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Non-trivial solutions for a discrete nonlinear boundary value problem with ϕ_c-Laplacian

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Abstract. In this paper, we prove the existence of at least one non-trivial solution for a discrete nonlinear boundary value problem with ϕ_c-Laplacian. The approach is based on variational methods.

Keywords: ϕ_c-Laplacian, variational methods, critical point theory.

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1. Introduction

It is well known that in fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modelling of important questions leads naturally to the consideration of nonlinear difference equations. For this reason, in recent years, many authors have widely developed various methods and techniques, such as fixed point theorems, upper and lower solutions, and Brouwer degree, to study discrete problems (see, e.g., [4, 5, 10, 11, 12, 16, 18] and references therein). Recently, also the critical point theory has aroused the

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attention of many authors in the study of these problems (see, e.g., [1, 2, 3, 6, 8, 9, 13, 14, 19]).

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, and $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$.

Consider the following boundary value problem of the second order nonlinear difference equation

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) = \lambda f(k, u_k), & k \in \mathbb{Z}(1, T), \\ u_0 = u_{T+1} = 0, \end{cases} \tag{1.1}$$

where T is a given positive integer, Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$, $\Delta^2 u_k = \Delta(\Delta u_k)$, ϕ_c is a special ϕ-Laplacian operator [15] defined by $\phi_c(s) = \frac{s}{\sqrt{1+s^2}}$, and $f(k, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for each $k \in \mathbb{Z}(1, T)$.

In the present paper, we obtain the existence of at least one solution for problem (1.1). It is worth noticing that, usually, to obtain the existence of one solution, asymptotic conditions both at zero and at infinity on the nonlinear term are requested, while, here, it is assumed only a unique algebraic condition (see (3.8) in Corollary 3.6). Our approach is variational and the main tool is a local minimum theorem established in [7].

2. Preliminaries

Our main tool is Theorem 2.1, a consequence of the existence result of a local minimum theorem [7, Theorem 3.1] which is inspired by the Ricceri Variational Principle [17].

For a given nonempty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(r_1, r_2)} \frac{\sup_{u \in \Phi^{-1}(r_1, r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_2(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r_1]} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}, r_1 < r_2$.

Theorem 2.1. [7, Theorem 5.1] *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}, r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$*, for each* $\lambda \in \left]\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$ *there is* $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ *such that* $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(r_1, r_2)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$.

Remark 2.2. It is worth noticing that whenever X is a finite dimensional Banach space, a careful reading of the proof of Theorem 2.1 shows that regarding to the regularity of the derivative of Φ and Ψ, it is enough to require only that Φ' and Ψ' are two continuous functionals on X^*.

Now we consider the T-dimensional Banach space

$$S := \{u : \mathbb{Z}(0, T+1) \to \mathbb{R} \,:\, u_0 = u_{T+1} = 0\}$$

endowed with the norm

$$\|u\| := \left(\sum_{k=1}^{T} u_k^2\right)^{\frac{1}{2}}.$$

In the sequel, we will use the inequality

$$\max_{k\in\mathbb{Z}(1,T)}\{|u_k|\} \leq \frac{\sqrt{T+1}}{2}\left(\sum_{k=0}^{T}(\Delta u_k)^2\right)^{\frac{1}{2}} \tag{2.1}$$

for every $u \in S$, where it immediately follows, for instance, from [13, Lemma 2.2]. Put

$$\Phi(u) := \sum_{k=0}^{T}(\sqrt{1+(\Delta u_k)^2} - 1), \qquad \Psi(u) := \sum_{k=1}^{T} F(k, u_k),$$

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \tag{2.2}$$

for every $u \in S$, where $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t) \in \mathbb{Z}(1,T) \times \mathbb{R}$. Standard argument shows that $I_\lambda \in C^1(S, \mathbb{R})$ and the critical points of I_λ are exactly the solutions of the problem(1.1). Finally, we recall the following strong maximum principle in order to get positive solutions to problem (1.1).

Theorem 2.3. [19, Theorem 2.1] *Assume* $u \in S$ *such that either*

$$u_k > 0 \quad \textit{or} \quad -\Delta(\phi_c(\Delta u_{k-1})) \geq 0. \tag{2.3}$$

for all $k \in \mathbb{Z}(1,T)$*. Then, either* $u > 0$ *in* $\mathbb{Z}(1,T)$ *or* $u \equiv 0$.

3. Main result

In this section, we formulate our main results as follow. For every two non-negative costants η and γ with

$$\sqrt{1+\frac{4\eta^2}{T+1}} \neq 2\sqrt{1+\gamma^2} - 1$$

we set

$$a_\eta(\gamma) := \frac{\sum_{k=1}^{T} \max_{|t|\leq \eta} F(k,t) - \sum_{k=1}^{T} F(k,\gamma)}{\sqrt{1+\frac{4\eta^2}{T+1}} - 2\sqrt{1+\gamma^2}+1}$$

We now present our main result as follows.

Theorem 3.1. *Assume that there exist three real constants η_1, η_2 and γ with*

$$\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\eta_1^2}{T+1}} < \sqrt{1+\gamma^2} < \frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\eta_2^2}{T+1}} \tag{3.1}$$

such that

$$a_{\eta_2}(\gamma) < a_{\eta_1}(\gamma). \tag{3.2}$$

Then, for each $\lambda \in \left]\frac{1}{a_{\eta_1}(\gamma)}, \frac{1}{a_{\eta_2}(\gamma)}\right[$, problem (1.1) *admits at least one non-trivial solution $\bar{u} \in S$, such that*

$$-1+\sqrt{1+\frac{4\eta_1^2}{T+1}} < \sum_{k=0}^{T}(\sqrt{1+(\Delta \bar{u}_k)^2}-1) < -1+\sqrt{1+\frac{4\eta_2^2}{T+1}}.$$

Proof. Our aim is to apply Theorem 2.1 to problem (1.1). Take the real Banach space S as defined in Section 2, and put Φ, Ψ, I_λ as in (2.2). An easy computation ensures the regularity assumptions required on Φ and Ψ. Note that the critical points of I_λ are exactly the solutions of problem (1.1). Put

$$r_1 = -1+\sqrt{1+\frac{4\eta_1^2}{T+1}} \quad \text{and} \quad r_2 = -1+\sqrt{1+\frac{4\eta_2^2}{T+1}}, \tag{3.3}$$

and pick $w \in S$, defined as follows:

$$w_k := \begin{cases} \gamma, & \text{if } k \in \mathbb{Z}(1,T), \\ 0, & \text{if } k=0, k=T+1. \end{cases} \tag{3.4}$$

So, from (2.2), we have

$$\Phi(w) = 2(\sqrt{1+\gamma^2}-1). \tag{3.5}$$

From the condition (3.1), we obtain $r_1 < \Phi(w) < r_2$. For all $u \in S$ such that $\Phi(u) < r_2$, let $v_k = \sqrt{1+(\Delta u_k)^2}-1$ for $k \in \mathbb{Z}(0,T)$, then $\sum_{k=0}^{T} v_k < r_2$ and

$$\sum_{k=0}^{T}(\Delta u_k)^2 = \sum_{k=0}^{T}(v_k^2+2v_k) \leq \left(\sum_{k=0}^{T} v_k\right)^2 + 2\sum_{k=0}^{T} v_k < r_2^2+2r_2 = \frac{4\eta_2^2}{T+1},$$

and taking (2.1) into account, one has $|u_k| < \eta_2$ for all $k \in \mathbb{Z}(1,T)$. Therefore, it follows

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r_2)} \sum_{k=1}^{T} F(k,u_k) \leq \sum_{k=1}^{T} \max_{|t|\leq\eta_2} F(k,t).$$

Arguing as before, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u) \leq \sum_{k=1}^{T} \max_{|t|\leq\eta_1} F(k,t).$$

Therefore, one has

$$\begin{aligned}
\beta(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\
&\leq \frac{\sum_{k=1}^{T} \max_{|t|\leq\eta_2} F(k,t) - \sum_{k=1}^{T} F(k,\gamma)}{-1+\sqrt{1+\frac{4\eta_2^2}{T+1}} - \left(2\sqrt{1+\gamma^2}-2\right)} \\
&= \frac{\sum_{k=1}^{T} \max_{|t|\leq\eta_2} F(k,t) - \sum_{k=1}^{T} F(k,\gamma)}{\sqrt{1+\frac{4\eta_2^2}{T+1}} - 2\sqrt{1+\gamma^2}+1} \\
&= a_{\eta_2}(\gamma).
\end{aligned}$$

On the other hand, arguing as before, one has

$$\begin{aligned}
\rho_2(r_1,r_2) &\geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u)}{\Phi(w) - r_1} \\
&\geq \frac{\sum_{k=1}^{T} F(k,\gamma) - \sum_{k=1}^{T} \max_{|t|\leq\eta_1} F(k,t)}{2\sqrt{1+\gamma^2}-2-\left(-1+\sqrt{1+\frac{4\eta_1^2}{T+1}}\right)} \\
&= \frac{\sum_{k=1}^{T} F(k,\gamma) - \sum_{k=1}^{T} \max_{|t|\leq\eta_1} F(k,t)}{2\sqrt{1+\gamma^2} - \sqrt{1+\frac{4\eta_1^2}{T+1}} - 1} \\
&= a_{\eta_1}(\gamma).
\end{aligned}$$

Hence, from assumption (3.2), one has $\beta(r_1,r_2) < \rho_2(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in \left]\frac{1}{a_{\eta_1}(\gamma)}, \frac{1}{a_{\eta_2}(\gamma)}\right[$, the functional I_λ admits at least one critical point $\bar{u}$ such that $r_1 < \Phi(\bar{u}) < r_2$, that is

$$-1+\sqrt{1+\frac{4\eta_1^2}{T+1}} < \sum_{k=0}^{T}(\sqrt{1+(\Delta\bar{u}_k)^2}-1) < -1+\sqrt{1+\frac{4\eta_2^2}{T+1}},$$

and the conclusion is achieved. □

Now, we point out an immediate consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist two positive constants η and γ, with*

$$\sqrt{1+\gamma^2} < \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\eta^2}{T+1}}$$

such that

$$\frac{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t)}{\sqrt{1+\frac{4\eta^2}{T+1}}-1} < \frac{\sum_{k=1}^{T} F(k,\gamma)}{2\sqrt{1+\gamma^2}-2}. \tag{3.6}$$

Then, for each

$$\lambda \in \left] \frac{2\sqrt{1+\gamma^2}-2}{\sum_{k=1}^{T} F(k,\gamma)}, \frac{\sqrt{1+\frac{4\eta^2}{T+1}}-1}{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t)} \right[,$$

problem (1.1) admits at least one non-trivial solution $\bar{u} \in S$ such that $|\bar{u}_k| < \eta$ for all $k \in \mathbb{Z}(1,T)$.

Proof. Our aim is to employ Theorem 3.1, by choosing $\eta_1 := 0$ and $\eta_2 := \eta$. Therefore, owing to the inequality (3.6), we see that

$$\begin{aligned} a_\eta(\gamma) &= \frac{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t) - \sum_{k=1}^{T} F(k,\gamma)}{\sqrt{1+\frac{4\eta^2}{T+1}} - 2\sqrt{1+\gamma^2}+1} \\ &< \frac{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t)\left(1-\frac{2\sqrt{1+\gamma^2}-2}{\sqrt{1+\frac{4\eta^2}{T+1}}-1}\right)}{\sqrt{1+\frac{4\eta^2}{T+1}} - 2\sqrt{1+\gamma^2}+1} \\ &= \frac{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t)}{\sqrt{1+\frac{4\eta^2}{T+1}}-1}. \end{aligned}$$

On the other hand, one has

$$a_0(\gamma) = \frac{\sum_{k=1}^{T} F(k,\gamma)}{2\sqrt{1+\gamma^2}-2}.$$

Now, inequality (3.6) yields $a_\eta(\gamma) < a_0(\gamma)$. Hence, taking (2.1) into account, Theorem 3.1 ensures the conclusion. □

Theorem 3.3. *Assume that*

$$\lim_{\xi\to 0^+} \frac{\sum_{k=1}^{T} F(k,\xi)}{\sqrt{1+\xi^2}-1} = +\infty. \tag{3.7}$$

Moreovere, for each $\eta > 0$, set

$$\lambda_\eta^* := \frac{\sqrt{1+\frac{4\eta^2}{T+1}}-1}{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t)}.$$

Then, for every $\lambda \in]0, \lambda_\eta^*[$, *problem* (1.1) *admits at least one non-trivial solution* $\bar{u} \in S$ *such that* $|\bar{u}_k| < \eta$ *for all* $k \in \mathbb{Z}(1, T)$.

Proof. Fix $\eta > 0$ and $\lambda \in]0, \lambda_\eta^*[$. From (3.7), there exist a positive constant γ with

$$\sqrt{1+\gamma^2} < \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\eta^2}{T+1}}$$

such that

$$\frac{2\sqrt{1+\gamma^2} - 2}{\sum_{k=1}^{T} F(k,\gamma)} < \lambda < \frac{\sqrt{1+\frac{4\eta^2}{T+1}} - 1}{\sum_{k=1}^{T} \max_{|t|\le\eta} F(k,t)}.$$

Hence, owing to Theorem 3.2, for every $\lambda \in]0, \lambda_\eta^*[$ problem (1.1) admits at least one non-trivial solution $\bar{u} \in S$, such that $|\bar{u}_k| < \eta$ for all $k \in \mathbb{Z}(1, T)$. The proof is complete. □

Remark 3.4. We claim that under the above assumptions, the mapping $\lambda \mapsto I_\lambda(\bar{u})$ is negative and strictly decreasing in $]0, \lambda_\eta^*[$. Indeed, the restriction of the functional I_λ to $\Phi^{-1}(0, r_2)$, where $r_2 = -1 + \sqrt{1 + \frac{4\eta_2^2}{T+1}}$, admits a global minimum, which is a critical point (local minimum) of I_λ in S. Moreover, since $w \in \Phi^{-1}(0, r_2)$ and

$$\frac{\Phi(w)}{\Psi(w)} = \frac{2\sqrt{1+\gamma^2} - 2}{\sum_{k=1}^{T} F(k,\gamma)} < \lambda,$$

we have

$$I_\lambda(\bar{u}) \le I_\lambda(w) = \Phi(w) - \lambda\Psi(w) < 0.$$

Next, we observe that

$$I_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right),$$

for every $u \in S$ and fix $0 < \lambda_1 < \lambda_2 < \lambda_\eta^*$. Set

$$m_{\lambda_1} := \left(\frac{\Phi(u_1)}{\lambda_1} - \Psi(u_1)\right) = \inf_{u \in \Phi^{-1}(0,r_2)} \left(\frac{\Phi(u)}{\lambda_1} - \Psi(u)\right),$$

$$m_{\lambda_2} := \left(\frac{\Phi(u_2)}{\lambda_2} - \Psi(u_2)\right) = \inf_{u \in \Phi^{-1}(0,r_2)} \left(\frac{\Phi(u)}{\lambda_2} - \Psi(u)\right).$$

Clearly, as claimed before, $m_{\lambda_i} < 0$ (for$i = 1, 2$), and $m_{\lambda_1} \le m_{\lambda_2}$ thanks to $\lambda_1 < \lambda_2$. Then the mapping $\lambda \mapsto I_\lambda(\bar{u})$ is strictly decreasing in $]0, \lambda_\eta^*[$ owing to

$$I_{\lambda_2}(\bar{u}_2) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(\bar{u}_1).$$

This concludes the proof of our claim.

Remark 3.5. In other word, Theorem 3.3 ensures that if the asymptotic condition at zero (3.7) is verified then, for every parameter λ belonging to the real interval $]0, \lambda^*[$, where

$$\lambda^* := \sup_{\eta>0} \frac{\sqrt{1+\frac{4\eta^2}{T+1}}-1}{\sum_{k=1}^T \max_{|t|\leq\eta} F(k,t)},$$

problem (1.1) admits at least one non-trivial solution $\bar{u} \in S$.

Corollary 3.6. *Let $\alpha : \mathbb{Z}(1,T) \to \mathbb{R}$ be a non-negative and non-zero function and let $g : [0,+\infty) \to \mathbb{R}$ be a continuous function such that $g(0) = 0$. Assume that there exist two positive costants γ, η, with*

$$\sqrt{1+\gamma^2} < \frac{1}{2} + \sqrt{\frac{1}{4}+\frac{\eta^2}{T+1}},$$

for which

$$\frac{\max_{0\leq t\leq\eta} G(t)}{\sqrt{1+\frac{4\eta^2}{T+1}}-1} < \frac{G(\gamma)}{2\sqrt{1+\gamma^2}-2}, \tag{3.8}$$

where $G(t) := \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$. Then, for each

$$\lambda \in \frac{1}{\sum_{k=1}^T a_k}\left]\frac{2\sqrt{1+\gamma^2}-2}{G(\gamma)}, \frac{\sqrt{1+\frac{4\eta^2}{T+1}}-1}{\max_{0\leq t\leq\eta} G(t)}\right[,$$

the problem

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) = \lambda\alpha_k g(u_k), & k \in \mathbb{Z}(1,T), \\ u_0 = u_{N+1} = 0, \end{cases}$$

admits at least one positive solution $\bar{u} \in S$, such that $\bar{u}_k < \eta$ for all $k \in \mathbb{Z}(1,T)$.

Proof. Put

$$f(k,t) := \begin{cases} \alpha_k g(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

for every $k \in \mathbb{Z}(1,T)$ and $t \in \mathbb{R}$. The conclusion follows from Theorem 3.2 owing to (3.8) and taking into account Theorem 2.3. □

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