Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(1)(2022), 1-14 (Research paper)

Solving fuzzy multi objective liner programming problems: an α-cut approach

Fatemeh Fayyaz Rouhbakhsh [1] , Hassan Hassanpour [1] and Sohrab Effati [2].
[1] Faculty of Mathematical Sciences and Statistics, University of Birjand, Birjand. Iran.
[2] Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.

Abstract. In this paper, as an extension of Pareto optimality concepts for multi objective programming problems to fuzzy multi objective linear programming (FMOLP) problems, different types of Pareto optimal solutions (POSs), namely, weakly, strictly, and properly POSs are defined on the basis of α-cuts of fuzzy numbers. Then a method for solving FMOLP problems is proposed to obtain them. It is shown that they can be obtained by solving some non fuzzy multi objective linear programming problems. A numerical example is solved to illustrate the method.

Keywords: Multi objective linear programming, Fuzzy number, Properly α-Pareto optimal solution, Weakly α-Pareto optimal solution, Strictly α-Pareto optimal solution .

2020 Mathematics subject classification: 94D05, Secondary 90C05.

1. Introduction

There exist different mathematical methods to solve a single objective programming problem. Whereas, in real world there exist many problems called multi objective programming (MOP) problems which

[1]Corresponding author: fateme_ fayyaz@birjand.ac.ir
Received: 01 November 2019
Revised: 06 january 2020
Accepted: 07 January 2020

contain two or more objective functions under given constraints. In these problems the objective functions often conflict each other i.e. improving one of them may result in worsening another. So, an optimal solution that optimizes simultaneously all the objectives may not exist. Then, the notion of Pareto optimality (efficiency) for an MOP problem is defined. Let us consider the following MOP problem.

$$\mathrm{minimize}_{\mathrm{x}\in\mathrm{S}}(\mathrm{f}_1(\mathrm{x}),\mathrm{f}_2(\mathrm{x}),\ldots,\mathrm{f}_\mathrm{p}(\mathrm{x})) \tag{1.1}$$

where S is a feasible set.

Definition 1.1. (Pareto optimal solution)[**?**, **?**, **?**]. $\bar{x} \in S$ is said to be a Pareto optimal solution (efficient solution) to the MOP problem if there dose not exist another $x \in S$ such that $f_i(x) \leq f_i(\bar{x})$ for all i and $f_j(x) \neq f_j(\bar{x})$ for at least one j.

In many fields of real world such as management, engineering, etc, MOP has many applications (see e.g. [**?**, **?**, **?**, **?**, **?**]). Because of the complexity of real world problems, the decision makers (DMs) and experts may not exactly determine the values of parameters in multi objective linear programming (MOLP) problems. For solving problems in which description of observations are imprecise and uncertain, fuzzy set theory has been expanded. Fuzzy set theory is suitable to handle the imprecision of information (parameters of the problem) as fuzzy data. There are many studies including various approaches to solve FMOLP problems in literature. In earlier works, general MOLP problem with fuzzy parameters was formulated by Orlovski [**?**] and Tanaka [**?**]. The notion of α-POS for MOP problems with fuzzy parameters was introduced by Sakawa[**?**] then several interactive methods were proposed. By using weighted max-min method and the deviation degree measures, a method was proposed by Chen[**?**] to solve FMOLP problem wherein all the coefficients are triangular fuzzy numbers (TFNs). An approximate algorithm for solving FMOLP problems wherein all the coefficients are fuzzy numbers with various types of membership functions was proposed by Wu [**?**]. By using the nearest interval approximation operator, an approach was proposed by Luhandjula [**?**] in which FMOP problem is converted into a nonfuzzy one. For a multi objective nonlinear programming problem which contains three types of fuzzy goals expressed as "fuzzy min", "fuzzy max" and "fuzzy equal", some new solution concepts were introduced by Rouhbakhsh et al. [**?**] then some methods to get on these solutions were provided. A genetic algorithm was proposed by Thapar [**?**] to find a satisfactory solution in the feasible set of the MOP problems under max-product fuzzy relation equations. Singh and Shiv [**?**] proposed a new approach for finding efficient solutions of an MOP problem in intuitionistic fuzzy environment based on different views of the decision makers. Bharati and Singh [?] gave a new computational algorithm for solving the multi-objective linear programming problem in interval-valued intuitionistic fuzzy environment. Stanojević et al. [?] considered FMOLP problems with fuzzy coefficients in the objective functions. They introduced a parametric approach that helps to compute the membership values of the extreme points in the fuzzy set solution to such problems. This paper considers the FMOLP problem in which all the parameters are fuzzy numbers. The aim of this paper is solving the FMOLP problems and obtaining satisfactory solution for the DM. To this end, some extensions of the POSs to the FMOLP problem are provided. To obtain these solutions, the FMOLP problem is converted to an MOLP problem which depends on α-cuts of fuzzy numbers. Sometimes, in the set of POSs, some POSs which have finite tradeoffs between objective functions, are desirable to the DM, called properly POSs. Also, the notions of strict and weak POSs of MOP problems are extended to the FMOLP problems, and a method is proposed to get on these solutions. The remainder of the paper is organized as follows. In Section 2, some preliminaries and basic definitions are presented, in Section 3 some new solution consepts are introduced, then a method is proposed to obtain them and the final section is devoted to some conclusions.

2. Preliminaries

In this section, we review some preliminaries used in this paper according to [?].
Let $\mathrm{w}^1, \mathrm{w}^2 \in \mathbb{R}^{\mathrm{p}}$. We define

$$\begin{aligned}
&\mathrm{w}^1 \leqq \mathrm{w}^2 \Longleftrightarrow \mathrm{w}^1_{\mathrm{k}} \leq \mathrm{w}^2_{\mathrm{k}}, && \mathrm{k} = 1, 2, \ldots, \mathrm{p},\\
&\mathrm{w}^1 \leq \mathrm{w}^2 \Longleftrightarrow \mathrm{w}^1_{\mathrm{k}} \leq \mathrm{w}^2_{\mathrm{k}} && \text{for all } \mathrm{k} \in \{1, 2, \ldots, \mathrm{p}\} \text{ and } \mathrm{w}^1_{\mathrm{j}} < \mathrm{w}^2_{\mathrm{j}} \text{ for one j},\\
&\mathrm{w}^1 < \mathrm{w}^2 \Longleftrightarrow \mathrm{w}^1_{\mathrm{k}} < \mathrm{w}^2_{\mathrm{k}}, && \mathrm{k} = 1, 2, \ldots, \mathrm{p}.
\end{aligned}$$

Also,

Definition 2.1. A fuzzy number is a fuzzy set $\tilde{M}$ on $\mathbb{R}$ with membership function that has the following properties:
(1) $\mu_{\tilde{M}}(x)$ is upper semi-continuous,
(2) $\mu_{\tilde{M}}(x) = 0$ outside some interval $[c, d]$,
(3) There are real numbers a, b such that $c \leq a \leq b \leq d$ and
(3.1) $\mu_{\tilde{M}}(x)$ is monotonic increasing on $[c, a]$,
(3.2) $\mu_{\tilde{M}}(x)$ is monotonic decreasing on $[b, d]$,
(3.3) $\mu_{\tilde{M}}(x) = 1$, $a \leq x \leq b$.

Also, the α-cut of the fuzzy number $\tilde{M}$ which is a closed interval, denoted by $\tilde{M}_\alpha = [a^l_\alpha, a^r_\alpha]$, is defined by

$$\tilde{M}_\alpha = [a^l_\alpha, a^r_\alpha] = \begin{cases} \{x \in \mathbb{R} \mid \mu_{\tilde{M}}(x) \geq \alpha\} & 0 < \alpha \leq 1 \\ cl(supp(\tilde{M})) & \alpha = 0 \end{cases}$$

where $supp(\tilde{M}) = \{x \in \mathbb{R} \mid \mu_{\tilde{M}}(x) > 0\}$, and $cl(supp(\tilde{M}))$ is its closure.

Definition 2.2. A fuzzy number with the membership function

$$\mu_{\tilde{N}}(x) = \begin{cases} \dfrac{x - n_1}{n_2 - n_1} & n_1 \leq x \leq n_2 \\ 1 & x = n_2 \\ \dfrac{x - n_3}{n_2 - n_3} & n_2 \leq x \leq n_3 \\ 0 & otherwise \end{cases}$$

is called a TFN and denoted by $\tilde{N} = (n_1, n_2, n_3)$, where n_1 and n_3 are the left and right end points of $supp(\tilde{N})$, respectively and n_2 is the center of $\tilde{N}$. It can be easily shown that the α-cut of the TFN is as follows:

$$\tilde{N}_\alpha = [n^l_\alpha, n^r_\alpha] = [n_1 + \alpha(n_2 - n_1), n_3 - \alpha(n_3 - n_2)].$$

3. MOLP problem with fuzzy parameters

In this section, we define various types of POSs, namely strictly, weakly, and properly α-POSs for the FMOLP problem on the basis of the α-cuts of fuzzy numbers, as an extension of POS for the MOP problem. Then, we propose a technique to get on them. Consider the FMOLP problem

$$\begin{aligned} \min \quad & \tilde{Z}_k = \tilde{C}_k x, \qquad k = 1, 2, \ldots, p \\ s.t. \quad & x \in S(\tilde{A}, \tilde{B}) \end{aligned} \tag{3.1}$$

where $S(\tilde{A}, \tilde{B}) = \{x \in \mathbb{R}^n | \, \tilde{A}_i x \geq \tilde{b}_i, \, i = 1, 2, \ldots, m; \; x \geq 0\}$ is the feasible set, and $\tilde{C}_k = (\tilde{c}_{k1}, \tilde{c}_{k2}, \ldots, \tilde{c}_{kn})$, $\tilde{A}_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in})$ and $\tilde{B} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)^T$ represent the vectors of fuzzy parameters in objective functions and constraints. Suppose that $\mu_{\tilde{c}_{k1}}(c_{k1}), \mu_{\tilde{c}_{k2}}(c_{k2}), \ldots, \mu_{\tilde{c}_{kn}}(c_{kn})$, $\mu_{\tilde{a}_{i1}}(a_{i1})$, $\mu_{\tilde{a}_{i2}}(a_{i1})$, $\ldots$, $\mu_{\tilde{a}_{in}}(a_{in})$ and $\mu_{\tilde{b}_i}(b_i)$ denote the membership functions of the fuzzy numbers $\tilde{c}_{k1}, \tilde{c}_{k2}, \ldots \tilde{c}_{kn}$, $\tilde{a}_{i1}$, $\tilde{a}_{i2}, \ldots, \tilde{a}_{in}$ and $\tilde{b}_i$, respectively. For this problem wherein parameters are fuzzy numbers, the notion of POS defined for MOLP can not be used directly. Therefore, Sakawa introduced fuzzy POS or α-POS for the first time and proposed a solution method to get on it [**?**]. For this purpose, the α-cut of the fuzzy numbers $\tilde{a}_{ij}, \tilde{b}_i$ and $\tilde{c}_{kj}$ are introduced.

Definition 3.1. [**?**] For simplicity in notation, suppose that $\tilde{C} = (\tilde{C}_1, \ldots, \tilde{C}_p)$, $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_m)^T$ and $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_m)^T$. The α-cut of the fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}_{kj}$ is defined as follows:

$$(\tilde{A}, \tilde{B}, \tilde{C})_\alpha = \{(a, b, c) \mid \mu_{\tilde{a}_{ij}}(a_{ij}) \geq \alpha,\ \mu_{\tilde{b}_i}(b_i) \geq \alpha,\ \mu_{\tilde{c}_{kj}}(c_{kj}) \geq \alpha,$$
$$i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, p\},$$

where $a = (a_1, a_2, \ldots, a_m)^T$, $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $c = (c_1, c_2, \ldots, c_p)$, $c_k = (c_{k1}, c_{k2}, \ldots, c_{kn})$ and $b = (b_1, b_2, \ldots, b_m)^T$.

Now suppose that the DM chooses a degree $\alpha \in [0, 1]$. Then the FMOLP problem can be interpreted as the following nonfuzzy multi objective linear programming problem which depends on the coefficients $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$:

$$\begin{aligned} \min \quad & z_k = c_k x \qquad k = 1, 2, \ldots, p \\ s.t. \quad & x \in S(a, b) = \{x \in \mathbb{R}^n | a_i x \geq b_i,\ i = 1, 2, \ldots, m;\ x \geq 0\} \end{aligned} \tag{3.2}$$

Observe that there exists an infinite number of such problems depending on the coefficients $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$, and the values of (a, b, c) are arbitrarily chosen from $(\tilde{A}, \tilde{B}, \tilde{C})_\alpha$, in the sense that the degree of all of the membership functions for the fuzzy numbers in the FMOLP exceeds the level α. However, if possible, it would be desirable for the DM to choose $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$ in the problem (**??**) to minimize the objective functions under the constraints. From such a point of view, for a certain degree α, it seems to be quite natural to have the FMOLP as the following nonfuzzy α-multi objective programming (α-MOP) problem [**?**].

$$\begin{aligned} & (\alpha - MOP): \\ & \quad \min \quad z_k = c_k x \qquad k = 1, 2, \ldots, p \\ & \quad s.t. \quad x \in S(a, b) = \{x \in \mathbb{R}^n | a_i x \geq b_i,\ i = 1, 2, \ldots, m;\ x \geq 0\} \\ & \qquad\qquad (a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha \end{aligned} \tag{3.3}$$

where $c_k = (c_{k1}, c_{k2}, \ldots, c_{kn})$ and $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. The parameters (a, b, c) are treated as decision variables [**?**]. Therefore, the problem (**??**) is nonlinear. Sakawa [**?**] introduced the concept of an α-POS to the (α-MOP) as a natural extension of the Pareto optimality concept for the MOP as follows.

Definition 3.2. [**?**] A feasible solution $\bar{x} \in S(\bar{a}, \bar{b})$ is called an α-POS to (α-MOP) if there does not exist another $x \in S(a, b)$, $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$ such that $c_k x \leq \bar{c}_k \bar{x}$ for all k and $c_j x < \bar{c}_j \bar{x}$ for at least one j. The parameters $(\bar{a}, \bar{b}, \bar{c})$ are called α-optimal parameters.

Sometimes an α-POS may not exist for the FMOLP problem. Therefore, we define a slightly weaker solution concept than α-POS called weakly α-POS to the FMOLP problem. The set of weakly α-POSs holds a larger solution set than the preceding set and the set of weakly α-POSs is not empty.

Definition 3.3. A feasible solution $\bar{x} \in S(\bar{a}, \bar{b})$ is called a weakly α-POS to (α-MOP) if there is no other $x \in S(a, b)$, $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$ such that $c_k x < \bar{c}_k \bar{x}$ for all k. The parameters $(\bar{a}, \bar{b}, \bar{c})$ are called weak α-optimal parameters.

In the following a special type of α-POSs is defined. In spite of weakly α-POS, this type of solution yields a restricted set of α-POSs to (α-MOP) problem.

Definition 3.4. A feasible solution $\bar{x} \in S(\bar{a}, \bar{b})$ is called a strictly α-POS to (α-MOP) if there does not exist another $x \in S(a, b)$, $x \neq \bar{x}$, $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$ such that $c_k x \leq \bar{c}_k \bar{x}$ for all k. The parameters $(\bar{a}, \bar{b}, \bar{c})$ are called strict α-optimal parameters.

We denote the sets of all α-POSs, strictly α-POSs and weakly α-POSs to (α-MOP) problem by $X_{\alpha P}$, $X_{\alpha SP}$ and $X_{\alpha WP}$ respectively. It is clear that (α-MOP) is nonlinear. Howevere, in the following, by solving an MOLP problem which depends on α, different types of α-POSs are obtained.

Proposition 3.5. *1. Suppose that $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ and $\tilde{B} = [\tilde{b}_{ij}]_{m\times 1}$ are fuzzy matrices consisting of fuzzy numbers. For a fixed value of degree α, denote the matrices containing α-cut of the elements of matrices $\tilde{A}$ and $\tilde{B}$ by $\tilde{A}_\alpha = [A^l_\alpha, A^r_\alpha]$ and $\tilde{B}_\alpha = [b^l_\alpha, b^r_\alpha]$ respectively. Then for any given $x \geq 0$, $A^l_\alpha x \geq b^r_\alpha$ and $A^r_\alpha x \geq b^l_\alpha$ are the smallest and largest feasible regions in the problem (**??**), respectively.*
2. Suppose that $z_k = c_k x$ is the k^{th} objective function such that $c_k \in \tilde{C}_{k\alpha} = [c^l_{k\alpha}, c^r_{k\alpha}]$. Then for any given $x \geq 0$ we have $c_k x \geq c^l_{k\alpha} x$.

Proof. The proof is straightforward. □

After taking the degree α from the DM, we construct a multi objective linear programming problem as follows:

$$\begin{aligned} \min \quad & z_k = c^l_{k\alpha} x \quad k = 1, 2, \ldots, p \\ s.t. \quad & x \in S(a^r_\alpha, b^l_\alpha) = \{x \in \mathbb{R}^n | a^r_{i\alpha} x \geq b^l_{i\alpha},\ i = 1, 2, \ldots, m;\ x \geq 0\} \end{aligned} \tag{3.4}$$

where $c^l_{k\alpha} = (c^l_{k1\alpha}, c^l_{k2\alpha}, \ldots, c^l_{kn\alpha})$ and $a^r_{i\alpha} = (a^r_{i1\alpha}, a^r_{i2\alpha}, \ldots, a^r_{in\alpha})$.
In fact, this problem has the largest feasible region, and lowest objective values among all possible situations of the problem (α-MOP). In

the following we have worthful theorems which allow us to solve the linear programming problem (**??**) instead of the nonlinear programming problem (**??**).

Theorem 3.6. *If $\bar{x}$ is a POS to the problem (**??**), then $\bar{x}$ is an α-POS to (α-MOP) with α-optimal parameters $(a^r_\alpha, b^l_\alpha, c^l_\alpha)$.*

Proof. Clearly, $\bar{x}$ is feasible to (α-MOP) with parameters (a^r_α, b^l_α). Now suppose that $\bar{x}$ is not an α-POS to (α-MOP). Then there is $x \in S(a,b)$ that $(a,b,c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$ such that $c_k x \leq c^l_{k\alpha}\bar{x}$ for all k and $c_j x < c^l_{j\alpha}\bar{x}$ for at least one j. Therefore,

$$c^l_{k\alpha}x \leq c_k x \leq c^l_{k\alpha}\bar{x}$$

for all k, and

$$c^l_{j\alpha}x \leq c_j x < c^l_{j\alpha}\bar{x}$$

for at least one j. Also x is a feasible solution for the problem (**??**), because the feasible set of the problem (**??**) is a subset of the feasible set of the problem (**??**). All these imply that $\bar{x}$ is not POS of the problem (**??**). This is a contradiction. □

In the following we prove the inverse of the above theorem.

Theorem 3.7. *If $\bar{x}$ is an α-POS to (α-MOP) with α-optimal parameters $(\bar{a}, \bar{b}, \bar{c})$, then $\bar{x}$ is POS to the problem (**??**).*

Proof. Clearly, $\bar{x}$ is feasible to the problem (**??**). Suppose that $\bar{x}$ is not POS to (**??**). So, there is x with $a^r_\alpha x \geq b^l_\alpha$ and $x \geq 0$ such that $c^l_{k\alpha}x \leq c^l_{k\alpha}\bar{x}$ for all k and $c^l_{j\alpha}x < c^l_{j\alpha}\bar{x}$ for at least one j. Therefore,

$$c^l_{k\alpha}x \leq c^l_{k\alpha}\bar{x} \leq \bar{c}_k\bar{x}$$

for all k, and

$$c^l_{j\alpha}x < c^l_{j\alpha}\bar{x} \leq \bar{c}_j\bar{x}$$

for at least one j. On the other hand, $x \in S(a^r_\alpha, b^l_\alpha)$ and $(a^r_\alpha, b^l_\alpha, c^l_\alpha) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$. All these imply that $\bar{x}$ is not α-POS of the problem (**??**). This is a contradiction. □

According to the above theorems, all α-POSs to (α-MOP) can be obtained by solving the problem (**??**), which is an MOLP problem. Some results hold for weakly and strictly α-POSs, which are stated in Theorems **??**-**??**.

Theorem 3.8. *If $\bar{x}$ is a weakly POS to the problem (**??**), then $\bar{x}$ is a weakly α-POS to (α-MOP) with parameters $(a^r_\alpha, b^l_\alpha, c^l_\alpha)$.*

Proof. Clearly, $\bar{x}$ is feasible to (α-MOP) with parameters $(a^r_\alpha, b^l_\alpha, c^l_\alpha)$. Suppose that $\bar{x}$ is not a weakly α-POS to (α-MOP). Then there is $x \in S(a,b)$ with $(a,b,c) \in (\tilde{A},\tilde{B},\tilde{C})_\alpha$ such that $c_k x < c^l_{k\alpha}\bar{x}$ for all k. Therefore,

$$c^l_{k\alpha}x \le c_k x < c^l_{k\alpha}\bar{x}$$

for all k. Also x is a feasible solution for the problem (**??**), because the feasible set of problem (**??**) is a subset of the feasible set of the problem (**??**). Therefore, $\bar{x}$ is not weakly POS of the problem (**??**). This is a contradiction. □

Now we prove the inverse of the above theorem.

Theorem 3.9. *If $\bar{x}$ is a weakly α-POS to (α-MOP) with α-optimal parameters $(\bar{a},\bar{b},\bar{c})$, then $\bar{x}$ is a weakly POS to the problem (**??**).*

Proof. Clearly, $\bar{x}$ is feasible to the problem (**??**). Suppose that $\bar{x}$ is not weakly POS to (**??**). So, there is x with $a^r_{i\alpha}x \ge b^l_{i\alpha}$ and $x \ge 0$ such that $c^l_{k\alpha}x < c^l_{k\alpha}\bar{x}$ for all k. Therefore,

$$c^l_{k\alpha}x < c^l_{k\alpha}\bar{x} \le \bar{c}_k\bar{x}$$

for all k. On the other hand, $x \in S(a^r_\alpha, b^l_\alpha)$, $(a^r_\alpha, b^l_\alpha, c^l_\alpha) \in (\tilde{A},\tilde{B},\tilde{C})_\alpha$. All these imply that $\bar{x}$ is not weakly α-POS of the problem (**??**). This is a contradiction. □

Theorem 3.10. *If $\bar{x}$ is a strictly POS to the problem (**??**), then $\bar{x}$ is a strictly α-POS to (α-MOP) with α-optimal parameters $(a^r_\alpha, b^l_\alpha, c^l_\alpha)$.*

Proof. The proof is similar to the proof of Theorems **??** and **??**. □

Theorem 3.11. *If $\bar{x}$ is a strictly α-POS to (α-MOP) with α-optimal parameters $(\bar{a},\bar{b},\bar{c})$, then $\bar{x}$ is a strictly POS to the problem (**??**).*

Proof. The proof is similar to the proof of Theorems **??** and **??**. □

Theorems **??**-**??** show that all weakly and strictly α-POSs of problem (α-MOP) can be obtained by solving the problem (**??**).

Sometimes inside the set of α-POSs of the FMOLP problem, the DM desires to has some solutions that bound the tradeoffs between the objective function values. In these solutions the DM can improve one of intended objective functions by admitting to worse some others. Such points called properly α-POSs, are defined below.

Definition 3.12. A feasible solution $\bar{x} \in S(\bar{a},\bar{b})$ is called a properly α-POS to (α-MOP) problem if it is α-POS and there is a real number $M > 0$ such that for all k and $x \in S(a,b)$ that $(a,b,c) \in (\tilde{A},\tilde{B},\tilde{C})_\alpha$ satisfying $c_k x < \bar{c}_k\bar{x}$ there exists an index j such that $c_j x > \bar{c}_j\bar{x}$ and

$\dfrac{\bar{c}_k\bar{x} - c_k x}{c_j x - \bar{c}_j\bar{x}} \leq M$. The parameters $(\bar{a}, \bar{b}, \bar{c})$ are called proper α-optimal parameters.

We denote the set of all properly α-POSs to (α-MOP) problem by $X_{\alpha PP}$. Clearly we have the the following relationships between $X_{\alpha P}, X_{\alpha SP}, X_{\alpha PP}$ and $X_{\alpha WP}$:

$$X_{\alpha SP} \subseteq X_{\alpha P} \subseteq X_{\alpha WP}, \quad X_{\alpha PP} \subseteq X_{\alpha P}.$$

In the following, we propose a theorem to determine properly α-POSs. First, Lemma **??** is presented which is used in the proof of Theorem **??**.

Lemma 3.13. [?] *Let $X \subseteq \mathbb{R}^n$ be convex and assume $f_k : X \to R$ are convex functions for $k = 1, \ldots, p$. Then $\hat{x} \in X$ is properly efficient for the MOP (**??**) if and only if $\hat{x}$ is an optimal solution of the problem*

$$\min_{x \in X} \sum_{k=1}^{p} \lambda_k f_k(x) \tag{3.5}$$

with positive weights λ_k, $k = 1, \ldots, p$.

Theorem 3.14. *Let $\bar{x}$ be a properly POS to the problem (**??**). Then $\bar{x}$ is a properly α-POS to (α-MOP) with α-optimal parameters $(a^r_\alpha, b^l_\alpha, c^l_\alpha)$.*

Proof. By Theorem **??**, $\bar{x}$ is α-POS. So, we only show that $\bar{x}$ is properly α-POS to (**??**). Since $S = S(a^r_\alpha, b^l_\alpha)$ is a convex set and the objective functions $c^l_{k\alpha}x$ for $k = 1, 2, \ldots, p$ are convex functions in the problem (**??**), according to Lemma **??** there are positive weights λ_k, $k = 1, 2, \ldots, p$ such that $\bar{x}$ is an optimal solution to the problem

$$\begin{array}{ll} \min & \displaystyle\sum_{k=1}^{p} \lambda_k c^l_{k\alpha} x \\ s.t. & x \in S. \end{array}$$

In other words,

$$\sum_{k=1}^{p} \lambda_k c^l_{k\alpha}\bar{x} \leq \sum_{k=1}^{p} \lambda_k c^l_{k\alpha} x \qquad \forall x \in S. \tag{3.6}$$

Now, suppose that $\bar{x}$ is not properly α-POS to (**??**). Then $\forall M > 0$, there exist $x \in S(a, b)$, $(a, b, c) \in (\tilde{A}, \tilde{B}, \tilde{C})_\alpha$ and $k \in \{1, 2, \ldots, p\}$ such that $c_k x < c^l_{k\alpha}\bar{x}$ and $\dfrac{c^l_{k\alpha}\bar{x} - c_k x}{c_j x - c^l_{j\alpha}\bar{x}} > M$ for all j with $c_j x > c^l_{j\alpha}\bar{x}$. In other words, $(c^l_{k\alpha}\bar{x} - c_k x) > M(c_j x - c^l_{j\alpha}\bar{x})$. Define $M := (p-1) max_{j \neq k} \dfrac{\lambda_j}{\lambda_k}$.

Therefore, we have

$$c^l_{k\alpha}\bar{x} - c_k x > (p-1)\frac{\lambda_j}{\lambda_k}(c_j x - c^l_{j\alpha}\bar{x}), \quad \forall j \neq k.$$

Multiplying each of these inequalities by $\dfrac{\lambda_k}{p-1}$ and summing them over $j,\ j \neq k$ implies

$$\sum_{\{j:j\neq k\}}^{p} \frac{\lambda_k}{p-1}(c^l_{k\alpha}\bar{x} - c_k x) > \sum_{\{j:j\neq k\}}^{p} \lambda_j(c_j x - c^l_{j\alpha}\bar{x})$$

$$\Rightarrow \ \lambda_k(c^l_{k\alpha}\bar{x} - c_k x) > \sum_{\{j:j\neq k\}}^{p} \lambda_j(c_j x - c^l_{j\alpha}\bar{x})$$

$$\Rightarrow \ \sum_{k=1}^{p} \lambda_k c^l_{k\alpha}\bar{x} > \sum_{k=1}^{p} \lambda_k c_k x. \tag{3.7}$$

On the other hand, we have $c_k x \geq c^l_{k\alpha}x$ and $\lambda_k > 0$ for all k. Therefore,

$$\sum_{k=1}^{p} \lambda_k c_k x \geq \sum_{k=1}^{p} \lambda_k c^l_{k\alpha} x. \tag{3.8}$$

Then, (**??**) and (**??**) imply that

$$\sum_{k=1}^{p} \lambda_k c^l_{k\alpha}\bar{x} > \sum_{k=1}^{p} \lambda_k c^l_{k\alpha} x. \tag{3.9}$$

Also x is a feasible solution for the problem (**??**), because the feasible set of problem (**??**) is a subset of the feasible set of problem (**??**). Inequality (**??**) contradicts (**??**). □

Theorem 3.15. *If $\bar{x}$ is a properly α-POS to (α-MOP) with α-optimal parameters $(\bar{a}, \bar{b}, \bar{c})$, then $\bar{x}$ is a properly POS to the problem (**??**).*

Proof. By Theorem **??**, $\bar{x}$ is a POS to (**??**). So, we only show that $\bar{x}$ is properly POS to (**??**). Suppose that $\bar{x}$ is not properly POS to (**??**). Then $\forall M > 0$, there exist $x \in S(a^r_\alpha, b^l_\alpha)$ and $k \in \{1, 2, \ldots, p\}$ such that $c^l_{k\alpha}x < c^l_{k\alpha}\bar{x}$ and $\dfrac{c^l_{k\alpha}\bar{x} - c^l_{k\alpha}x}{c^l_{j\alpha}x - c^l_{j\alpha}\bar{x}} > M$ for all j with $c^l_{j\alpha}x > c^l_{j\alpha}\bar{x}$. Since $\bar{c}_k x \geq c^l_{k\alpha}x$ for all $x \geq 0$ we have

$$\frac{\bar{c}_k\bar{x} - c^l_{k\alpha}x}{c^l_{j\alpha}x - c^*_j\bar{x}} \geq \frac{c^l_{k\alpha}\bar{x} - c^l_{k\alpha}x}{c^l_{j\alpha}x - c^l_{j\alpha}\bar{x}} > M$$

Therefore, according to Definition **??**, $\bar{x}$ is not α-POS to (α-MOP). This is a contradiction. □

Remark 3.16. The problem (**??**) is a usual MOLP problem. Therefore, it can be solved by existing methods to solve MOP problems such as the elastic constraint method, the ε-constraint, the weighted sum, the hybrid, the lexicographic methods, etc [**?**]. So, we have various options to obtain solutions. Since the problem (**??**) is convex, solving the weighted sum method can give us the different types of α-POSs [**?**]. Also, the weighted sum method is an easy method to use. So let us use the weighted sum method to solve the problem (**??**).

$$\begin{array}{lll} \min & W = \sum_{k=1}^{p} w_k c^l_{k\alpha} x & \\ s.t. & a^r_{i\alpha} x \geq b^l_{i\alpha} & i = 1, 2, \ldots, m \\ & x \geq 0 & \end{array} \tag{3.10}$$

where $w = (w_1, w_2, \ldots, w_p) \in \mathbb{R}^p$ is a vector of nonnegative weights. The component w_k in the problem (**??**) can be interpreted as the relative importance of the k^{th} objective function. The weights $w_k,\ k = 1, 2, \ldots, p$ in the problem (**??**) which indicate the preference degrees of different objective functions are given by the DM. If the DM can not give the weights $w_k,\ k = 1, 2, \ldots, p$ in the problem (**??**), multi attribute decision making methods such as AHP and TOPSIS can be used to determine these weights [**?**].

Theorem 3.17. [**?**] *Suppose that $\bar{x}$ be an optimal solution to the problem (**??**). Then the following results hold:*
1. If $w \in \mathbb{R}^p_{\geq}$ then $\bar{x}$ is weakly POS.
2. If $w \in \mathbb{R}^p_{>}$ then $\bar{x}$ is properly POS.
*3. If $w \in \mathbb{R}^p_{\geq}$ and $\bar{x}$ is a unique optimal solution of (**??**), then $\bar{x}$ is strictly POS.*

In the following we illustrate the proposed method by a practical example.

Example 3.18. Suppose that a company is going to produce two kinds of goods, namely $P1$ and $P2$. Let x_1 and x_2 represent the value of goods $P1$ and $P2$, respectively. The production of x_1 and x_2 require two kinds of energy resoures, namely R_1 and R_2. Let us assume that all parameters of the problem containing total necessary time, total energy resource, storage, profit of each unit, and market conditions are considered as TFNs, given in Tables 1 and 2. The aim is to minimize total pollution of production and total time of working and to maximize total profit.

Table 1. Given data

	P1	P2	supply
The requirements of R_1	(9,11,12)	(5,7,9)	(2100,2300,2400)
The requirements of R_2	(7,9,11)	(12,15,17)	(2500, 2700, 3000)

Table 2. Given data

	P1	P2
unit profit	$(45, 47, 48)$	$(49, 50, 52)$
unit necessary time	$(20, 27, 30)$	$(10, 13, 15)$
unit pollution	$(5, 8, 10)$	$(18, 21, 25)$

So, the problem can be modeled as the following FMOLP problem:

$$\begin{aligned}
\max\ & \mathrm{Z}_1(\mathrm{x}) = (45, 47, 48)\mathrm{x}_1 + (49, 50, 52)\mathrm{x}_2 \\
\min\ & \mathrm{Z}_2(\mathrm{x}) = (20, 27, 30)\mathrm{x}_1 + (10, 13, 15)\mathrm{x}_2 \\
\min\ & \mathrm{Z}_3(\mathrm{x}) = (5, 8, 10)\mathrm{x}_1 + (18, 21, 25)\mathrm{x}_2 \\
s.t.\ & \quad (9, 11, 12)x_1 + (5, 7, 9)x_2 \leq (2100, 2300, 2400) \\
& \quad (7, 9, 11)x_1 + (12, 15, 17)x_2 \leq (2500, 2700, 3000) \\
& \quad x_1 \geq 0, x_2 \geq 0.
\end{aligned}$$

For a given α the problem (**??**) is as follows:

$$\begin{aligned}
\min\ & -\mathrm{Z}_1(\mathrm{x}) = (-48 + \alpha)\mathrm{x}_1 + (-54 + 2\alpha)\mathrm{x}_2 \\
\min\ & \mathrm{Z}_2(\mathrm{x}) = (20 + 7\alpha)\mathrm{x}_1 + (10 + 3\alpha)\mathrm{x}_2 \\
\min\ & \mathrm{Z}_3(\mathrm{x}) = (5 + 3\alpha)\mathrm{x}_1 + (18 + 3\mathrm{r})\mathrm{x}_2 \\
s.t.\ & \quad (9 + 2\alpha)x_1 + (5 + 2\alpha)x_2 \leq (2400 - 100\alpha) \\
& \quad (7 + 2\alpha)x_1 + (12 + 3\alpha)x_2 \leq (3000 - 300\alpha) \\
& \quad x_1 \geq 0, x_2 \geq 0.
\end{aligned}$$

Now assume that $\alpha = 0.5$. Then, we have

$$\begin{aligned}
\min\ & -\mathrm{Z}_1(\mathrm{x}) = -75.5\mathrm{x}_1 - 53\mathrm{x}_2 \\
\min\ & \mathrm{Z}_2(\mathrm{x}) = 23.5\mathrm{x}_1 + 11.5\mathrm{x}_2 \\
\min\ & \mathrm{Z}_3(\mathrm{x}) = 6.5\mathrm{x}_1 + 19.5\mathrm{x}_2 \\
s.t.\ & \quad 10x_1 + 6x_2 \leq 2350 \\
& \quad 8x_1 + 13.5x_2 \leq 3150 \\
& \quad x_1 \geq 0, x_2 \geq 0.
\end{aligned}$$

Assume that the importance of objective functions Z_2, Z_3 are the same and the half of the first objective function. So we set $\lambda_2 = \lambda_3 = 0.25$ and $\lambda_1 = 0.5$, then by using the weighted sum method and applying LINGO software, the optimal solution $\bar{x} = (\bar{x}_1, \bar{x}_2) = (152.1739, 138.0435)$ with objective function values $\mathrm{Z}_1 = 18874.4567$, $\mathrm{Z}_2 = 5163.5869$ and $Z_3 =$

3680.9786 is obtained. By considering Theorem **??** we can conclude that $\bar{x}$ is properly α-POS.

4. Discussion

For the fuzzy multi objective linear programming (FMOLP) problem the notions of strict, weak and properly Pareto optimal solutions (POSs) of multi objective programming (MOP) problems was extended, then a method was proposed to get on these solutions. In the proposed method the FMOLP problem is converted to an MOLP problem which depends on α-cuts of fuzzy numbers. Before solving the problem (**??**), the value of α is specified by the decision maker (DM). Therefore, two approach may used: 1) one can get α from DM, solve the problem (**??**), and give the solution to DM. If DM is not satisfied with the solution, change α, and repeat the process. 2) solve the problem (**??**) for different values of α, give the solutions to DM, to select his desirable solution. Note that by considering a level α, we try to obtain solutions whose membership degrees are at least α. This means that the minimum satisfaction degree of DM from the solution is α. So, $\alpha = 0$ is not so desirable. On the other hand, using $\alpha = 1$ (which means the most possible satisfaction degree) has two shortcoming: 1) may cause infeasibility; 2) the fuzzy nature of the problem is lost. Therefore, the values $\alpha \in (0, 1)$ are recommended, of course, the values closer to 1 are preferred.

Conclution

The current research focused on fuzzy multi objective linear programming (FMOLP) problem where the parameters in both constraints and objective functions are fuzzy numbers. On the basis of α-cut of fuzzy parameters, some new solutions to the FMOLP problem have been proposed namely weakly, properly and strictly α-Pareto optimal solutions (α-POSs). In the set of POSs, DM is often interested in the properly POSs that have finite tradeoffs between objective functions. Therefore, in this article, the notions of properly POSs of multi objective problems have been extended to the FMOLP problem. To the best knowledge of the authors, this is the first paper which defines different types of α-POSs to the FMOLP problem and obtains them by maintaining the fuzzy nature of the problem. In this method, problem's model (multi objective) does not change. So, we can use different methods to solve multi objective programming problems such as the weighted sum, the ε-constraint, the elastic constraint, the hybrid, the lexicographic methods, etc. Since the related non fuzzy problem is convex, all POSs can be obtained by solving the weighted sum method. On the other hand, the weighted sum method is an easy method to solve multi objective programming problems. So, we used the weighted sum method. The main advantage of this method is that different types of α-POSs (especially, properly α-POS) can be obtained to the FMOLP problem by maintaining the fuzzy nature of the problem. In this method, no ranking function is used to convert the FMOLP problem to an MOLP one. Note that, when a defuzzification method is used, the fuzzy aspect of the problem is actually lost, which is not desirable. Because of efficiency and satisfactory of the obtained new solutions for DM, this research undoubtedly can be useful to solve the FMOLP problems.

References

[1] Amid A, Ghodsypour SH, Christopher O, *A weighted additive fuzzy multi objective model for the supplier selection problem under price breaks in a supply Chain*,International Journal of Production Economics, Vol. 121, 323-332, (2009).

[2] Bharati S.K., Singh S. R., *Solution of multi objective linear programming problem in interval-valued intuitionistic fuzzy environment*, Soft Computing, Vol. 23(1), 77-84, (2019).

[3] Cava WL, Danai K, Spector L, Fleming P, A. Wright and M. Lacknera, *Automatic identification of wind turbine models using evolutionary multi objective optimization*, Renewable Energy, Vol. 87, 892-902, (2016).

[4] Charitopoulos VM, and Dua V, *A unified framework for model-based multi-objective linear process and energy optimisation under uncertainty*, Applied Energy, Vol. 186, 539-548, (2017).

[5] Chenga H, Huanga W, Zhoub Q, Caic J, *Solving fuzzy multi-objective linear programming problems using deviation degree measures and weighted max–min method*, Applied Mathematical Modelling, Vol. 37, 6855-6869 , (2013).

[6] Chinneck JW, Ramadan K, *Linear programming with interval coeffietiont*, Journal of the operational research society 51, 209-220, (2002).

[7] Ehrgott M, *Multicriteria optimization,* Springer Berlin, (2005).

[8] Gleeson MP, Montanari D, *Optimization of Biological and Physical Properties in Drug Discovery*, Reference Module in Chemistry, Molecular Sciences and Chemical Engineering, 64-93, (2017).

[9] Hwang CL, Yoon K, *Multiple attribute decision making: methods and applications, a state of the art survey*, Springer-Verlag, Berlin, (1981)

[10] Iqbal N, Naeema M, Anpalagan A, Qadri NN, Imran M, *Multi-objective optimization in sensor networks: Optimization classification, applications and solution approaches*, Computer Networks, Vol. 99, 134-161, (2016).

[11] Orlovski S, *Multi-objective programming problems with fuzzy parameters*, International institute for applied systems analysis, Austria, (1983).

[12] Rouhbakhsh FF, Hassanpour H, Effati S, *Some new solution concepts in generalized fuzzy multi objective optimization problems*, Soft Computing, 1-10, (2017).

[13] Sakawa M, *Fuzzy sets and Interactive Multi objective Optimization*, Plenum Press, New York, (1993).

[14] Singh M and Dhillon JS,*Multi objective thermal power dispatch using opposition-based greedy heuristic search*, International Journal of Electrical Power and Energy Systems, Vol. 82, 339-353, (2016).

[15] V. Singh and Shiv P. Y, *Modeling and optimization of multi-objective programming problems in intuitionistic fuzzy environment*, Optimistic, pessimistic and mixed approaches, Expert Syst. 143–157, (2018).

[16] Stanojević, B., and Stanojević, M, *On Fuzzy Solutions to a Class of Fuzzy Multi-objective Linear Optimization Problems*, In Advances in Operational Research in the Balkans, 63–76, (2020).

[17] Luhandjula MK, Rangoaga MJ, *An approach for solving a fuzzy multi objective programming problem,* European Journal of Operational Research 232, 249-255, (2014).

[18] Pareto V, *Manual di economie politique (in French)*, F. Rouge, Lausanne, (1896).

[19] Tanaka H, Ichihashi H, Asai K, *Formulation of fuzzy linear programming problem by fuzzy objective function*, Journal of the Operational Research Society, (1984).

[20] Thapar A, Pandey D, Gaur SK, *Satisficing solutions of multi-objective fuzzy optimization problems using genetic algorithm,* Applied Soft Computing, Vol. 12, 2178-2187, (2012).

[21] Wu FJ, Lu J, Zhang GQ, *A new approximate algorithm for solving multiple objective linear programming problems with fuzzy parameters,* Applied Mathematics and Computation, Vol. 174, 524-544, (2006).