Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260

CJMS. **9**(1)(2020), 137-150

Weighted slant Toep-Hank operators

Gopal Datt[1] and Anshika Mittal[2]
[1] Department of Mathematics,PGDAV College, University of Delhi, Delhi - 110065 (India).
[2] Department of Mathematics, University of Delhi, Delhi - 110007 (India).

Abstract. A *weighted slant Toep-Hank* operator L_ϕ^β with symbol $\phi \in L^\infty(\beta)$ is an operator on $L^2(\beta)$ whose representing matrix consists of all even (odd) columns from a weighted slant Hankel (slant weighted Toeplitz) matrix, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$. A matrix characterization for an operator to be *weighted slant Toep-Hank* operator is also obtained.

Keywords: Weighted slant Hankel operators, Slant weighted Toeplitz operators, slant Toep-Hank operator.

2000 Mathematics subject classification: Primary 47B35; Secondary 45B37.

1. Introduction

Let $\mathbb{C}$ and $\mathbb{Z}$ denote the set of all complex numbers and integers respectively. Throughout this paper, the spaces are considered, unless otherwise stated, under the assumption that the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence of positive numbers (that is $\beta_n = \beta_{-n}$ for each $n \in \mathbb{Z}$) with $\beta_0 = 1$ and $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, for some $r > 0$.

[1]Corresponding author: gopal.d.sati@gmail.com
Received: 17 January 2018
Accepted: 29 January 2020

Consider the spaces

$$L^2(\beta) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \big| a_n \in \mathbb{C},\ \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2 < \infty\}$$

and

$$H^2(\beta) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \big| a_n \in \mathbb{C},\ \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty\}.$$

The space $(L^2(\beta), \| \ . \ \|_\beta)$ is a Hilbert space with the inner product defined by

$$\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \rangle = \sum_{n=-\infty}^{\infty} a_n \ \bar{b}_n {\beta_n}^2.$$

The set $\{e_n : e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for the space $L^2(\beta)$ and $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\| \cdot \|_\infty$. Also, $L^\infty(\beta) \subseteq L^2(\beta)$. $H^\infty(\beta)$ denotes the set of formal power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. These weighted sequence spaces cover Bergman, Hardy, Dirichlet and Fischer spaces for specifically designed sequences $\beta = \{\beta_n\}$ and thus become more demanding. For the detailed study of these spaces, we refer [11] and the references therein. If $\phi \in L^\infty(\beta)$, then the weighted Laurent operator M_ϕ^β on $L^2(\beta)$ is given by

$$M_\phi^\beta e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z).$$

If we put $\phi(z) = z$, then the operator $M_z^\beta e_k(z) = \frac{\beta_{k+1}}{\beta_k} e_{k+1}(z)$, for all $k \in \mathbb{Z}$, and is known as a weighted shift [11].

The Hankel and Toeplitz operators arise in plenty of applications like stationary processes, perturbation theory, wavelet analysis and many more. For the detailed study of these operators and their applications, we refer [[5, 6, 7, 10]] and the references therein. Over the years, many generalizations of these operators also came up including slant Hankel [1] and slant Toeplitz [8] operators on the space $L^2(\mathbb{T})$, $\mathbb{T}$ being the unit circle. Meanwhile, the weighted sequence spaces $L^2(\beta)$, $H^2(\beta)$ and their generalizations came up and gained popularity with the work of Shields

[11] and Lauric [9]. Further, the notions of slant Toeplitz (slant Hankel) operators were lifted to slant weighted Toeplitz (weighted slant Hankel) operators on the space $L^2(\beta)$ (see [2] and [3]).

Motivated by all these developments and the study of a *slant Toep-Hank* operator L_ϕ on $L^2(\mathbb{T})$ discussed in [4] (whose matrix representation provides a slant Hankel (slant Toeplitz) matrix if only even (odd) columns are considered), we now introduce and study the notion of a *weighted slant Toep-Hank* operator L_ϕ^β on the space $L^2(\beta)$. In the second section, we obtain various characterizations for an operator to be *weighted slant Toep-Hank* operator. In the third section, we obtain symbols for any weighted slant Hankel or slant weighted Toeplitz operator to be a *weighted slant Toep-Hank* operator.

2. Main Result

The main aim of this section is to find characterizations for *weighted slant Toep-Hank* operators in terms of matrices and operator equations. We begin with the following definitions of operators frequently used in the paper.

Definition 2.1. [2] For $\phi \in L^\infty(\beta)$, a slant weighted Toeplitz operator U_ϕ^β on the space $L^2(\beta)$ is an operator given by $U_\phi^\beta = W^\beta M_\phi^\beta$, where W^β be the operator on $L^2(\beta)$ given by

$$W^\beta e_n(z) = \begin{cases} \frac{\beta_m}{\beta_{2m}} e_m(z) & \text{if n=2m for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

Definition 2.2. [3] A weighted slant Hankel operator K_ϕ^β induced by ϕ in $L^\infty(\beta)$ is an operator on $L^2(\beta)$ given by $K_\phi^\beta = J^\beta W^\beta M_\phi^\beta$, where J^β is the reflection operator on $L^2(\beta)$ given by $J^\beta(e_n) = e_{-n}$ for $n \in \mathbb{Z}$.

It is known [3] that if the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual, then the expression $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$ is in $L^\infty(\beta)$ for each $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^\infty(\beta)$.

For $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, the operators U_ϕ^β and K_ϕ^β satisfy that

$$U_\phi^\beta e_j(z) = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{2n-j} \beta_n e_n(z)$$

and

$$K_\phi^\beta e_j(z) = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{-2n-j} \beta_{-n} e_n(z)$$

for each $j \in \mathbb{Z}$.

Definition 2.3. [4] Let $\phi \in L^\infty(\mathbb{T})$. A *slant Toep-Hank* operator L_ϕ on $L^2(\mathbb{T})$ induced by ϕ is given by $L_\phi = K_\phi W + U_{z\tilde{\phi}}\mathcal{M}$, where W and $\mathcal{M}$ are operators on $L^2(\mathbb{T})$ given by $W(e_{2n}) = e_n$, otherwise zero and $\mathcal{M}(e_{2n+1}) = e_n$, otherwise zero.

We now extend the notion of *slant Toep-Hank* operator to $L^2(\beta)$ as follows.

Definition 2.4. Let $\phi \in L^\infty(\beta)$. A *weighted slant Toep-Hank* operator L_ϕ^β on $L^2(\beta)$ is given by $L_\phi^\beta = K_\phi^\beta \hat{W}^\beta + U^\beta_{z\tilde{\phi}}\hat{\mathcal{M}}^\beta$, where we define operators $\hat{\mathcal{M}}^\beta$ and $\hat{W}^\beta$ on $L^2(\beta)$ as

$$\hat{\mathcal{M}}^\beta e_n = \begin{cases} e_{(\frac{n-1}{2})} & \text{if n is odd,} \\ 0 & \text{if n is even} \end{cases} \text{ and } \hat{W}^\beta e_n = \begin{cases} e_{(\frac{n}{2})} & \text{if n is even,} \\ 0 & \text{if n is odd} \end{cases}$$

for each $n \in \mathbb{Z}$.

It is worth noticing that weighted slant Hankel and slant weighted Toeplitz operators are linear with respect to their symbols. Thus, the class $\{L_\phi^\beta | \phi \in L^\infty(\beta)\}$ is a linear subspace of $\mathfrak{B}(L^2(\beta))$, the space of all bounded operators on $L^2(\beta)$, that is, $L_\phi^\beta + L_\psi^\beta = L_{\phi+\psi}^\beta$ and $\alpha L_\phi^\beta = L_{\alpha\phi}^\beta$. Furthermore, $\|L_\phi^\beta\| \leq 2\|\phi\|_\infty$ and $L_\phi^\beta = 0$ if and only if $\phi = 0$. One can also observe that the correspondence $\phi \to L_\phi^\beta$ is an injective linear mapping from $L^\infty(\beta)$ into $\mathfrak{B}(L^2(\beta))$. The matrix of L_ϕ^β with respect to the orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of $L^2(\beta)$ is of the form

$$\left[\begin{array}{cc|cccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & a_6\frac{\beta_{-3}}{\beta_{-1}} & a_6\frac{\beta_3}{\beta_0} & a_7\frac{\beta_{-3}}{\beta_0} & a_5\frac{\beta_3}{\beta_1} & a_8\frac{\beta_{-3}}{\beta_1} & a_4\frac{\beta_3}{\beta_2} & \cdots \\
\cdots & a_4\frac{\beta_{-2}}{\beta_{-1}} & a_4\frac{\beta_2}{\beta_0} & a_5\frac{\beta_{-2}}{\beta_0} & a_3\frac{\beta_2}{\beta_1} & a_6\frac{\beta_{-2}}{\beta_1} & a_2\frac{\beta_2}{\beta_2} & \cdots \\
\cdots & a_2\frac{\beta_{-1}}{\beta_{-1}} & a_2\frac{\beta_1}{\beta_0} & a_3\frac{\beta_{-1}}{\beta_0} & a_1\frac{\beta_1}{\beta_1} & a_4\frac{\beta_{-1}}{\beta_1} & a_0\frac{\beta_1}{\beta_2} & \cdots \\
\cdots & a_0\frac{\beta_0}{\beta_{-1}} & a_0\frac{\beta_0}{\beta_0} & a_1\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_2\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & \cdots \\
\hline
\cdots & a_{-2}\frac{\beta_1}{\beta_{-1}} & a_{-2}\frac{\beta_{-1}}{\beta_0} & a_{-1}\frac{\beta_1}{\beta_0} & a_{-3}\frac{\beta_{-1}}{\beta_1} & a_0\frac{\beta_1}{\beta_1} & a_{-4}\frac{\beta_{-1}}{\beta_2} & \cdots \\
\cdots & a_{-4}\frac{\beta_2}{\beta_{-1}} & a_{-4}\frac{\beta_{-2}}{\beta_0} & a_{-3}\frac{\beta_2}{\beta_0} & a_{-5}\frac{\beta_{-2}}{\beta_1} & a_{-2}\frac{\beta_2}{\beta_1} & a_{-6}\frac{\beta_{-2}}{\beta_2} & \cdots \\
\cdots & a_{-6}\frac{\beta_3}{\beta_{-1}} & a_{-6}\frac{\beta_{-3}}{\beta_0} & a_{-5}\frac{\beta_3}{\beta_0} & a_{-7}\frac{\beta_{-3}}{\beta_1} & a_{-4}\frac{\beta_3}{\beta_1} & a_{-8}\frac{\beta_{-3}}{\beta_2} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right].$$

As is observed in the case of *slant Toep-Hank* operators, the matrix of *weighted slant Toep-Hank* operator L_ϕ^β provides the matrix of weighted slant Hankel operator K_ϕ^β if only even columns are considered and the matrix of slant weighted Toeplitz operator $U^\beta_{z\widetilde{\phi}}$ if only odd columns are considered. Further, if $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ is the Fourier expansion of ϕ and $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ denotes the matrix of the operator L_ϕ^β, then the $(i,j)^{th}$ entry is given by $\langle \alpha_{i,j} \rangle = \langle a_{-2i-n} \frac{\beta_{-i}}{\beta_n} \rangle$, if $j = 2n$ and $\langle \alpha_{i,j} \rangle = \langle a_{-2i+n+1} \frac{\beta_i}{\beta_n} \rangle$, if $j = 2n+1$, $n \in \mathbb{Z}$. Clearly, $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ satisfies the following relations:

$$\begin{cases} \frac{\beta_{2j-1}}{\beta_{k+j}}\alpha_{k+j,4j-1} = \frac{\beta_{2j}}{\beta_{-(k-j)}}\alpha_{k-j,4j} = \frac{\beta_0}{\beta_{-k}}\alpha_{k,0} & \text{for } k,j \in \mathbb{Z} \\ \frac{\beta_{2j}}{\beta_{k+j}}\alpha_{k+j,4j+1} = \frac{\beta_{2j-1}}{\beta_{-(k-j)}}\alpha_{k-j,4j-2} = \frac{\beta_0}{\beta_k}\alpha_{k,1} & \text{for } k,j \in \mathbb{Z}. \end{cases} \tag{2.1}$$

In [2] and [3], the matrix characterizations for slant weighted Toeplitz and weighted slant Hankel operators are obtained. Similarly, one can expect a matrix characterization for *weighted slant Toep-Hank* operators. For that purpose, we introduce the following notion.

A doubly infinite matrix $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ is said to be a *weighted slant Toep-Hank* matrix if it satisfies the relation (2.1).
We begin with the result which serve as a great tool for our study.

Lemma 2.5. *If A is any bounded linear operator on $L^2(\beta)$ such that its matrix $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ is a weighted slant Toep-Hank matrix, then the following holds:*

(1) $\frac{\beta_{j+2k}}{\beta_{-(i-k)}}\alpha_{i-k,2j+4k} = \frac{\beta_j}{\beta_{-i}}\alpha_{i,2j}$ *for* $i,j,k \in \mathbb{Z}$.

(2) $\frac{\beta_{j+2}}{\beta_{i+1}}\alpha_{i+1,2j+5} = \frac{\beta_j}{\beta_i}\alpha_{i,2j+1}$ *for* $i,j \in \mathbb{Z}$.

Proof. We first prove (1). Let i', k' and j' be any integers. First we consider the case when j' is an odd integer. As $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ is a *weighted slant Toep-Hank* matrix, it satisfies the relation (2.1). Hence, for each $k, j \in \mathbb{Z}$,

$$\frac{\beta_{2j-1}}{\beta_{-(k-j)}}\alpha_{k-j,4j-2} = \frac{\beta_0}{\beta_k}\alpha_{k,1}.$$

On substituting $k = i' + (\frac{j'+1}{2}), j = \frac{j'+2k'+1}{2}$ and $k = i' + (\frac{j'+1}{2}), j = \frac{j'+1}{2}$ successively in the above equation, we get

$$\frac{\beta_{j'+2k'}}{\beta_{-(i'-k')}}\alpha_{i'-k',2j'+4k'} = \frac{\beta_0}{\beta_{i'+(\frac{j'+1}{2})}}\alpha_{i'+(\frac{j'+1}{2}),1} = \frac{\beta_{j'}}{\beta_{-i'}}\alpha_{i',2j'}.$$

Now consider the case when j' is an even integer. Again from relation (2.1), for each $k, j \in \mathbb{Z}$,

$$\frac{\beta_{2j}}{\beta_{-(k-j)}}\alpha_{k-j,4j} = \frac{\beta_0}{\beta_{-k}}\alpha_{k,0}.$$

This equation, on substituting $k = i' + (\frac{j'}{2}), j = \frac{j'+2k'}{2}$ and $k = i' + (\frac{j'}{2}), j = \frac{j'}{2}$ successively, gives that

$$\frac{\beta_{j'+2k'}}{\beta_{-(i'-k')}}\alpha_{i'-k',2j'+4k'} = \frac{\beta_0}{\beta_{-(i'+\frac{j'}{2})}}\alpha_{i'+\frac{j'}{2},0} = \frac{\beta_{j'}}{\beta_{-i'}}\alpha_{i',2j'}.$$

This completes the proof of (1). Similarly, (2) can be obtained using the equations

$$\frac{\beta_{2j}}{\beta_{k+j}}\alpha_{k+j,4j+1} = \frac{\beta_0}{\beta_k}\alpha_{k,1} \text{ and } \frac{\beta_{2j-1}}{\beta_{k+j}}\alpha_{k+j,4j-1} = \frac{\beta_0}{\beta_{-k}}\alpha_{k,0}$$

from (2.1). □

In [12], Zorboska discussed the notion of composition operator C_ϕ^β $(f \to f \circ \phi)$ on the weighted sequence spaces. It is evident from here that if the sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ be such that the sequence $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, then the composition operator $C_{z^2}^\beta$ is a bounded operator on $L^2(\beta)$. For $\beta_n = 1$ for each n, the operator $C_{z^2}^\beta$ coincides with the composition operator C_{z^2} on $L^2(\mathbb{T})$. Further, it is proved in [4] that AC_{z^2} is a slant Hankel operator and $AM_zC_{z^2}$ is a slant Toeplitz operator for every *slant Toep-Hank* operator A on $L^2(\mathbb{T})$. However, we will see that this is not the situation in case of *weighted slant Toep-Hank* operator.

It is known that corresponding to the weight sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ of positive real numbers, a doubly infinite matrix $\{\lambda_{i,j}\}_{i,j\in\mathbb{Z}}$ is called

(1) slant weighted Toeplitz matrix if $\frac{\beta_j}{\beta_i}\lambda_{i,j} = \frac{\beta_{j+2}}{\beta_{i+1}}\lambda_{i+1,j+2}$ for each $i, j \in \mathbb{Z}$.
(2) weighted slant Hankel matrix if $\frac{\beta_j}{\beta_{-i}}\lambda_{i,j} = \frac{\beta_{j+2k}}{\beta_{-(i-k)}}\lambda_{i-k,j+2k}$ for each $i, j, k \in \mathbb{Z}$.

Under the assumptions of $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ being bounded, it is shown in [2] ([3]) that an operator on $L^2(\beta)$ is slant weighted Toeplitz (weighted slant Hankel) operator if and only if its matrix is a slant weighted Toeplitz (weighted slant Hankel) matrix. We show through next example that for a *weighted slant Toep-Hank* operator A on $L^2(\beta)$, $AC_{z^2}^\beta$ and $AM_z^\beta C_{z^2}^\beta$ need not be weighted slant Hankel and slant weighted Toeplitz operator respectively.

Example 2.6. Let $\phi(z) = z^{-3} + 1$ and $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be defined as

$$\beta_n = \begin{cases} 1 & if\ n = 0, 1, -1 \\ 2 & otherwise \end{cases}.$$

Then $\{\beta_n\}$ is a bounded semi-dual sequence such that $\frac{1}{2} \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. We see that $\phi \in L^\infty(\beta)$. Consider the *weighted slant Toep-Hank* operator A $(= L^\beta_\phi)$ on $L^2(\beta)$. Let $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$, $\{\lambda_{i,j}\}_{i,j\in\mathbb{Z}}$ and $\{\gamma_{i,j}\}_{i,j\in\mathbb{Z}}$ be the matrices of A, $AC^\beta_{z^2}$ and $AM^\beta_z C^\beta_{z^2}$ respectively with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$. Now using Lemma 2.5, we find that the matrix $\{\lambda_{i,j}\}_{i,j\in\mathbb{Z}}$ of $AC^\beta_{z^2}$ satisfies

$$\begin{aligned} \frac{\beta_{j+2k}}{\beta_{-(i-k)}}\lambda_{i-k,j+2k} &= \frac{\beta_{j+2k}}{\beta_{-(i-k)}}\langle AC^\beta_{z^2}e_{j+2k}, e_{i-k}\rangle \\ &= \frac{\beta_{j+2k}}{\beta_{-(i-k)}}\langle \frac{\beta_{2j+4k}}{\beta_{j+2k}}Ae_{2j+4k}, e_{i-k}\rangle \\ &= \frac{\beta_{2j+4k}}{\beta_{-(i-k)}}\alpha_{i-k,2j+4k} = \frac{\beta_{2j+4k}}{\beta_{j+2k}}\frac{\beta_j}{\beta_{-i}}\alpha_{i,2j} \end{aligned}$$

and $\frac{\beta_j}{\beta_{-i}}\lambda_{i,j} = \frac{\beta_{2j}}{\beta_{-i}}\alpha_{i,2j}$ for each $i, j, k \in \mathbb{Z}$. Thus for $i = j = k = 1$, we find that $\frac{\beta_{j+2k}}{\beta_{-(i-k)}}\lambda_{i-k,j+2k} = 1 \neq 2 = \frac{\beta_j}{\beta_{-i}}\lambda_{i,j}$. Hence, $AC^\beta_{z^2}$ can not be a weighted slant Hankel operator.

On the similar lines of computation, we obtain that

$$\frac{\beta_{j+2}}{\beta_{i+1}}\gamma_{i+1,j+2} = \frac{\beta_{j+2}}{\beta_{i+1}}\langle AM^\beta_z C^\beta_{z^2}e_{j+2}, e_{i+1}\rangle = \frac{\beta_{2j+5}}{\beta_{j+2}}\frac{\beta_j}{\beta_i}\alpha_{i,2j+1}$$

and $\frac{\beta_j}{\beta_i}\gamma_{i,j} = \frac{\beta_{2j+1}}{\beta_i}\alpha_{i,2j+1}$ for each $i, j \in \mathbb{Z}$. These for $i = j = 1$ show that $\frac{\beta_{j+2}}{\beta_{i+1}}\gamma_{i+1,j+2} = 1 \neq 2 = \frac{\beta_j}{\beta_i}\gamma_{i,j}$. This shows that $AM^\beta_z C^\beta_{z^2}$ can not be a slant weighted Toeplitz operator.

In order to derive a weighted slant Hankel operator (slant weighted Toeplitz operator) from a given *weighted slant Toep-Hank* operator, we proceed to define the following operators.

Definition 2.7. Consider the following operators defined for $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n \in L^2(\beta)$,

(1) An operator $\hat{C}^\beta_{z^2}$ on $L^2(\beta)$ is defined as

$$\hat{C}^\beta_{z^2}(f(z)) = \sum_{n=-\infty}^{\infty} a_n \frac{\beta_n}{\beta_{2n}} z^{2n}.$$

(2) An operator $\hat{M}_z^\beta$ on $L^2(\beta)$ is defined as

$$\hat{M}_z^\beta(f(z)) = \sum_{n=-\infty}^{\infty} a_n \frac{\beta_n}{\beta_{n+1}} z^{n+1}.$$

Clearly $\hat{C}_{z^2}^\beta$ and $\hat{M}_z^\beta$ are bounded linear operators on $L^2(\beta)$. Further, $\hat{C}_{z^2}^\beta(e_n) = \hat{C}_{z^2}^\beta(\frac{z^n}{\beta_n}) = \frac{z^{2n}}{\beta_{2n}} = e_{2n}$ and $\hat{M}_z^\beta(e_n) = e_{n+1}$ for each $n \in \mathbb{Z}$.

The following result is now immediate.

Proposition 2.8. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence such that $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. If matrix of any bounded linear operator A defined on $L^2(\beta)$ is a weighted slant Toep-Hank matrix, then $A\hat{C}_{z^2}^\beta$ is a weighted slant Hankel operator and $A\hat{M}_z^\beta\hat{C}_{z^2}^\beta$ is a slant weighted Toeplitz operator on $L^2(\beta)$.*

It is clear that the matrix representation of a *weighted slant Toep-Hank* operator on $L^2(\beta)$ is always (without $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being semi-dual or $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded) a *weighted slant Toep-Hank* matrix. However, these additional assumptions on the sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ along with Proposition 2.8 help us to prove the main result of this section as follows.

Theorem 2.9. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence such that $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then for a bounded operator A on $L^2(\beta)$, the following are equivalent.*

(1) *A is a weighted slant Toep-Hank operator.*
(2) *Matrix of A with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$ is a weighted slant Toep-Hank matrix.*
(3) *A satisfies the following equations:*
   (a) $M_{z^{-1}}^\beta A\hat{C}_{z^2}^\beta = A\hat{C}_{z^2}^\beta M_{z^2}^\beta$.
   (b) $M_z^\beta A\hat{M}_z^\beta\hat{C}_{z^2}^\beta = A\hat{M}_z^\beta\hat{C}_{z^2}^\beta M_{z^2}^\beta$.
   (c) $M_{z^{-1}}^\beta AM_{z^{-1}}^\beta C_{z^2}^\beta e_1 = A\hat{M}_{z^2}^\beta M_z^{\beta *} e_1$.
   (d) $A\hat{M}_z^\beta\hat{C}_{z^2}^\beta e_1 = M_z^\beta AM_{z^{-1}}^\beta e_1$.

Proof. Let $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ denotes the matrix of operator A with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$.

(1) implies (2) is obvious. For the reverse, assume that the matrix $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ of A is a *weighted slant Toep-Hank* matrix and hence satisfies the relation (2.1). Using Proposition 2.8, $A\hat{C}_{z^2}^\beta$ is a weighted slant Hankel operator and $A\hat{M}_z^\beta\hat{C}_{z^2}^\beta$ is a slant weighted Toeplitz operator on $L^2(\beta)$. Let $A\hat{M}_z^\beta\hat{C}_{z^2}^\beta = U_\psi^\beta$ and $A\hat{C}_{z^2}^\beta = K_\zeta^\beta$ for some $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$

and $\zeta(z)=\sum_{n=-\infty}^{\infty} c_n z^n$ in $L^\infty(\beta)$. Let $\{\gamma_{i,j}\}_{i,j\in\mathbb{Z}}$ and $\{\lambda_{i,j}\}_{i,j\in\mathbb{Z}}$ be the matrices of $A\hat{M}_z^\beta\hat{C}'^\beta_{z^2}$ and $A\hat{C}'^\beta_{z^2}$ respectively. Then using the definition of slant weighted Toeplitz operator, we have $\gamma_{i,j}=\frac{\beta_i}{\beta_j}b_{2i-j}$ for $i,j\in\mathbb{Z}$. This fact along with the equations in (2.1) yields that for each $n\in\mathbb{Z}$

$$\langle\psi,e_n\rangle = \begin{cases} b_{2k}=\frac{\beta_0}{\beta_k}\gamma_{k,0}=\frac{\beta_0}{\beta_k}\langle A\hat{M}_z^\beta\hat{C}^\beta_{z^2}e_0,e_k\rangle & \text{if } n=2k\\ b_{2k-1}=\frac{\beta_1}{\beta_k}\gamma_{k,1}=\frac{\beta_1}{\beta_k}\langle A\hat{M}_z^\beta\hat{C}^\beta_{z^2}e_1,e_k\rangle & \text{if } n=2k-1\end{cases}$$

$$= \begin{cases} \frac{\beta_0}{\beta_k}\alpha_{k,1} & \text{if } n=2k\\ \frac{\beta_1}{\beta_k}\alpha_{k,3}=\frac{\beta_0}{\beta_{-(k-1)}}\alpha_{k-1,0} & \text{if } n=2k-1\end{cases}.$$

Define a complex valued function $\phi(z)=\sum_{n=-\infty}^{\infty}a_nz^n$, where $a_n=b_{-n+1}$ for $n\in\mathbb{Z}$. Then $\psi=z\widetilde{\phi}$ so that $\phi\in L^\infty(\beta)$ (since $\{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual). Now by the definition of weighted slant Hankel operator, we have $\lambda_{i,j}=\frac{\beta_{-i}}{\beta_j}c_{-2i-j}=\frac{\beta_i}{\beta_j}c_{-2i-j}$ for $i,j\in\mathbb{Z}$. This gives that for $n\in\mathbb{Z}$

$$\begin{aligned} c_n=\langle\zeta,e_n\rangle &= \begin{cases}\frac{\beta_0}{\beta_k}\lambda_{k,0}=\frac{\beta_0}{\beta_k}\langle A\hat{C}^\beta_{z^2}e_0,e_k\rangle & \text{if } n=-2k\\ \frac{\beta_1}{\beta_k}\lambda_{k,1}=\frac{\beta_1}{\beta_k}\langle A\hat{C}^\beta_{z^2}e_1,e_k\rangle & \text{if } n=-2k-1\end{cases}\\ &= \begin{cases}\frac{\beta_0}{\beta_k}\alpha_{k,0}=b_{2k+1}=a_{-2k} & \text{if } n=-2k\\ \frac{\beta_1}{\beta_k}\alpha_{k,2}=\frac{\beta_0}{\beta_{k+1}}\alpha_{k+1,1}=a_{-2k-1} & \text{if } n=-2k-1\end{cases}\\ &= a_n.\end{aligned}$$

This provides that $\zeta=\phi$. Hence, $A\hat{C}'^\beta_{z^2}$ is the weighted slant Hankel operator K^β_ϕ and $A\hat{M}_z^\beta\hat{C}^\beta_{z^2}$ is the slant weighted Toeplitz operator $U^\beta_{z\widetilde{\phi}}$.

Now each $h(z)\in L^2(\beta)$ can be written as $h(z)=h_1(z^2)+zh_2(z^2)$ with $h_1,h_2\in L^2(\beta)$. Say, $h_1(z)=\sum_{n=-\infty}^{\infty}a_nz^n$ and $h_2(z)=\sum_{n=-\infty}^{\infty}b_nz^n$. Then, $h_1(z^2)=\sum_{n=-\infty}^{\infty}a_nz^{2n}$, $zh_2(z^2)=\sum_{n=-\infty}^{\infty}b_nz^{2n+1}$ and

$$\begin{aligned} L^\beta_\phi h(z) &= (K^\beta_\phi\hat{W}^\beta+U^\beta_{z\widetilde{\phi}}\hat{\mathcal{M}}^\beta)(h_1(z^2)+zh_2(z^2))\\ &= K^\beta_\phi(\sum_{n=-\infty}^{\infty}a_n\beta_{2n}e_n)+U^\beta_{z\widetilde{\phi}}(\sum_{n=-\infty}^{\infty}b_n\beta_{2n+1}e_n)\\ &= A\hat{C}^\beta_{z^2}(\sum_{n=-\infty}^{\infty}a_n\beta_{2n}e_n)+A\hat{M}_z^\beta\hat{C}^\beta_{z^2}(\sum_{n=-\infty}^{\infty}b_n\beta_{2n+1}e_n)\end{aligned}$$

$$= A\Big(\sum_{n=-\infty}^{\infty} a_n \beta_{2n} e_{2n} + \sum_{n=-\infty}^{\infty} b_n \beta_{2n+1} e_{2n+1}\Big) = Ah(z)$$

for each $h \in L^2(\beta)$. This implies that $A = L_\phi^\beta$ and hence (1) and (2) are equivalent.

Now we prove the equivalency of (1) and (3). To obtain (3) from (1), suppose that A is a *weighted slant Toep-Hank* operator. Then, its matrix $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ satisfies (2.1). As any bounded operator B on $L^2(\beta)$ is weighted slant Hankel (slant weighted Toeplitz) if and only if $M_{z^{-1}}^\beta B = BM_{z^2}^\beta$ ($M_z^\beta B = BM_{z^2}^\beta$), we obtain (a) and (b) using Proposition 2.8.

From (2.1), $\frac{\beta_{2j-1}}{\beta_{-(k-j)}}\alpha_{k-j,4j-2} = \frac{\beta_0}{\beta_k}\alpha_{k,1}$ for each $k, j \in \mathbb{Z}$. It provides on replacing k by $k+1$ and j by 1, that $\frac{\beta_1}{\beta_{-k}}\alpha_{k,2} = \frac{\beta_0}{\beta_{k+1}}\alpha_{k+1,1}$ for each $k \in \mathbb{Z}$. Then $\big\langle M_{z^{-1}}^\beta A M_{z^{-1}}^\beta C_{z^2}^\beta e_1, e_k\big\rangle = \frac{\beta_k}{\beta_{k+1}}\big\langle Ae_1, e_{k+1}\big\rangle = \frac{\beta_k}{\beta_{k+1}}\alpha_{k+1,1} = \frac{\beta_1}{\beta_0}\alpha_{k,2} = \frac{\beta_1}{\beta_0}\big\langle Ae_2, e_k\big\rangle = \big\langle A\hat{M}_{z^2}^\beta M_z^{\beta^*} e_1, e_k\big\rangle$ for each $k \in \mathbb{Z}$. Thus we have (c).

Now on replacing k by $k-1$ and j by 1 in the equation $\frac{\beta_{2j-1}}{\beta_{k+j}}\alpha_{k+j,4j-1} = \frac{\beta_0}{\beta_{-k}}\alpha_{k,0}$ of (2.1) and applying the same arguments as earlier, we obtain (d).

In order to obtain (1) from (3), suppose that A satisfies (a), (b), (c) and (d). Then, (a) and (b) respectively provide that $A\hat{C}_{z^2}^\beta$ is the weighted slant Hankel operator and $A\hat{M}_z^\beta \hat{C}_{z^2}^\beta$ is the slant weighted Toeplitz operator. The equations (c) and (d) respectively gives $\frac{\beta_1}{\beta_k}\alpha_{k,2} = \frac{\beta_0}{\beta_{k+1}}\alpha_{k+1,1}$ and $\frac{\beta_1}{\beta_k}\alpha_{k,3} = \frac{\beta_0}{\beta_{-(k-1)}}\alpha_{k-1,0}$ for each $k \in \mathbb{Z}$.

On using these facts and applying the arguments as in the proof of (2) implies (1), we obtain that A is a *weighted slant Toep-Hank* operator. This completes the proof. □

The adjoint $L_\phi^{\beta^*}$ of a *weighted slant Toep-Hank* operator L_ϕ^β is nothing but an operator on $L^2(\beta)$ satisfying $L_\phi^{\beta^*} = \hat{W}^{\beta^*} K_\phi^{\beta^*} + \hat{\mathcal{M}}^{\beta^*} U_{z\widetilde{\phi}}^{\beta^*}$, where $\hat{W}^{\beta^*}$ and $\hat{\mathcal{M}}^{\beta^*}$ on $L^2(\beta)$ are defined as $\hat{W}^{\beta^*}(e_n) = e_{2n}$ and $\hat{\mathcal{M}}^{\beta^*}(e_n) = e_{2n+1}$ for $n \in \mathbb{Z}$. Further, if $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n \in L^\infty(\beta)$, then for each $j \in \mathbb{Z}$

$$K_\phi^{\beta^*} e_j = \beta_{-j} \sum_{n=-\infty}^{\infty} \frac{\overline{a}_{-2j-n}}{\beta_n} e_n \text{ and } U_{z\widetilde{\phi}}^{\beta^*} e_j = \beta_j \sum_{n=-\infty}^{\infty} \frac{\overline{a}_{-2j+n+1}}{\beta_n} e_n.$$

It is now natural from Theorem 2.9 that the adjoint of a *weighted slant Toep-Hank* operator need not be a *weighted slant Toep-Hank* operator on $L^2(\beta)$. This can be verified from the following example.

Example 2.10. Let $\phi(z) = z^{-1}$ and $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be the sequence as in Example 2.6. Clearly, $\phi \in L^\infty(\beta)$. But the matrix of $L_\phi^{\beta^*}$ is not a *weighted slant Toep-Hank* matrix as, if we take $k = 2$ and $j = 2$ then the condition

$$\frac{\beta_{2j-1}}{\beta_{k+j}}\big\langle L_\phi^{\beta^*} e_{4j-1}, e_{k+j}\big\rangle = \frac{\beta_0}{\beta_{-k}}\big\langle L_\phi^{\beta^*} e_0, e_k\big\rangle$$

implies that $0 = \frac{1}{2}$. Hence $L_\phi^{\beta^*}$ is not a *weighted slant Toep-Hank* operator on $L^2(\beta)$.

However if $L_\phi^{\beta^*}$, for $\phi \in L^\infty(\beta)$, is a *weighted slant Toep-Hank* operator on $L^2(\beta)$, then from relation (2.1), for each $k, j \in \mathbb{Z}$,

$$\frac{\beta_{2j}}{\beta_{-(k-j)}}\big\langle L_\phi^{\beta^*} e_{4j}, e_{k-j}\big\rangle = \frac{\beta_0}{\beta_{-k}}\big\langle L_\phi^{\beta^*} e_0, e_k\big\rangle.$$

In particular, for $k = j = 2l$ $(l \in \mathbb{Z})$,

$$\frac{\beta_{4l}}{\beta_0}\big\langle e_{8l}, K_\phi^\beta e_0\big\rangle = \frac{\beta_0}{\beta_{-2l}}\big\langle e_0, K_\phi^\beta e_l\big\rangle.$$

This implies that $a_{-l} = \beta_l\ \beta_{-2l}\ \beta_{4l}\ \beta_{-8l}\ a_{-16l}$ for each $l \in \mathbb{Z}$. From here we observe that if the sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\beta_n = 1$ for each n, then using the fact that $\lim_{n\to\infty} a_n = 0$, we have $a_n = 0$ for all $n \in \mathbb{Z}$. So $\phi = 0$.

The above observation can be summed up in the following form.

Remark 2.11. The only self-adjoint *slant Toep-Hank* operator on $L^2(\mathbb{T})$ is the zero operator.

3. Connection among various classes

Proposition 2.8 helps us to obtain a weighted slant Hankel operator or slant weighted Toeplitz operator from a given *weighted slant Toep-Hank* operator. In the present section, our aim is to compute the intersection of the class of *weighted slant Toep-Hank* operators with the classes of weighted slant Hankel and slant weighted Toeplitz operators. Let the classes of *weighted slant Toep-Hank*, weighted slant Hankel and slant weighted Toeplitz operators be denoted by $\mathcal{C}_{wst-h}$, $\mathcal{C}_{wsh}$ and $\mathcal{C}_{swt}$ respectively. Now we have the following.

Theorem 3.1. *A weighted slant Hankel operator on $L^2(\beta)$ is a weighted slant Toep-Hank operator if and only if it is a zero operator.*

Proof. Let a weighted slant Hankel operator A be a *weighted slant Toep-Hank* operator on $L^2(\beta)$, say $A = L^\beta_\phi$ for some $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. Thus, its matrix satisfies equations in (2.1). The operator A, being a weighted slant Hankel operator, satisfies

$$Ae_j(z) = L^\beta_\phi e_j(z) = \frac{1}{\beta_j}\sum_{n=-\infty}^{\infty} a_{-2n-j}\beta_{-n}e_n(z).$$

Now using relation (2.1), for all $k, j \in \mathbb{Z}$, we have

$$\frac{\beta_{2j}}{\beta_{-(k-j)}}\langle L^\beta_\phi e_{4j}, e_{k-j}\rangle = \frac{\beta_0}{\beta_{-k}}\langle L^\beta_\phi e_0, e_k\rangle.$$

Hence

$$\frac{\beta_{2j}}{\beta_{-(k-j)}}\Big\langle \frac{1}{\beta_{4j}}\sum_{n=-\infty}^{\infty} a_{-2n-4j}\beta_{-n}e_n, e_{k-j}\Big\rangle = \frac{\beta_0}{\beta_{-k}}\Big\langle \frac{1}{\beta_0}\sum_{n=-\infty}^{\infty} a_{-2n}\beta_{-n}e_n, e_k\Big\rangle.$$

Therefore $a_{-2k-2j} = a_{-2k}\frac{\beta_{4j}}{\beta_{2j}}$ for all $k, j \in \mathbb{Z}$. Putting $k = 0$, we get $a_{-2j} = a_0\frac{\beta_{4j}}{\beta_{2j}}$ for all $j \in \mathbb{Z}$.

Again from (2.1) for $k, j \in \mathbb{Z}$,

$$\frac{\beta_{2j-1}}{\beta_{k+j}}\langle L^\beta_\phi e_{4j-1}, e_{k+j}\rangle = \frac{\beta_0}{\beta_{-k}}\langle L^\beta_\phi e_0, e_k\rangle.$$

Using the same computations as above, we get $a_{-2j+1} = a_0\frac{\beta_{4j}}{\beta_{2j}}$ for all $j \in \mathbb{Z}$. Now, since $|a_0|^2\sum_{n=-\infty}^{\infty}(1)^n \leq \sum_{n=-\infty}^{\infty}\frac{\beta_{4n}^2}{\beta_{2n}^2}|a_0|^2{\beta_n}^2 = \sum_{n=-\infty}^{\infty}|a_n|^2{\beta_n}^2 < \infty$, hence we must have $a_0 = 0$ so that $\phi = 0$. This provides that A is a zero operator. The converse is obvious. Hence the theorem. □

The next result, proof of which follows almost along the same arguments as made in Theorem 3.1, is the following.

Theorem 3.2. *A non-zero slant weighted Toeplitz operator cannot be a weighted slant Toep-Hank operator on $L^2(\beta)$.*

From Theorem 3.1 and Theorem 3.2, we conclude that

$$\mathcal{C}_{wst-h} \cap \mathcal{C}_{wsh} = \{0\} = \mathcal{C}_{wst-h} \cap \mathcal{C}_{swt}.$$

Our next result discusses the product of *weighted slant Toep-Hank* operator with the operator W^β as well as with the weighted Laurent operator. We recall that the adjoint of W^β is given by $W^{\beta^*}e_n(z) = \frac{\beta_n}{\beta_{2n}}e_{2n}(z)$, $n \in \mathbb{Z}$.

Theorem 3.3. *For ϕ, ψ in $L^\infty(\beta)$,*

(1) $W^\beta L_\phi^\beta$ *is a weighted slant Toep-Hank operator if and only if* $\phi = 0$.
(2) $M_{z^m}^\beta L_\psi^\beta$ *and* $M_\phi^\beta L_{z^m}^\beta$, $m \in \mathbb{Z}$, *are weighted slant Toep-Hank operators.*

Proof. For the necessary part of (1), let $W^\beta L_\phi^\beta$ be a *weighted slant Toep-Hank* operator and $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ be its matrix. Then relation (2.1) gives that for $k, j \in \mathbb{Z}$,

$$\frac{\beta_{2j-1}}{\beta_{k+j}}\alpha_{k+j,4j-1} = \frac{\beta_0}{\beta_{-k}}\alpha_{k,0},$$

which implies that

$$\frac{\beta_{2j-1}}{\beta_{(k+j)}}\langle W^\beta L_\phi^\beta e_{4j-1}, e_{k+j}\rangle = \frac{\beta_0}{\beta_{-k}}\langle W^\beta L_\phi^\beta e_0, e_k\rangle.$$

Hence,

$$\frac{\beta_{2j-1}}{\beta_{(k+j)}}\langle U_{z\widetilde{\phi}}^\beta e_{2j-1}, W^{\beta^*} e_{k+j}\rangle = \frac{\beta_0}{\beta_{-k}}\langle K_\phi^\beta e_0, W^{\beta^*} e_k\rangle.$$

This equality for $k = 0$ provides that $a_0 = a_{-2j}$ for all $j \in \mathbb{Z}$. Similarly, using the equality $\frac{\beta_{2j}}{\beta_{k+j}}\alpha_{k+j,4j+1} = \frac{\beta_0}{\beta_k}\alpha_{k,1}$ of (2.1), we obtain that $a_{-2} = a_{-2j+1}$ for $j \in \mathbb{Z}$. As $\phi \in L^\infty(\beta) \subseteq L^2(\beta)$, we must have $a_0 = a_{-2} = 0$ so that $\phi = 0$. Sufficient part is obvious. Hence the result.

Now we prove (2). Let $\psi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^\infty(\beta)$ and $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ be the matrix representation of $M_{z^m}^\beta L_\psi^\beta$. Without much computations, we obtain that

$$\frac{\beta_{2j-1}}{\beta_{k+j}}\alpha_{k+j,4j-1} = \frac{\beta_{2j}}{\beta_{-(k-j)}}\alpha_{k-j,4j} = \frac{\beta_0}{\beta_{-k}}\alpha_{k,0} = a_{-2k+2m}$$

and

$$\frac{\beta_{2j}}{\beta_{k+j}}\alpha_{k+j,4j+1} = \frac{\beta_{2j-1}}{\beta_{-(k-j)}}\alpha_{k-j,4j-2} = \frac{\beta_0}{\beta_k}\alpha_{k,1} = a_{-2k+2m+1}$$

for each $k, j \in \mathbb{Z}$. Thus $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}}$ is a *weighted slant Toep-Hank* matrix. Similarly, if $\{\gamma_{i,j}\}_{i,j\in\mathbb{Z}}$ denotes the matrix of $M_\phi^\beta L_{z^m}^\beta$, $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, then for each $k, j \in \mathbb{Z}$

$$\frac{\beta_{2j-1}}{\beta_{k+j}}\gamma_{k+j,4j-1} = \frac{\beta_{2j}}{\beta_{-(k-j)}}\gamma_{k-j,4j} = \frac{\beta_0}{\beta_{-k}}\gamma_{k,0} \quad = \begin{cases} \overline{a}_{k+\frac{m}{2}} & \text{if } m \textit{ is even} \\ 0 & \text{if } m \textit{ is odd} \end{cases}$$

and

$$\frac{\beta_{2j}}{\beta_{k+j}}\gamma_{k+j,4j+1} = \frac{\beta_{2j-1}}{\beta_{-(k-j)}}\gamma_{k-j,4j-2} = \frac{\beta_0}{\beta_k}\gamma_{k,1} \quad = \quad \begin{cases} 0 & \text{if } m \textit{ is even} \\ \overline{a}_{k+(\frac{m-1}{2})} & \text{if } m \textit{ is odd.} \end{cases}$$

This proves that $\{\gamma_{i,j}\}_{i,j\in\mathbb{Z}}$ is a *weighted slant Toep-Hank* matrix. This completes the proof.

□

References

[1] S. C. Arora, R. Batra and M. P. Singh, Slant Hankel Operators, *Archivium Mathematicum(BRNO) Tomus* **42***(2006), 125-133.*

[2] S. C. Arora and R. Kathuria, Properties of the slant weighted Toeplitz operator, *Ann. Funct. Analy.* **2(1)***(2011), 19-30.*

[3] G. Datt and D. K. Porwal, On weighted slant Hankel operators, *Filomat* **27(2)***(2013), 227-243.*

[4] G. Datt and A. Mittal, An extension of the notion induced by Toeplitz and Hankel operators, *International journal of Mathematical Analysis* **12, (3)***(2018), 145-156.*

[5] T. Goodman, C. Micchelli and J. Ward, Spectral radius formula for subdivision operators, Recent advances in Wavelet Analysis, Edited by L. Schumaker and G. Webb, Acedemic Press, 1994.

[6] H. Guediri, Dual Toeplitz operators on the sphere, *Acta Math. Sin. (Engl. Ser.)* **29(9)***(2013), 1791-1808.*

[7] H. Guediri, Product of Toeplitz and Hankel operators on the Hardy space of the unit sphere, *Oper. Theory Adv. Appl.* **236***(2014), 243-256.*

[8] M. C. Ho, Properties of Slant Toeplitz operators, *Indiana Univ. Math. J.* **45***(1996), 843-862.*

[9] V. Lauric, On a Weighted Toepliz operator and its commutant, *International Jr. Math. and Math. Sc.* **6***(2005), 823-835.*

[10] V. Peller and S. V. Khrushchev, Hankel operators, best approximatins and stationary Gaussian processes, *Russian Math. Surveys* **37(1)***(1982), 61-144.*

[11] A. L. Shields, Weighted shift operators and analytic function theory, Topics in Operator Theory, Math. Surveys, no.13, American Mathematical Society, Rhode Ireland, 1974.

[12] N. Zorboska, Compact composition operators on some weighted Hardy spaces, *Journal of operator theory* **22***(1989), 233-241.*