left|\frac{\partial^{M+1}u(x,t)}{\partial x^{M+1}}\right| \leq M_1, \tag{5.2}$$

$$\max_{(x,t)\in\Omega}\left|\frac{\partial^{N+1}u(x,t)}{\partial t^{N+1}}\right| \leq M_2, \tag{5.3}$$

$$\max_{(x,t)\in\Omega}\left|\frac{\partial^{M+N+2}u(x,t)}{\partial x^{M+1}\partial t^{N+1}}\right| \leq M_3. \tag{5.4}$$

By use (5.1), (5.2), (5.3), (5.4) and estimation of Chebyshev interpolation (See [9]) we obtain

$$\left|u(x,t) - P_{M,N}(x,t)\right| \leq M_1\frac{(l/2)^{M+1}}{(M+1)!2M} + M_2\frac{(T/2)^{N+1}}{(N+1)!2N} + M_3\frac{(l/2)^{M+1}(T/2)^{N+1}}{(M+1)!(N+1)!2^{M+N}} \tag{5.5}$$

Following results are obtain by using (5.5) as follows (See [25]).

Theorem 5.1. *Assume that* $u_{M,N}(x,t) = U^T_{M,N}\Psi(x,t)$ *is the 2D shifted Legendre function of* $u(x,t)$ *which is sufficient smooth on* Ω*, and*

$$U = [u_{00}, \cdots, u_{0N}, u_{10}, \cdots, u_{1N}, \cdots, u_{M0}, \cdots, u_{MN}]^T,$$

and

$$u_{mn} = \frac{(2m+1)(2n+1)}{lT}\int_0^T\int_0^l u(x,t)\Phi(x,t)dxdt,$$

then, there are M'_1, M'_2, M'_3 *such that:*

$$\begin{aligned}\left|\left|u(x,t) - P_{M,N}(x,t)\right|\right|_2 \quad \leq \quad & M'_1\frac{(l/2)^{M+1}}{(M+1)!2M} \\ & +M'_2\frac{(T/2)^{N+1}}{(N+1)!2N} + M'_3\frac{(l/2)^{M+1}(T/2)^{N+1}}{(M+1)!(N+1)!2^{M+N}}.\end{aligned}$$

Proof. Suppose that $X_{M,N}$ be a subset of two variables polynomials of degree $\leq M$ for x and degree $\leq N$ for t. Let $w(x,t)$ be an arbitrary polynomial in $X_{M,N}$, the best approximation of u is $u_{M,N}$ such that:

$$\left|\left|u(x,t) - u_{M,N}(x,t)\right|\right|_2 \quad \leq \quad \left|\left|u(x,t) - w(x,t)\right|\right|_2 \qquad (5.6)$$

We can derive that

$$\begin{aligned}\left|\left|u(x,t) - u_{M,N}(x,t)\right|\right|_2^2 \quad & \leq \quad \int_0^T\int_0^l \left|u(x,t) - u_{M,N}(x,t)\right|^2 dxdt \\ & \leq \quad \int_0^T\int_0^l \left|u(x,t) - P_{M,N}(x,t)\right|^2 dxdt \qquad (5.7)\end{aligned}$$

where $P_{M,N}(x,t)$ is the approximation of u. Now from (5.5) and (5.7) we have

$$\begin{aligned}\left|\left|u(x,t) - u_{M,N}(x,t)\right|\right|_2^2 \quad \leq \quad & \int_0^T\int_0^l \Bigg[M'_1\frac{(l/2)^{M+1}}{(M+1)!2M} \\ & +M'_2\frac{(T/2)^{N+1}}{(N+1)!2N} + M'_3\frac{(l/2)^{M+1}(T/2)^{N+1}}{(M+1)!(N+1)!2^{M+N}}\Bigg]^2 dxdt \\ = \quad & lT\Bigg[M'_1\frac{(l/2)^{M+1}}{(M+1)!2M} + M'_2\frac{(T/2)^{N+1}}{(N+1)!2N} \\ & +M'_3\frac{(l/2)^{M+1}(T/2)^{N+1}}{(M+1)!(N+1)!2^{M+N}}\Bigg]^2.\end{aligned}$$

By substitution $M'_1 = \sqrt{lT}M_1$, $M'_2 = \sqrt{lT}M_2$ and $M'_3 = \sqrt{lT}M_3$ in previous expression (5.8) we can see that hypothesis (5.6) is true. □

Theorem 5.2. *Assume that* $u_{M,N}(x,t)$ *and* $\overline{u}_{M,N}(x,t) \subseteq X_{M,N}$ *and* $u_{M,N}$ *and* $\overline{u}_{M,N}$ *are two variables smooth functions such as* $u_{M,N} =$

$U_{M,N}\Psi(x,t)$ and $\overline{u}_{M,N} = \overline{U}_{M,N}\Psi(x,t)$, then there is a positive number $\beta_{M,N} \geq 0$ such that

$$\big|\big|u_{M,N}(x,t) - u_{M,N}(x,t)\big|\big|_2 \;\leq\; \big|\big|u_{M,N}(x,t) - w_{M,N}(x,t)\big|\big|_2 \quad (5.8)$$

Proof. To prove the hypothesis of theorem we can construct the solution from left side such as:

$$\begin{aligned}
\big|\big|u_{M,N}(x,t) - \overline{u}_{M,N}(x,t)\big|\big|_2^2 &= \int_0^T \int_0^l \big|u(x,t)_{M,N} - \overline{u}_{M,N}(x,t)\big|^2 dxdt \\
&= \int_0^T \int_0^l \left|\sum_{m=0}^{M}\sum_{n=0}^{N}(u_{mn} - \overline{u}_{mn})\Psi_{mn}(x,t)\right|^2 dxdt \\
&\leq \int_0^T \int_0^l \left(\sum_{m=0}^{M}\sum_{n=0}^{N}|u_{mn} - \overline{u}_{mn}|^2\right)\left(\sum_{m=0}^{M}\sum_{n=0}^{N}|\Psi_{mn}(x,t)|^2\right) dxdt \\
&= \sum_{m=0}^{M}\sum_{n=0}^{N}|u_{mn} - \overline{u}_{mn}|^2 \times \sum_{m=0}^{M}\sum_{n=0}^{N}\int_0^T \int_0^l |\Psi_{mn}(x,t)|^2 dxdt \\
&= ||U_{M,N} - \overline{U}_{M,N}||_2^2 \times \sum_{m=0}^{M}\sum_{n=0}^{N}||\Psi_{mn}(x,t)||_2^2 \\
&= ||U_{M,N} - \overline{U}_{M,N}||_2^2 \times \sum_{m=0}^{M}\sum_{n=0}^{N}||\frac{lT}{(2m+1)(2n+1)} \\
&= lT||U_{M,N} - \overline{U}_{M,N}||_2^2 \left(\sum_{m=0}^{M}\frac{1}{2m+1}\right)\left(\sum_{n=0}^{N}\frac{1}{2n+1}\right). \qquad (5.9)
\end{aligned}$$

Taking the squared root from both sides of (5.9) conclude that

$$\big|\big|u(x,t)_{M,N} - \overline{u}_{M,N}(x,t)\big|\big|_2^2 \;\leq\; \sqrt{lT\left(\sum_{m=0}^{M}\frac{1}{2m+1}\right)\left(\sum_{n=0}^{N}\frac{1}{2n+1}\right)}||U_{M,N} - \overline{U}_{M,N}||_2.$$

Finally from (5.10), we conclude that (5.8) is valid with

$$\beta_{M,N} = \sqrt{lT\left(\sum_{m=0}^{M}\frac{1}{2m+1}\right)\left(\sum_{n=0}^{N}\frac{1}{2n+1}\right)}, \qquad (5.10)$$

the proof is complete. □

Lemma 5.3. *In Theorem (5.2) if $M = N$ and $l = T = 1$ then*

$$\big|\big|u(x,t) - \overline{u}_{M,M}(x,t)\big|\big|_2 \;\leq\; \left(M_1' + M_2' + \frac{M_3'}{(M+1)!2^{2M+1}}\right)\frac{1}{(M+1)!2^{2M+1}},$$

therefore

$$\big|\big|u(x,t)-\overline{u}_{M,M}(x,t)\big|\big|_2 \quad = \quad O\bigg(\frac{1}{(M+1)!2^{2M+1}}\bigg),$$

so $u_{M,M}(x,t)$ *is the best approximation for two variables the smooth function* $u(x,t)$*, It means that when calculating the upper band for error in* (5.5)*, the expression containing* M_3' *could be eliminate.*

6. Numerical illustration

In this section, some experiments of 2DNVIDEs are given to illustrate our results. The supplementary initial conditions from the exact solution are taking into account for all examples. The presented method in this paper has been used to find the solution of two examples. The numerical results are compared with the exact solutions by the next error function:

$$e(x,t) =\mid u(x,t) - \bar{u}_{m_1,m_2}(x,t)\mid,$$

where $u(x,t)$ is exact soulution and $\bar{u}_{m_1,m_2}(x,t)$ represent the approximate solution of the integral equation. The error estimation for different values of m_1 and m_2 as $e(x,t)$ for the next set

$$D_{grids} = \{(0.0,0.0),(0.1,0.1),(0.2,0.2),\cdots,(0.9,0.9)\}, \qquad (6.1)$$

are evaluated, and the results represented in Tables 1-2.

Example 6.1. The first example represent the following equation [29],

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^2 u(x,t)}{\partial x\partial t}+xu^3(x,t)+\int_0^x\int_0^t u^2(t,\tau)dtd\tau = f(x,t), \quad x,t\in[0,1],$$

and

$$f(x,t) = 2y+\frac{1}{15}x^3t^5 - x^4t^5+2x,$$

where $u(x,t) = xt^2$ is the exact solution of this problem. With supplementary initial conditions.

$$B.Cs: \quad u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0. \qquad (6.2)$$

The numerical results for different values of m_1 and m_2 are shown in Table 1.

Example 6.2. For second example, we consider a 2DVIDE as follows

$$u(x,t)\frac{\partial^2 u(x,t)}{\partial t^2}-4u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}+4\int_0^x\int_0^t u^2u(t,\tau)dtd\tau = f(x,t), \quad x,t\in[0,1],$$

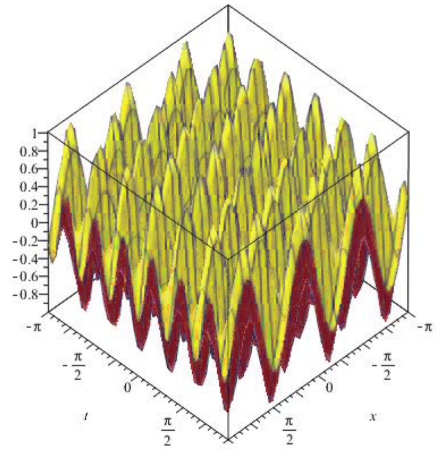

FIGURE 1. Exact and approximate solution with $m =$ 32of Example 1.

TABLE 1. Absolute errors of Example 1

$x = t$	e(x,t)	e(x,t)	e(x,t)
	$m_1 = m_2 = 4$	$m_1 = m_2 = 8$	$m_1 = m_2 = 32$
0	3.2612×10^{-3}	4.1230×10^{-3}	1.1905×10^{-5}
0.1	2.2941×10^{-3}	9.2341×10^{-3}	2.6508×10^{-5}
0.2	2.2154×10^{-3}	1.7014×10^{-3}	3.2548×10^{-5}
0.3	3.2541×10^{-3}	4.4011×10^{-3}	1.2884×10^{-4}
0.4	3.2522×10^{-3}	5.1472×10^{-3}	2.1524×10^{-4}
0.5	2.2589×10^{-3}	7.1203×10^{-3}	6.8459×10^{-4}
0.7	2.2542×10^{-3}	9.1014×10^{-3}	2.9528×10^{-4}
0.8	3.2514×10^{-3}	3.2458×10^{-2}	9.1215×10^{-4}
0.9	2.2254×10^{-3}	2.9872×10^{-2}	2.2547×10^{-3}

and

$$f(x,t) = \left(x - \frac{1}{2\pi} sin(2\pi x)\right)\left(t - \frac{1}{4\pi} sin(4\pi t)\right)$$

where $u(x,t) = sin(\pi x)cos(2\pi t)$ is the exact solution of this problem [29]. With supplementary boundary and initial conditions such as:

$$B.Cs: \quad u(0,t) = u(1,t) = 0,$$

$$I.Cs: \quad u(x,0) = sin(\pi x), \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \le x \le 1,$$

the numerical results and error estimation are presented in Table 2.

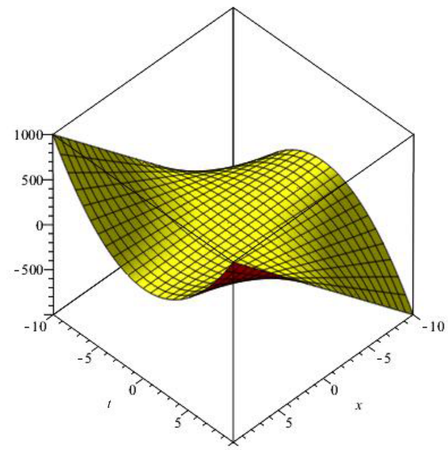

FIGURE 2. Comparison of the exact solution and present method with $m = 32$ of Example 2.

TABLE 2. Absolute errors of Example 2

$x = t$	e(x,t)	e(x,t)	e(x,t)
	$m_1 = m_2 = 4$	$m_1 = m_2 = 8$	$m_1 = m_2 = 32$
0	2.0352×10^{-2}	7.1215×10^{-3}	5.1963×10^{-5}
0.1	1.5322×10^{-2}	4.2509×10^{-3}	3.6591×10^{-5}
0.2	8.4454×10^{-2}	1.2145×10^{-3}	1.2548×10^{-5}
0.3	7.2840×10^{-2}	1.2547×10^{-3}	2.2124×10^{-4}
0.4	5.9522×10^{-2}	2.5804×10^{-3}	2.2524×10^{-4}
0.5	8.7589×10^{-2}	3.2154×10^{-3}	2.1063×10^{-4}
0.6	1.1850×10^{-2}	1.2152×10^{-3}	3.2562×10^{-4}
0.7	2.3698×10^{-2}	3.1002×10^{-2}	2.5098×10^{-4}
0.8	2.5874×10^{-2}	7.2425×10^{-2}	7.5214×10^{-3}
0.9	3.4512×10^{-2}	1.9272×10^{-2}	6.1251×10^{-3}

7. CONCLUSION

This paper implies the two-dimensional Legendre operational matrix to approximate the numerical solution of two-dimensional nonlinear Volterra integro-differential equation. The Legendre operational matrix has been used to convert the 2DNVIDE to an algebraic system, which could be easily solved to find the approximate solution. Furthermore, the error estimation for the proposed method is analyzed. Some examples and results examined the effectiveness and accuracy of the method have shown remarkable performance.

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