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Some results on Hermite-Hadamard type inequalities for fractional integrals

Hasan Barsam[1] and Sayyed Mehrab Ramezani[2]
[1] Department of Mathematics, Faculty of Science, University of Jiroft, P.O. Box 78671-61167, Jiroft, Iran
[2] Faculty of Technology and Mining, Yasouj University, Choram 75761-59836, Iran

Abstract. In this paper, we establish Hermite-Hadamard type inequalities for uniformly p-convex functions. Also, a new fractional Hermite-Hadamard type inequality for convex functions is obtained by using only the left Riemann-Liouville fractional integral. Finally some estimation of left fractional integration studies for Hermite-Hadamard type inequalities.

Keywords: Hermite-Hadamard inequalities, Uniformly p-convex functions, Hölder inequality.

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1. Introduction

In the field of mathematics inequalities, Hermite-Hadamard's inequality has been the subject of much attention by many mathematicians because of its usefulness. Many researchers have extended the Hermite-Hadamard's inequality, to different forms, using the classical convex function. For further details involving Hermite-Hadamard's type inequality on a different concept of convex function and generalizations, the interested reader is referred to [1, 3, 4, 11].

[1]Corresponding author: hasanbarsam@ujiroft.ac.ir
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The theory of convex functions has been widely studied and applied to various fields of science. Due to its close relation to the theory of inequalities, a rich literature on inequalities can be found in the study of convex functions [9, 12].

Many important integral inequalities are based on a convexity assumption of a certain function. Furthermore, theory of inequality is one of the most important application fields of convex and abstract analysis, while the common usage within inequalities in convex analysis is Hermite-Hadamard inequality.

These inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature. Hermite-Hadamard inequality state that if $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$,then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

We consider the basic concepts and results, which are needed to obtain our main results.

Definition 1.1. ([2]) A mapping $f : \mathbb{R} \to \mathbb{R}$ is called uniformly p-convex function with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if ψ is increasing, ψ vanishes only at 0, and

$$f(tx + (1-t)y) + t(1-t)\psi(|x-y|) \leq f(x) + f(y), \tag{1.1}$$

for each $x, y \in [0, +\infty)$ and $t \in [0, 1]$.

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 1.2. The left-sided and right-sided Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$, for $f \in L[a, b]$ of order $\alpha > 0$ with $b \geq a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \text{ with } x > a,$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \text{ with } x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and its definition is

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt.$$

It is to be noted that $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Also, recall that the Beta function which is defined by

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}:=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x>0,\ y>0.$$

For more details see ([7], [11]).

In [10], M. Z. Sarikaya et al. presented the Hermite-Hadamard's inequalities for fractional integrals as follows.

Theorem 1.3. *([10]) Let $f: I \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals holds.*

$$f(\frac{a+b}{2}) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)] \leq \frac{f(a)+f(b)}{2}.$$

In [5],[6],[8] the authors used the following equality to obtain some inequalities with respect to Hermite-Hadamard inequality.

Lemma 1.4. *([8]) Let $f: I^o \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o and let $a,b \in I^o$ with $a<b$ and $f' \in L^1[a,b]$, then*

$$\frac{\alpha f(a)+f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{a^+}f(b) = \frac{b-a}{\alpha+1}\int_0^1[1-(\alpha+1)t^\alpha]f'(ta+(1-t)b)dt.$$

Lemma 1.5. *([6]) Let $f: I^o \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I^o and let $a,b \in I^o$ with $a<b$ and $f'' \in L^1[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

This paper aims to show that Hermite-Hadamard type inequalities are established for uniformly p-convex functions. Moreover, a new fractional Hermite-Hadamard type inequality for convex functions is deduced by using only the left Riemann-Liouville fractional integral. Finally we obtain some estimation of left fractional integration with respect Hermite-Hadamard type inequalities.

2. Hermite-Hadamard's inequality for uniformly p-convex functions

In this section we give a new result of the Hermite-Hadamard inequalities for uniformly p-convex functions.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a uniformly p-convex function with modulus ψ. Then for each $\alpha > 0$ the following inequalities for fractional integrals hold:*

$$\begin{aligned}
&f(\frac{a+b}{2}) + \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^{\alpha}} J^{\alpha}_{(a-b)^+}\psi(|a-b|) \\
&\le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)] \\
&\le 2(f(a)+f(b)) - 2\alpha\beta(\alpha+1,2)\psi(|a-b|).
\end{aligned}$$

Proof. In Equation (1.1), set $t := \frac{1}{2}$, then

$$f(\frac{x+y}{2}) + \frac{1}{4}\psi(|x-y|) \le f(x) + f(y). \tag{2.1}$$

Now, taking $x := ta + (1-t)b$ and $y := (1-t)a + tb$ in Equation (2.1) and multiplying both sides of this equation by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over [0, 1], we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f(\frac{a+b}{2})dt + \frac{1}{4}\int_0^1 t^{\alpha-1}\psi(|(2t-1)(a-b)|)dt \\
&\le \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt,
\end{aligned}$$

by making the change of variables $ta + (1-t)b := x$, $(1-t)a + tb := y$ and $(2t-1)(a-b) := z$, then

$$\begin{aligned}
&\frac{f(\frac{a+b}{2})}{\alpha} + \frac{1}{4}\int_{b-a}^{a-b} (\frac{b-a-z}{2(b-a)})^{\alpha-1}\psi(|z|)\frac{dz}{2(a-b)} \le \\
&\int_b^a (\frac{b-x}{b-a})^{\alpha-1} f(x)\frac{dx}{a-b} + \int_a^b (\frac{y-a}{b-a})^{\alpha-1} f(y)\frac{dy}{b-a},
\end{aligned}$$

therefore

$$\frac{f(\frac{a+b}{2})}{\alpha} + \frac{\Gamma(\alpha)}{2^{\alpha+2}(b-a)^{\alpha}} J^{\alpha}_{(a-b)^+}\psi(|a-b|) \le \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)].$$

Conversely, since f is uniformly p-convex we have

$$f(tx + (1-t)y) + t(1-t)\psi(|x-y|) \le f(x) + f(y), \tag{2.2}$$

now, replace x by y then

$$f(ty + (1-t)x) + t(1-t)\psi(|x-y|) \le f(y) + f(x), \tag{2.3}$$

by adding Equation (2.2) to Equation (2.3), we arrive at the following equation:

$$f(tx + (1-t)y) + f((1-t)x + ty) + 2t(1-t)\psi(|x-y|) \le 2f(x) + 2f(y). \tag{2.4}$$

Put $x := a$ and $y := b$ in Equation (2.4) and also multiplying both sides of this equation by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over [0, 1], we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt \\ &+ \int_0^1 2t^{\alpha}(1-t)\psi(|a-b|)dt \\ &\leq \int_0^1 2t^{\alpha-1} f(a)dt + \int_0^1 2t^{\alpha-1} f(b)dt,\end{aligned}$$

so

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] \\ &\leq \frac{2f(a)+2f(b)}{\alpha} - 2\beta(\alpha+1,2)\psi(|a-b|),\end{aligned}$$

as asserted. □

3. Fractional Hermite-Hadamard type inequality for convex functions

In this section we will prove the identity related to Lemma 1.5 and deduce a new fractional Hermite-Hadamard type inequality for convex functions by using only the left Riemann-Liouville fractional integral.

Lemma 3.1. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I^o and let $a, b \in I^o$ with $a < b$ and $f'' \in L^1[a, b]$, then*

$$\begin{aligned}&\frac{\alpha f(a) + f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{a^+}^{\alpha} f(b) \\ &\qquad = \frac{(b-a)^2}{\alpha+1}\int_0^1 [t - t^{\alpha+1}] f''(ta+(1-t)b)dt. \qquad (3.1)\end{aligned}$$

Proof. By applying the integration by parts on the right hand side of Equation (3.1), we have

$$\begin{aligned}&\int_0^1 [t - t^{\alpha+1}] f''(ta+(1-t)b)dt \\ &= \frac{t - t^{\alpha+1}}{a-b} f'(ta+(1-t)b)|_0^1 - \frac{1}{a-b}\int_0^1 [1-(1+\alpha)t^{\alpha}] f'(ta+(1-t)b)dt \\ &= \frac{1-(1+\alpha)t^{\alpha}}{(b-a)^2} f(ta+(1-t)b)|_0^1 - \frac{\alpha(1+\alpha)}{(b-a)^2}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt \\ &= \frac{\alpha f(a)+f(b)}{(b-a)^2} - \frac{\alpha(1+\alpha)}{(b-a)^2}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt.\end{aligned}$$

□

Corollary 3.2. *Let $\alpha = 1$ in Lemma 3.1, then the inequality in Lemma 1.5 is obtained.*

Theorem 3.3. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o and let $a, b \in I^o$ with $a < b$. If the function $|f''|$ is a convex function on $[a, b]$, then*

$$\begin{aligned} I(f) &= \left|\frac{\alpha f(a) + f(b)}{\alpha + 1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a^+}^\alpha f(b)\right| \\ &\leq \frac{(b-a)^2}{\alpha+1}\left[\frac{\alpha}{3(\alpha+3)}\right][|f''(a)| + |f''(b)|]. \end{aligned}$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned} I(f) &= \frac{(b-a)^2}{\alpha+1}\int_0^1 |t - t^{\alpha+1}||f''(ta + (1-t)b)|dt \\ &\leq \frac{(b-a)^2}{\alpha+1}\int_0^1 |t - t^{\alpha+1}||t||f''(a)|dt + \frac{(b-a)^2}{\alpha+1}\int_0^1 |t - t^{\alpha+1}||1-t||f''(b)|dt \\ &\leq \frac{(b-a)^2}{\alpha+1}|f''(a)|\int_0^1 t^2(1-t^\alpha)dt + \frac{(b-a)^2}{\alpha+1}|f''(b)|\int_0^1 |t - t^{1+\alpha}|(1-t)dt \\ &\leq \frac{(b-a)^2}{\alpha+1}|f''(a)|\left[\int_0^1 t^2 dt - \int_0^1 t^{\alpha+2}dt\right] + \frac{(b-a)^2}{\alpha+1}|f''(b)|\left[\int_0^1 t^2 dt - \int_0^1 t^{\alpha+2}dt\right] \\ &\leq \frac{(b-a)^2}{\alpha+1}|f''(a)|\left[\frac{1}{3} - \frac{1}{\alpha+3}\right] + \frac{(b-a)^2}{\alpha+1}|f''(b)|\left[\frac{1}{3} - \frac{1}{\alpha+3}\right] \\ &\leq \frac{(b-a)^2}{\alpha+1}\left[\frac{\alpha|f''(a)| + \alpha|f''(b)|}{3(\alpha+3)}\right]. \end{aligned}$$

□

Theorem 3.4. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I^o and let $a, b \in I^o$ with $a < b$. If the function $|f''|^q$ is a convex function on $[a, b]$ for some $q > 1$, then the following inequality holds:*

$$\begin{aligned} I(f) &= \left|\frac{\alpha f(a) + f(b)}{\alpha + 1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a^+}^\alpha f(b)\right| \\ &\leq \frac{(b-a)^2}{\alpha(\alpha+1)}\left(\beta\left(\frac{p+1}{\alpha}, p+1\right)\right)^{\frac{1}{p}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{\frac{1}{q}}, \end{aligned}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 3.1, power mean inequality and the convexity of $|f''|^q$, we have

$$\begin{aligned} I(f) &= \frac{(b-a)^2}{\alpha+1}\int_0^1 |t-t^{\alpha+1}||f''(ta+(1-t)b)|dt \\ &\leq \frac{(b-a)^2}{\alpha+1}(\int_0^1 |t|^p|1-t^{\alpha}|^p dt)^{\frac{1}{p}}(\int_0^1 |f''(ta+(1-t)b)|^q dt)^{\frac{1}{q}} \\ &\leq \frac{(b-a)^2}{\alpha+1}(\int_0^1 t^p(1-t^{\alpha})^p dt)^{\frac{1}{p}}(\int_0^1 t|f''(a)|^q+(1-t)|f''(b)|^q dt)^{\frac{1}{q}} \\ &\leq \frac{(b-a)^2}{\alpha(\alpha+1)}(\beta(\frac{p+1}{\alpha},p+1))^{\frac{1}{p}}[\frac{|f''(a)|^q+|f''(b)|^q}{2}]^{\frac{1}{q}}. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^1 t^p(1-t^{\alpha})^p dt &= \int_0^1 u^{\frac{p}{\alpha}}(1-u)^p\frac{\sqrt[\alpha]{u}}{\alpha u}du \\ &= \frac{1}{\alpha}\int_0^1 u^{\frac{p}{\alpha}+\frac{1}{\alpha}}u^{-1}(1-u)^p du \\ &= \frac{1}{\alpha}\int_0^1 u^{\frac{p-\alpha+1}{\alpha}}(1-u)^p du \\ &= \frac{1}{\alpha}\beta(\frac{p+1}{\alpha},p+1). \end{aligned}$$

□

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