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Group analysis of time-fractional equation with Riemann-Liouville derivative

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Abstract. Finding Lie symmetries of nonlinear fractional differential equations play an important role in studying fractional differential equations. The purpose of this manuscript is to find the Lie point symmetries of the time-fractional Buckmaster equation. After that we use the infinitesimal generators for obtaining their corresponding invariant solutions.
Keywords: Fractional differential equation; Lie group; Time-fractional equation; Riemann-Liouville derivative; Group-invariant solutions.

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1. Introduction

The group analysis method of differential equations was introduced by Sophus Lie about one hundred years ago. Lie symmetries method is an effective method to solve the problems of mathematical physics, and researchers use it for analysis of partial differential equations (PDEs). The fractional differential equations (FDEs) have been studied by scientists about thirty years ago. There are many phenomena in the nature which can be described by making use of FDEs. The fractional

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differential equations arise in many fields of sciences such as: electrochemistry, physics, biology, mechanics, signal processing, viscoelastic materials [2, 15, 18, 22, 24]. Many articles have been presented on different definitions of fractional derivatives. The greatest numbers are the Caputo and the Riemann-Liouville derivatives. Each fractional derivative has some advantages and disadvantages. The Caputo derivative of a constant is zero, but Riemann-Liouville derivative of a constant is not zero. Many researchers in different articles have tried to find the exact solutions of FDEs. In recent years, there has been significant progression in the development of finding effective methods for obtaining exact solutions of FDEs. These methods include the separating variables method [7], the homotopy analysis method [8], the variational iteration method [10], the fractional complex transform [11], the homotopy perturbation Pade technique [16], the $\frac{G'}{G^2}$-expansion method[19], the first integral method[20], and so on. Many researchers obtained the exact solutions of many nonlinear PDEs by utilizing Lie group theory, but the question then arises: can we use this method for FDEs? There is a scarce of literature related to FDEs up to now [6, 9, 13, 14]. One of the difficulties of this type of problems originate from the non-local type of the fractional operators. In this manuscript, we study the time-fractional Buckmaster equation, namely,

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2(u^4)}{\partial x^2} - \frac{\partial(u^3)}{\partial x} = 0, \quad t > 0, x \in R. \tag{1.1}$$

Where $0 < \alpha < 1$, u is a function of (x, t). Thin viscous fluid sheet flows is described using the Buckmaster equation, and the exact solutions of this equation have been presented in many articles[1, 25].

The rest of our work is organized as follows. In Section 2 we present the analysis of the Lie Symmetry group of FDEs. After that in Section 3 we obtain the Lie point symmetries of the time-fractional Buckmaster equation. Finally, we obtain invariant solutions and reduced equations of this equation in Section 4. Discussion and conclusions are summarized in Section 5.

2. Description of the symmetry group analysis of FDEs

Finding the exact solutions of the fractional differential equations is an important and difficult task. Therefore, much effort has been made to obtain the exact solutions of them. We recall that symmetry is one of the most important concepts in studying the differential equations. Finding the exact solutions of differential equations using the fundamental method of the Lie symmetries was used by many researchers. Invariance of the equations under transformation groups is the basic concept of the Lie theory. Some researchers have studied this topic, Baumann

[3], Bluman [4], Ibragimov [12] and Olver [23]. Now we express the fractional Lie group method for finding infinitesimal functions of FPDEs. Let us assume a FPDE of form:

$$D_t^\alpha u = F(x, t, u_{(1)}, \ldots), \quad \alpha > 0, \tag{2.1}$$

where u is a function of independent variables x, t, and D_t^α can be defined as below.

Definition 2.1. D_t^α is the Riemann-Liouville fractional derivative operator which is defined by:

$$D_t^\alpha u = \begin{cases} \frac{\partial^m u}{\partial t^m}; \quad \alpha = m \in N, \\ \\ \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{u(\tau,x)}{(t-\tau)^{\alpha+1-m}} d\tau; \quad m-1 < \alpha < m,\ m \in N. \end{cases} \tag{2.2}$$

In a similar way of PDEs[5, 23], we can write

$$D_t^\alpha \bar{u} = D_t^\alpha u + \varepsilon[\eta_t^{(\alpha)}(x, t, u, u_{(\alpha)}, u_{(1)}, \ldots)] + O(\varepsilon^2), \tag{2.3}$$

here $\eta_t^{(\alpha)}$ is given by the prolongation formula [9]

$$\begin{aligned} \eta_t^{(\alpha)} &= D_t^\alpha(\eta) + \xi^x D_t^\alpha(u_x) - D_t^\alpha(\xi^x u_x) + D_t^\alpha(D_t(\xi^t)u) \\ &-D_t^{\alpha+1}(\xi^t u) + \xi^t D_t^{\alpha+1} u, \end{aligned} \tag{2.4}$$

where D_t is the total derivative operator defined as

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xxt} \frac{\partial}{\partial u_{xx}} + \cdots. \tag{2.5}$$

Simplifying (2.4) using the Leibnitz formula [26]

$$D_t^\alpha[f(t)g(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \quad \alpha > 0, \tag{2.6}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1} \alpha \Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}, \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{2.7}$$

it can be written [17]:

$$\begin{aligned} \eta_t^{(\alpha)} &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\xi^t)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} \\ &+ \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n(\eta_u)}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^t) \right] D_t^{\alpha-n}(u) \\ &- \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}(u_x) D_t^n(\xi^x). \end{aligned} \tag{2.8}$$

We have a definition as the following.

Definition 2.2. The equations for finding coefficients of the infinitesimal operator X are given below:

$$X^{(\alpha)}[D_t^\alpha u - F(x,t,u,u_{(1)},\dots)]_{D_t^\alpha u = F(x,t,u_{(1)},\dots)} = 0, \tag{2.9}$$

where

$$\begin{aligned} X^{(\alpha)} &= \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \\ &+\eta_i^{(1)}(x,t,u,u_{(1)})\frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}_{i_1 i_2\cdots,i_k}(x,t,u,u_{(1)},\dots u_{(k)}) \\ &\frac{\partial}{\partial u_{i_1 i_2 \cdots, i_k}} + \eta_t^{(\alpha)}(x,t,u,\dots,u_{(\alpha),\dots})\frac{\partial}{\partial u_t^{(\alpha)}}. \end{aligned} \tag{2.10}$$

Expanding the (2.9) using (2.10) and preceding relations, we obtain the determining equations. As a result, these obtained equations yields Lie symmetries.

3. Application of fractional Lie symmetries of the time-fractional Buckmaster equation

Here we employ this method to the time-fractional Buckmaster equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2(u^4)}{\partial x^2} - \frac{\partial(u^3)}{\partial x} = 0, \quad t>0,\ 0<\alpha<1. \tag{3.1}$$

We search the infinitesimal generators of (3.1).

Theorem 3.1. *Lie symmetries of the time fractional Buckmaster Equation* (3.1) *are*

1. *If* $\alpha \neq \frac{1}{2}, \frac{4}{5}$*, then we have:*

$$\xi^x = c_1\alpha x + c_2, \quad \xi^t = -c_1 t, \quad \eta_u = c_1\alpha u.$$

Where c_1 *and* c_2 *are two arbitrary constants. Therefore, the infinitesimal generators are given by*

$$X_{1.1} = \frac{\partial}{\partial x}, \quad X_{1.2} = \alpha x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u}.$$

2. *If* $\alpha = \frac{1}{2}$*, then we have:*

$$\xi^x = c_1 x + c_2, \quad \xi^t = -2c_1 t, \quad \eta_u = c_1 u.$$

Where c_1 *and* c_2 *are two arbitrary constants. Therefore, the infinitesimal generators are given by*

$$X_{2.1} = \frac{\partial}{\partial x}, \quad X_{2.2} = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

3. *If* $\alpha = \frac{4}{5}$, *then we have:*

$$\xi^x = 4c_1 x + c_2, \quad \xi^t = -5c_1 t, \quad \eta_u = 4c_1 u.$$

Where c_1 *and* c_2 *are three arbitrary constants. Therefore, the infinitesimal generators are given by*

$$X_{3.1} = \frac{\partial}{\partial x}, \quad X_{3.2} = 4x\frac{\partial}{\partial x} - 5t\frac{\partial}{\partial t} + 4u\frac{\partial}{\partial u}.$$

Proof. *Let us assume the one-parameter Lie group of infinitesimal transformation in* x, t, u *given by*

$$x^* = x + \varepsilon\xi^x(x,t,u) + O(\varepsilon^2),$$
$$t^* = t + \varepsilon\xi^t(x,t,u) + O(\varepsilon^2),$$
$$u^* = u + \varepsilon\eta_u(x,t,u) + O(\varepsilon^2),$$

where ε *is the group parameter, and the Lie algebra of Buckmaster equation is spanned by vector fields*

$$X = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta_u(x,t,u)\frac{\partial}{\partial u}, \tag{3.2}$$

where

$$\xi^x = \frac{dx^*}{d\varepsilon}|_{\varepsilon=0}, \quad \xi^t = \frac{dt^*}{d\varepsilon}|_{\varepsilon=0}, \quad \eta_u = \frac{du^*}{d\varepsilon}|_{\varepsilon=0}. \tag{3.3}$$

Applying the $X^{(\alpha)}$ *to* (3.1), *we have*

$$X^{(\alpha)}\left[\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2(u^4)}{\partial x^2} - \frac{\partial(u^3)}{\partial x}\right]_{\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2(u^4)}{\partial x^2} - \frac{\partial(u^3)}{\partial x}=0} = 0. \tag{3.4}$$

Where $X^{(\alpha)}$ *is given by* (2.10). *Expanding the* (3.4), *and solving obtained system using the Maple, we obtain the Lie point symmetries for the time-fractional Buckmaster equation. If* $\alpha \neq \frac{1}{2}, \frac{4}{5}$, *then we have:*

$$\xi^x = c_1\alpha x + c_2, \quad \xi^t = -c_1 t, \quad \eta_u = c_1\alpha u.$$

Therefore, the infinitesimal generators are given by

$$X_{1.1} = \frac{\partial}{\partial x}, \quad X_{1.2} = \alpha x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u}.$$

We now apply this argument again, with $\alpha = \frac{1}{2}$, *to obtain:*

$$\xi^x = c_1 x + c_2, \quad \xi^t = -2c_1 t, \quad \eta_u = c_1 u.$$

Therefore, the infinitesimal generators are given by

$$X_{2.1} = \frac{\partial}{\partial x}, \quad X_{2.2} = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

In the same manner for $\alpha = \frac{4}{5}$, *we can obtain:*

$$\xi^x = 4c_1 x + c_2, \quad \xi^t = -5c_1 t, \quad \eta_u = 4c_1 u.$$

Therefore, the infinitesimal generators are given by

$$X_{3.1} = \frac{\partial}{\partial x}, \quad X_{3.2} = 4x\frac{\partial}{\partial x} - 5t\frac{\partial}{\partial t} + 4u\frac{\partial}{\partial u}.$$

The proof is completed.

4. Invariant solutions and the reduced equations of the time-fractional Buckmaster equation

The time-fractional Buckmaster equation is expressed in the coordinates (x, t, u), so we want to reduce it by using new coordinates. By introducing invariants (r,z), we obtain the new coordinates corresponding of the infinitesimal symmetry generator, and we can reduce the mentioned equation [21]. Consider a Lie point symmetry

$$X = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta_u(x,t,u)\frac{\partial}{\partial u},$$

of the time-fractional Buckmaster equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2(u^4)}{\partial x^2} - \frac{\partial(u^3)}{\partial x} = 0, \quad t > 0,\ 0 < \alpha < 1.$$

Under the one-parameter group generated by X, the invariant solutions are obtained as follows. Two linearly independent invariants $r = \varphi(x,t)$ and $z = \psi(x,t)$ can be calculated by solving the first-order quasi-linear PDE

$$X(J) = \xi^x(x,t,u)\frac{\partial(J)}{\partial x} + \xi^t(x,t,u)\frac{\partial(J)}{\partial t} + \eta_u(x,t,u)\frac{\partial(J)}{\partial u} = 0,$$

or its characteristic equations

$$\frac{dx}{\xi^x(x,t,u)} = \frac{dt}{\xi^t(x,t,u)} = \frac{du}{\eta_u(x,t,u)}.$$

Then we write one of the invariants as a function of the other, for example

$$z = f(r), \tag{4.1}$$

and solve (4.1) for u. Finally, the expression of u is substituted in equation (3.1) and a fractional ODE is obtained for the unknown function f. With this procedure, we can reduce the number of independent variables by one. Now we obtain corresponding invariants, and present the reduced nonlinear fractional ordinary differential equations. Finally, we obtain the corresponding group invariant solutions of the fractional Buckmaster equation as follow.

Case1: $0 < \alpha < 1,\ X_{1.1} = \partial_x$.

In this case the corresponding invariants are given by:

$$r = t, \quad z = u. \tag{4.2}$$

A solution of our equation becomes

$$z = f(r) \Rightarrow u = f(t), \tag{4.3}$$

substitute (4.3) into (3.1) in order to determine the $f(r)$. Then $f(r)$ fulfills the following differential equation:

$$\frac{d^\alpha f(t)}{dt^\alpha} = 0. \tag{4.4}$$

The solution of the Eq.(4.4), by using the Laplace transform, is given by [24]

$$f(t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}. \tag{4.5}$$

Where k is a constant and $\Gamma(\alpha)$ is given by (2.7).

Case2: $0 < \alpha < 1,\ \alpha \neq \frac{1}{2}, \frac{4}{5},\ X_{1.2} = \alpha x \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \alpha u \frac{\partial}{\partial u}$.
In this case the corresponding invariants are given below:

$$r = t x^{\frac{1}{\alpha}}, \quad xz = u. \tag{4.6}$$

Then, a solution of our equation has the form

$$z = f(r) \Rightarrow u = x f(t x^{\frac{1}{\alpha}}), \tag{4.7}$$

and we substitute it into (3.1) to determine the $f(r)$. Then $f(r)$ has to satisfy in the following differential equation:

$$\alpha^2 \frac{\partial^\alpha f}{\partial r^\alpha} - 12\alpha^2 f(r)^4 - 3\alpha^2 f(r)^3 - 4r f'(r) f(r)^3 - 28 r\alpha f'(r) f(r)^3$$
$$-4r^2 f''(r) f(r)^3 - 12 r^2 f'(r)^2 f(r)^2 - 3r\alpha f'(r) f(r)^2 = 0.$$

In the same manner, we have

Case3: $\alpha = \frac{1}{2}, \quad X_{2.2} = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$.
The corresponding invariants for $\alpha = \frac{1}{2}$ and $X_{2.2}$ can be obtained as:

$$r = t x^2, \quad xz = u. \tag{4.8}$$

As a result, we obtain:

$$z = f(r) \Rightarrow u = x f(t x^2), \tag{4.9}$$

where $f(r)$ as solution of the following differential equation

$$\frac{\partial^\alpha f}{\partial r^\alpha} - 12 f(r)^4 - 72 r f'(r) f(r)^3 - 16 r^2 f''(r) f(r)^3 - 3 f(r)^3$$
$$-48 r^2 f'(r)^2 f(r)^2 - 6 r f'(r) f(r)^2 = 0.$$

Case4: $\alpha = \frac{4}{5}, \quad X_{3.2} = 4x\frac{\partial}{\partial x} - 5t\frac{\partial}{\partial t} + 4u\frac{\partial}{\partial u}$.
The invariants in this case have the following form:

$$r = t x^{5/4}, \quad xz = u. \tag{4.10}$$

As a result, we obtain:

$$z = f(r) \Rightarrow u = xf(tx^{5/4}), \tag{4.11}$$

by substituting it into (3.1), we conclude that $f(r)$ has to satisfy the following differential equation:

$$4\frac{\partial^\alpha f}{\partial r^\alpha} - 48f(r)^4 - 165rf'(r)f(r)^3 - 25r^2f''(r)f(r)^3 - 12f(r)^3$$
$$-75r^2f'(r)^2f(r)^2 - 15rf'(r)f(r)^2 = 0.$$

5. Conclusion

In the present study, we investigated the efficiency of the classical Lie symmetry group analysis to the fractional differential equations. The application of the fractional Lie symmetries method is considered to the time-fractional Buckmaster equation with the Riemann-Liouville derivative, and we found the Lie point symmetry group of this equation. As an application of the infinitesimal symmetries, we have presented that time-fractional Buckmaster equation can be obtained as a nonlinear ODE of fractional order. Finally, some group invariant solutions in an explicit form are obtained as well.

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