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Weighted composition operators on weighted Bergman spaces and weighted Bloch spaces

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Abstract. In this paper, we characterize the bonudedness and compactness of weighted composition operators from weighted Bergman spaces to weighted Bloch spaces. Also, we investigate weighted composition operators on weighted Bergman spaces and extend the obtained results in the unit ball of $\mathbb{C}^n$.

Keywords: Weighted composition operators, weighted Bergman spaces, weighted Bloch spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit ball in $\mathbb{C}$ and $H(\mathbb{D})$ denotes the class of all analytic functions on $\mathbb{D}$. Let w and v be strictly positive bounded continuous functions on $\mathbb{D}$. These types of functions are called *weights*. For $p > 0$ and a weight w the weighted Bergman space $A_{w,p}$ consists of all $f \in H(\mathbb{D})$ for which

$$||f||_{w,p}^p = \int_{\mathbb{D}} w(z)|f(z)|^p \, dA(z) < \infty,$$

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where $dA(z)$ is the normalized area measure on $\mathbb{D}$. Also the weighted Bloch space is defined by

$$B_v = \{f \in H(\mathbb{D}) : ||f||_{B_v} = \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty\}.$$

By letting $w_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$ and $v_\alpha(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, we get to the standard weighted Bergman space $A_{w_\alpha,p} = A^p_\alpha$ (if $\alpha = 0$, then $A^p_\alpha = A^p$ and $||.||_{w_\alpha,p} = ||.||_p$) and α-Bloch space $\mathcal{B}^\alpha$. Moreover the little weighted Bloch space defined as follows

$$B^0_v = \{f \in B_v : \lim_{|z|\to 1} v(z)|f'(z)| = 0\}.$$

A weight v is called *radial* if $v(z) = v(|z|)$. In the sequel we will consider the following weights. Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing, strictly positive, decreasing on $[0, 1)$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the corresponding weight v as follows

$$v(z) = \nu(|z|^2) \tag{1.1}$$

for every $z \in \mathbb{D}$. Here are some examples:

- **i:** If $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$ then $v(z) = (1-|z|^2)^\alpha$ (standard weight).
- **ii:** If $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$ then $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.
- **iii:** If $\nu(z) = \sin(1-z)$, then $v(z) = \sin(1-|z|^2)$.
- **iv:** If $\nu(z) = \frac{1}{1-\log(1-z)}$, then $v(z) = \frac{1}{1-\log(1-|z|^2)}$

For a fixed point $a \in \mathbb{D}$ we introduce a function $v_a(z) := \nu(\overline{a}z)$ for every $z \in \mathbb{D}$. Since ν is holomorphic on $\mathbb{D}$, so is the function v_a. In particular we assume that there is a positive constant c such that for $\varphi \in H(\mathbb{D})$

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)}{\nu(\overline{\varphi(a)}z)} \leq c. \tag{1.2}$$

See [13], for more examples about these weights.

For two analytic functions ψ and φ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, the well-known weighted composition operator ψC_φ is defined by $\psi C_\varphi(f) = \psi f \circ \varphi$, for any $f \in H(\mathbb{D})$, which is the generalization of composition operator and multiplication operator. Good references for composition operators are excellent books [1, 7]. Weighted composition operators are a general class of operators and they appear naturally in the study of surjective isometries on most of the function spaces, semigroup theory, dynamical systems, Brennan's conjecture, etc. There are lots of papers concerning the relation between operator theoretic properties of this operator acting on spaces of analytic functions with function theoretic properties of the symbols ψ and φ (see, e.g. [2, 4, 5, 8, 9, 10, 11, 12, 13, 14]).

Motivated by the results of [8, 9, 10, 11, 12], we are going to investigate weighted composition operators on weighted Bergman spaces and weighted Bloch spaces. We give a characterization for boundedness and compactness of the weighted composition operators from (weighted) Bergman spaces to weighted Bloch spaces and also between weighted Bergman spaces. The same work has been done between classical Bergman and Bloch spaces in [9]. In case of weighted Bergman spaces, we extend the obtained results to the unit ball of $\mathbb{C}^n$.

Throughout the paper c denotes a positive constant not necessarily the same at each occurrence. The notation $A \approx B$ means that there is a positive constant c such that $A/c \leq B \leq cA$.

2. Between (weighted) Bergman spaces and weighted Bloch spaces

The proof of the following theorems is similar to the one in Theorems 3.1 and 3.2 in [9], so we omit it.

Theorem 2.1. Let $p \geq 1$. Then ψC_φ is bounded from $A_{w_\alpha,p}$ into B_v if and only if

(1) $\sup_{z\in\mathbb{D}} \dfrac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} < \infty$

(2) $\sup_{z\in\mathbb{D}} \dfrac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+(2+\alpha)/p}} < \infty.$

For the compactness, we have the following theorem.

Theorem 2.2. Let $p \geq 1$ and ψC_φ is bounded from $A_{w_\alpha,p}$ into B_v. Then $\psi C_\varphi : A_{w_\alpha,p} \longrightarrow B_v$ is a compact operator if and only if

(1) $\lim_{|\varphi(z)|\to 1} \dfrac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} = 0$

(2) $\lim_{|\varphi(z)|\to 1} \dfrac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+(2+\alpha)/p}} = 0.$

If $\alpha = 0$ and $v(z) = 1-|z|^2$, then the results of Sharma and Kumari in [9] are derived. If $\alpha = 0$ and $v(z) = (1-|z|^2)\log\frac{2}{1-|z|^2}$, the results of Li in [4] are derived.

Lemma 2.3. [11] *Let w be a weight of the form $w = |u|$ where u is an analytic function without any zeros on $\mathbb{D}$. Then*

$$|f(z)| \leq \frac{||f||_{w,p}}{(1-|z|^2)^{2/p} w(z)^{1/p}}, \tag{2.1}$$

for all $z \in \mathbb{D}$ and $f \in A_{w,p}$.

Lemma 2.4. [14] *Let w be a weight of the form $w = |u|$ where u is an analytic function without any zeros on $\mathbb{D}$. Then there exists a positive constat c such that*

$$|f'(z)| \leq c\frac{||f||_{w,p}}{(1-|z|^2)^{1+2/p}w(z)^{1/p}} + \frac{1}{p}\frac{|u'(z)|\ ||f||_{w,p}}{(1-|z|^2)^{2/p}w(z)^{1+1/p}} \qquad (2.2)$$

Theorem 2.5. [14] Let w be a weight of the form $w = |u|$ where u is an analytic function without any zeros on $\mathbb{D}$. If

(1) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p}w(\varphi(z))^{1/p}} < \infty$,

(2) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} < \infty$,

(3) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|\ |u'(\varphi(z))|}{(1-|\varphi(z)|^2)^{2/p}w(\varphi(z))^{1+1/p}} < \infty$,

then the weighted composition operator $\psi C_\varphi : A_{w,p} \longrightarrow B_v$ is bounded.

Theorem 2.6. Suppose that w is a weight that satisfies conditions (1.1) and (1.2), ($w(z) = \mu(|z|^2)$) and v be a weight. If $\psi C_\varphi : A_{w,p} \longrightarrow B_v$ be bounded and

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|\ |\mu'(|\varphi(z)|^2)|}{(1-|\varphi(z)|^2)^{2/p}w(\varphi(z))^{1+1/p}} < \infty, \qquad (2.3)$$

then

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p}w(\varphi(z))^{1/p}} < \infty$$

and

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} < \infty. \qquad (2.4)$$

Proof. suppose that $\psi C_\varphi : A_{w,p} \longrightarrow B_v$ be bounded and the condition (2.3) holds. By letting f as a constant function we have $\psi \in B_v$ and letting $f(z) = z$, we have $\sup_{z\in\mathbb{D}} v(z)|\psi(z)\varphi'(z)| < \infty$. Fix $a \in \mathbb{D}$ and define f_a as follows

$$f_a(z) = \left(\frac{(1-|\varphi(a)|^2)^2}{\mu(\overline{\varphi(a)}z)(1-\overline{\varphi(a)}z)^4}\right)^{1/p}.$$

Then $f_a \in A_{w,p}$ and

$$f'_a(\varphi(a)) = \frac{1}{p}\frac{-\overline{\varphi(a)}\mu'(|\varphi(a)|^2)\ (1-|\varphi(a)|^2) + 4\overline{\varphi(a)}\ w(\varphi(a))}{w(\varphi(a))^{1+1/p}(1-|\varphi(a)|^2)^{1+2/p}}.$$

Since $\psi C_\varphi : A_{w,p} \longrightarrow B_v$ is bounded, there exists a positive constant M such that

$$M \geq ||\psi C_\varphi f_a||_{B_v} \geq v(a)|\psi'(a)f_a(\varphi(a)) + \psi(a)\varphi'(a)f'_a(\varphi(a))|.$$

Therefore

$$M+\frac{1}{p}\frac{v(a)|\psi(a)\varphi'(a)||\varphi(a)|\ |\mu'(|\varphi(a)|^2)|}{w(\varphi(a))^{1+1/p}(1-|\varphi(a)|^2)^{2/p}}+\frac{1}{p}\frac{4v(a)|\psi(a)\varphi'(a)|\ |\varphi(a)|}{w(\varphi(a))^{1/p}(1-|\varphi(a)|^2)^{1+2/p}}$$
$$\geq\frac{v(a)|\psi'(a)|}{w(\varphi(a))^{1/p}(1-|\varphi(a)|^2)^{2/p}}.$$

So it will be sufficient to prove that the condition (2.4) holds. We define another function g_a by

$$g_a(z)=\frac{(1-|\varphi(a)|^2)^{4/p}}{\mu(\overline{\varphi(a)}z)^{1/p}(1-\overline{\varphi(a)}z)^{6/p}}-\frac{(1-|\varphi(a)|^2)^{2/p}}{\mu(\overline{\varphi(a)}z)^{1/p}(1-\overline{\varphi(a)}z)^{4/p}}.$$

Then $g_a\in A_{w,p}$, $g_a(\varphi(a))=0$ and

$$g_a'(\varphi(a))=\frac{2}{p}\frac{\overline{\varphi(a)}}{w(\varphi(a))^{1/p}(1-|\varphi(a)|^2)^{1+2/p}}.$$

Since $\psi C_\varphi : A_{w,p}\longrightarrow B_v$ is bounded, there exists a positive constant M' such that

$$\begin{aligned}M'\geq ||\psi C_\varphi g_a||_{B_v}\geq & v(a)|\psi'(a)g_a(\varphi(a))+\psi(a)\varphi'(a)g_a'(\varphi(a))|\\ =&\frac{2}{p}\frac{v(a)\ |\varphi(a)|\ |\psi(a)\varphi'(a)|}{w(\varphi(a))^{1/p}(1-|\varphi(a)|^2)^{1+2/p}}.\end{aligned}$$

Therefore

$$\sup_{a\in\mathbb{D}}\frac{v(a)|\varphi(a)|\ |\psi(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{1+2/p}w(\varphi(a))^{1/p}}<\infty,$$

because a is arbitrary. Now fix $0<\delta<1$. Then

$$\sup_{\{a\in\mathbb{D},\ |\varphi(a)|>\delta\}}\frac{v(a)|\psi(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{1+2/p}w(\varphi(a))^{1/p}}<\infty, \tag{2.5}$$

and if $|\varphi(a)|\leq\delta$, then

$$\begin{aligned}&\sup_{\{a\in\mathbb{D},\ |\varphi(a)|\leq\delta\}}\frac{v(a)|\psi(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{1+2/p}w(\varphi(a))^{1/p}}\\ &\leq\frac{1}{(1-\delta^2)^{1+2/p}\mu(\delta^2)^{1/p}}\sup_{a\in\mathbb{D}}v(a)|\psi(a)\varphi'(a)|<\infty.\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\sup_{z\in\mathbb{D}}\frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}}<\infty,$$

so the condition (2.4) follows and proof will be complete. □

Theorem 2.7. Let w be a weight of the form $w=|u|$ where u is an analytic function without any zeros on $\mathbb{D}$ and $\psi C_\varphi : A_{w,p}\to B_v$ be bounded. If

$$(1)\ \lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1/p}} = 0,$$
$$(2)\ \lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p} w(\varphi(z))^{1/p}} = 0,$$
$$(3)\ \lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|\ |u'(\varphi(z))|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1+1/p}} = 0,$$

then the weighted composition operator $\psi C_\varphi : A_{w,p} \to B_v$ is compact.

Proof. Suppose that the conditions (1), (2) and (3) are true and $\{f_n\}$ be a bounded sequence in $A_{w,p}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. We want to prove $||\psi C_\varphi f_n||_{B_v} \to 0$. For $\epsilon > 0$, there exists $\delta > 0$ such that if $|\varphi(z)| > \delta$, then

$$\frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1/p}} < \epsilon,$$
$$\frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p} w(\varphi(z))^{1/p}} < \epsilon,$$
$$\frac{v(z)|\psi(z)\varphi'(z)|\ |u'(\varphi(z))|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1+1/p}} < \epsilon.$$

For $z \in \mathbb{D}$ and $|\varphi(z)| > \delta$, from (2.1) and (2.2) we get

$$\begin{aligned} v(z)|(\psi C_\varphi f_n)'(z)| =& v(z)|\psi'(z) f_n(\varphi(z)) + \psi(z)\varphi'(z) f_n'(\varphi(z))| \\ \leq& \frac{v(z)|\psi'(z)|\ ||f_n||_{w,p}}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1/p}} \\ &+ c\frac{v(z)|\psi(z)\varphi'(z)|\ ||f_n||_{w,p}}{(1-|\varphi(z)|^2)^{1+2/p} w(\varphi(z))^{1/p}} \\ &+ \frac{1}{p}\frac{v(z)|\psi(z)\varphi'(z)|\ |u'(\varphi(z))|\ ||f_n||_{w,p}}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1+1/p}} \\ <& (1+c+\frac{1}{p})\epsilon M \end{aligned}$$

where $M = \sup_{n\in\mathbb{N}} ||f_n||_{w,p}$. On the other hand $f_n \to 0$ and $f_n' \to 0$ uniformly on the compact set $\{t : |t| \leq \delta\}$. Thus there exists n_0 such that for $|\varphi(z)| \leq \delta$ and $n \geq n_0$, we have $|f_n(\varphi(z))| < \epsilon$ and $|f_n'(\varphi(z))| < \epsilon$.

Hence for $n \geq n_0$

$$\begin{aligned}||\psi C_\varphi f_n||_{B_v} =& \sup_{z\in\mathbb{D}} v(z)|(\psi C_\varphi f_n)'(z)| \\ \leq& \sup_{\{z:|\varphi(z)|>\delta\}} v(z)|(\psi C_\varphi f_n)'(z)| + \sup_{\{z:|\varphi(z)|\leq\delta\}} v(z)|(\psi C_\varphi f_n)'(z)| \\ <& (1+c+\frac{1}{p})\epsilon M + \epsilon \sup_{\{z:|\varphi(z)|\leq\delta\}} v(z)|\psi'(z)| \\ &+ \epsilon \sup_{\{z:|\varphi(z)|\leq\delta\}} v(z)|\psi(z)\varphi'(z)|,\end{aligned}$$

two last supremum are finite, because $\psi C_\varphi : A_{w,p} \to B_v$ is bounded. It follows that $||\psi C_\varphi f_n||_{B_v} \to 0$. Therefore ψC_φ is compact. □

Theorem 2.8. Suppose that w is a weight as in Theorem 2.6. If $\psi C_\varphi : A_{w,p} \to B_v$ be compact and

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|\ |\mu'(|\varphi(z)|^2)|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1+1/p}} = 0,$$

then

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1/p}} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p} w(\varphi(z))^{1/p}} = 0. \tag{2.7}$$

Proof. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$. Define

$$f_n(z) = \left(\frac{(1-|\varphi(z_n)|^2)^2}{\mu(\overline{\varphi(z_n)}z)(1-\overline{\varphi(z_n)}z)^4}\right)^{1/p}.$$

Then $f_n \in A_{w,p}$, $||f_n||_{w,p} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\psi C_\varphi : A_{w,p} \to B_v$ is compact, then $||\psi C_\varphi f_n||_{B_v} \to 0$. On the other hand

$$||\psi C_\varphi f_n||_{B_v} \geq v(z_n)|\psi'(z_n) f_n(\varphi(z_n)) + \psi(z_n)\varphi'(z_n) f_n'(\varphi(z_n))|.$$

So

$$\begin{aligned}||\psi C_\varphi f_n||_{B_v} &+ \frac{1}{p} \frac{v(z_n)|\psi(z_n)\varphi'(z_n)|\ |\mu'(|\varphi(z_n)|^2)|}{w(\varphi(z_n))^{1+1/p}(1-|\varphi(z_n)|^2)^{2/p}} \\ &+ \frac{1}{p} \frac{4v(z_n)|\psi(z_n)\varphi'(z_n)|\ |\varphi(z_n)|}{w(\varphi(z_n))^{1/p}(1-|\varphi(z_n)|^2)^{1+2/p}} \\ &\geq \frac{v(z_n)|\psi'(z_n)|}{w(\varphi(z_n))^{1/p}(1-|\varphi(z_n)|^2)^{2/p}}.\end{aligned}$$

It will be enough to prove the condition (2.7) holds. Define another function

$$g_n(z) = \frac{(1-|\varphi(z_n)|^2)^{4/p}}{\mu(\overline{\varphi(z_n)}z)^{1/p}(1-\overline{\varphi(z_n)}z)^{6/p}} - \frac{(1-|\varphi(z_n)|^2)^{2/p}}{\mu(\overline{\varphi(z_n)}z)^{1/p}(1-\overline{\varphi(z_n)}z)^{4/p}}.$$

Then $\{g_n\}$ is a bounded sequence in $A_{w,p}$, converges to 0 uniformly on compact subsets of $\mathbb{D}$, $g_n(\varphi(z_n)) = 0$ and

$$g_n'(\varphi(z_n)) = \frac{2}{p}\frac{\overline{\varphi(z_n)}}{w(\varphi(z_n))^{1/p}(1-|\varphi(z_n)|^2)^{1+2/p}}.$$

Since $\psi C_\varphi : A_{w,p} \to B_v$ is compact,

$$0 \leftarrow ||\psi C_\varphi g_n||_{B_v} \geq \frac{2}{p}\frac{v(z_n)\ |\varphi(z_n)|\ |\psi(z_n)\varphi'(z_n)|}{w(\varphi(z_n))^{1/p}(1-|\varphi(z_n)|^2)^{1+2/p}}.$$

Thus

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} = 0.$$

□

Corollary 2.9. *Putting* $\mu(z) = (1-z)^\alpha$, $\alpha \geq 1$, *we have* $|\mu'(|\varphi(z)|^2)| = \alpha(1-|\varphi(z)|^2)^{\alpha-1}$. *Now if* $\psi C_\varphi : A_{w,p} \to B_v$ *is bounded and*

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|\ |\mu'(|\varphi(z)|^2)|}{(1-|\varphi(z)|^2)^{2/p}w(\varphi(z))^{1+1/p}} = \alpha \sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+(2+\alpha)/p}} < \infty, \tag{2.8}$$

then by Theorem 2.6,

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} < \infty$$

and

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+(2+\alpha)/p}} < \infty.$$

So we can omit the condition (2.8) and the "only if" part of Theorem 2.1 is derived. Also by these setting we can obtain the "only if" part of Theorem 2.2.

In the same way as in main results, we can prove some results for the weighted composition operators from weighted Bergman spaces into little weighted Bloch spaces.

3. Weighted composition operators on weighted Bergman spaces

Lemma 3.1. [10] *Let v be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in \mathbb{D}$) such that*

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)|v_a(\sigma_a(z))|}{v(\sigma_a(z))} \leq C < \infty.$$

Then

$$|f(z)| \leq \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}||f||_{v,p}$$

for every $z \in \mathbb{D}$ and $f \in A_{v,p}$.

Here $\sigma_a(z)$ is the Mobius transformation of $\mathbb{D}$ which interchanges 0 and a, that is

$$\sigma_a(z) = \frac{a-z}{1-\overline{a}z}, \qquad a, z \in \mathbb{D}.$$

Lemma 3.2. [10] *Let w be a weight and v be a weight as in Lemma 3.1. If*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)^{\frac{1}{p}}}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \infty, \tag{3.1}$$

then the operator $\psi C_\varphi : A_{v,p} \longrightarrow A_{w,p}$ is bounded.

We want to prove converse of this lemma.

Theorem 3.3. *Let u and μ be two analytic functions on $\mathbb{D}$ such that $u(z) \neq 0$ on $\mathbb{D}$. Put $v(z) = |u(z)|$ and $w(z) = |\mu(z)|$, $z \in \mathbb{D}$. If $\psi C_\varphi : A_{v,p} \longrightarrow A_{w,p}$ be a bounded operator, then*

$$\sup_{z\in\mathbb{D}} \frac{w(z)^{\frac{1}{p}}|\psi(z)|(1-|z|^2)^{\frac{2}{p}}}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \infty. \tag{3.2}$$

Proof. Fix $a \in \mathbb{D}$ and define

$$f_a(z) = \frac{1}{u(z)^{\frac{1}{p}}}(-\sigma'_{\varphi(a)}(z))^{\frac{2}{p}}.$$

Then $f_a \in A_{v,p}$ and $||f_a||_{v,p} = 1$. Since $\psi C_\varphi : A_{v,p} \longrightarrow A_{w,p}$ is bounded and $||f_a||_{v,p} \leq 1$, there exists a positive constant M such that

$$M \geq ||\psi C_\varphi(f_a)||^p_{w,p} = \int_{\mathbb{D}} w(z)|\psi(z)|^p|f_a(\varphi(z))|^p \, dA(z).$$

Subharmonicity of the function $|\mu(z)||\psi(z)|^p|f_a(\varphi(z))|^p$ implies that

$$\begin{aligned}\frac{w(a)|\psi(a)|^p}{(1-|\varphi(a)|^2)^2 v(\varphi(a))} =&|\mu(a)||\psi(a)|^p|f_a(\varphi(a))|^p\\ \leq&\frac{4}{(1-|a|^2)^2}\int_{E(a)}|\mu(z)||\psi(z)|^p|f_a(\varphi(z))|^p\ dA(z)\\ \leq&M\frac{4}{(1-|a|^2)^2},\end{aligned}$$

where $E(a)=\{z\in\mathbb{D}:\ |z-a|\leq\frac{1-|a|^2}{2}\}$. Hence, we have

$$\frac{w(a)|\psi(a)|^p(1-|a|^2)^2}{(1-|\varphi(a)|^2)^2 v(\varphi(a))}<4M.$$

Since a is arbitrary, the condition (3.2) follows. □

Theorem 3.4. *Let w be a weight and v be a weight as in Lemma 3.1, and $\psi C_\varphi : A_{v,p}\longrightarrow A_{w,p}$ be a bounded operator. If*

$$\lim_{|\varphi(z)|\to 1}\frac{w(z)^{\frac{1}{p}}|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0, \tag{3.3}$$

then $\psi C_\varphi : A_{v,p}\longrightarrow A_{w,p}$ is compact.

Proof. Suppose that the condition (3.3) holds. Let $\{f_n\}$ be a bounded sequence in $A_{v,p}$ convergent to zero uniformly on compact subsets of $\mathbb{D}$. Set $M=\sup_n||f_n||_{v,p}$. For every $\epsilon>0$, there exists $0<r<1$, such that $r<|\varphi(z)|<1$ implies

$$\frac{w(z)^{\frac{1}{p}}|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}<\epsilon.$$

So

$$\int_{\{z:|\varphi(z)|>r\}}\frac{w(z)|\psi(z)|^p||f_n||^p_{v,p}}{(1-|\varphi(z)|^2)^2 v(\varphi(z))}\ dA(z)<M\epsilon^p. \tag{3.4}$$

On the other hand $f_n\to 0$ uniformly on the compact set $\{t:|t|\leq r\}$. Then there exists $n_0\in\mathbb{N}$ such that if $|\varphi(z)|\leq r$ and $n\geq n_0$, $|f_n(\varphi(z))|<\epsilon$. Then

$$\int_{\{z:|\varphi(z)|\leq r\}}w(z)|\psi(z)|^p|f_n(\varphi(z))|^p\ dA(z)<\epsilon^p\int_{\mathbb{D}}w(z)|\psi(z)|^p\ dA(z). \tag{3.5}$$

The last integral is finite, since $\psi C_\varphi : A_{v,p}\longrightarrow A_{w,p}$ is bounded. It follows from (3.4) and (3.5) that $||\psi C_\varphi(f_n)||_{w,p}\to 0$ and $\psi C_\varphi : A_{v,p}\to A_{w,p}$ is compact. □

The converse of the above theorem is not true. Take $\psi(z) = v = w \equiv 1$ and φ be a lens map as defined on page 27 of [7]. Then $\psi C_\varphi = C_\varphi$ is compact on A^p. But the condition (3.3) does not hold.

Theorem 3.5. *Let u and μ be two analytic functions on $\mathbb{D}$ such that $u(z) \neq 0$ on $\mathbb{D}$. Put $v(z) = |u(z)|$ and $w(z) = |\mu(z)|$, $z \in \mathbb{D}$. If $\psi C_\varphi : A_{v,p} \longrightarrow A_{w,p}$ be a compact operator, then*

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)^{\frac{1}{p}}|\psi(z)|(1-|z|^2)^{\frac{2}{p}}}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} = 0, \tag{3.6}$$

Proof. Suppose that $\psi C_\varphi : A_{v,p} \longrightarrow A_{w,p}$ be compact and condition (3.6) doesn't hold. Then there exist $\epsilon > 0$ and sequence $\{z_n\}$ in $\mathbb{D}$ with $|\varphi(z_n)| \to 1$ such that for any n

$$\frac{w(z_n)^{\frac{1}{p}}|\psi(z_n)|(1-|z_n|^2)^{\frac{2}{p}}}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}} \geq \epsilon.$$

Since $|\varphi(z_n)| \to 1$, there exists a sequence $\{\alpha_n\}$ of natural numbers such that $\lim_{n\to\infty} \alpha_n = \infty$ and $|\varphi(z_n)|^{\alpha_n} \geq \frac{1}{2}$. Letting

$$g_n(z) = \frac{1}{u(z)^{\frac{1}{p}}}(-\sigma'_{\varphi(z_n)}(z))^{\frac{2}{p}} z^{\alpha_n}$$

this is a norm bounded sequence in $A_{v,p}$ and $g_n \to 0$, pointwise on $\mathbb{D}$. So there exists a subsequence of $\{\psi C_\varphi g_n\}$ convergent to 0. Putting

$$E(z_n) = \{z \in \mathbb{D} : \ |z - z_n| \leq \frac{1-|z_n|^2}{2}\}$$

for any $n \in \mathbb{N}$, by subharmonicity property we have

$$\begin{aligned}
||\psi C_\varphi g_n||^p_{w,p} &= \int_{\mathbb{D}} w(z)|\psi(z)|^p |g_n(\varphi(z))|^p \; dA(z) \\
&\geq \int_{E(z_n)} w(z)|\psi(z)|^p |g_n(\varphi(z))|^p \; dA(z) \\
&\geq |\mu(z_n)||\psi(z_n)|^p |g_n(\varphi(z_n))|^p \frac{(1-|z_n|^2)^2}{4} \\
&= \frac{w(z_n)|\psi(z_n)|^p |\varphi(z_n)|^{\alpha_n}(1-|z_n|^2)^2}{4(1-|\varphi(z_n)|^2)^2 v(\varphi(z_n))} \\
&\geq \frac{1}{8}\epsilon^p,
\end{aligned}$$

which is a contradiction. □

In Theorem 3.3, if we take $p = 2$ and $v(z) = w(z) = 1$, then

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)|}{1-|\varphi(z)|^2} < \infty.$$

So we obtain to the following result of Čučković and Zhao [2].

Corollary 3.6. *If the weighted composition operator* ψC_φ *is bounded on the Bergman spaces, then*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)|}{1-|\varphi(z)|^2} < \infty.$$

Here we have some examples for the main results:

Example 3.7. For $\psi(z) = v = w \equiv 1$ and $\varphi(z) = id$, the identity map on $\mathbb{D}$, the corresponding weighted composition operator equals the identity in A^p, which is bounded. Also the condition (3.2) is satisfied, since

$$\sup_{z\in\mathbb{D}} \frac{w(z)^{\frac{1}{p}}|\psi(z)|(1-|z|^2)^{\frac{2}{p}}}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} = 1 < \infty.$$

Example 3.8. If $p = 1$, $\psi(z) = 1$, $\varphi(z) = \frac{z+1}{2}$, $w(z) = |1-z|$ and $v(z) = |1-z|^2$, then for every real number $z = t \in \mathbb{D}$

$$\frac{w(t)|\psi(t)|(1-t^2)^2}{(1-|\varphi(t)|^2)^2 v(\varphi(t))} = \frac{(1-t)(1-t^2)^2}{(1-(\frac{t+1}{2})^2)^2(1-\frac{t+1}{2})^2},$$

which tends to ∞, if t tends to 1. It follows from Theorem 3.3 that the weighted composition operator ψC_φ is not bounded.

Example 3.9. For $p = 1$, $\psi(z) = 1-z$, $\varphi(z) = z$, $v(z) = |1-z|^2$ and $w(z) = |1-z|$, we have

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|\psi(z)|(1-|z|^2)^2}{(1-|\varphi(z)|^2)^2 v(\varphi(z))} = 1 \neq 0.$$

So the weighted composition operator ψC_φ is not compact (by Theorem 3.5).

4. Generalization to the unit ball of $\mathbb{C}^N$

In this section we generalize the main results to the previous section to the unit ball of $\mathbb{C}^N$.

For $z = (z_1, \cdots, z_N)$ and $a = (a_1, \cdots, a_N)$ in $\mathbb{C}^N$, we define $\langle z, a\rangle = z_1\overline{a_1} + \cdots + z_N\overline{a_N}$, where $\overline{a_k}$ is the complex conjugate of a_k. We also write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_N|^2}$. Let B_N denotes the open unit ball of $\mathbb{C}^N$, that is

$$B_N = \{z \in \mathbb{C}^N : |z| < 1\},$$

and $H(B_N)$ be the set of all analytic functions on B_N. For any $a \in B_N - \{0\}$, we define

$$\sigma_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle} \qquad z \in B_N,$$

where $s_a = \sqrt{1-|a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^N$ onto the subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^N$ onto $\mathbb{C}^N - [a]$. When $a = 0$, write $\varphi_a(z) = -z$. These functions are called involutions.

Let ψ and φ be analytic functions on B_N such that $\varphi(B_N) \subset B_N$. Then the weighted composition operator ψC_φ is defined by $\psi C_\varphi(f) = \psi f \circ \varphi$, for any $f \in H(B_N)$. For $1 \leq p < \infty$ and a weight (strictly positive bounded continuous function) w on B_N, the weighted Bergman space $A_{w,p}(B_N)$ consists of all $f \in H(B_N)$ for which

$$||f||_{w,p}^p = \int_{B_N} w(z)|f(z)|^p \, dV(z) < \infty,$$

where $dV(z)$ is the normalized Lebesgue measure on B_N such that $V(B_N) = 1$.

As in the first section, we use weights of the following type. Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing, strictly positive on $[0,1)$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the corresponding weight v as follows

$$v(z) := \nu(\langle z, z\rangle) = \nu(|z|^2) \quad z \in B_N.$$

For a fixed point $a \in B_N$ we introduce a function $v_a(z) := \nu(\langle z, a\rangle)$ for every $z \in B_N$. Since ν is holomorphic on B_N, so is the function v_a. (see [13])

Lemma 4.1. [12] *Let v be a weight as we defined above such that $\sup_{z\in B_N} \sup_{a\in B_N} \frac{v(z)|v_a(\sigma_a(z))|}{v(\sigma_a(z))} \leq C < \infty$. Then*

$$|f(z)| \leq \frac{C^{\frac{1}{p}}}{\left(\int_{B_N} v(t)dV(t)\right)^{\frac{1}{p}} (1-|z|^2)^{\frac{N+1}{p}} v(z)^{\frac{1}{p}}} ||f||_{v,p}$$

for every $z \in B_N$ and every $f \in A_{v,p}(B_N)$.

Proposition 4.2. [12] *Let w be a weight and v be a weight as in Lemma 4.1. If*

$$\sup_{z\in B_N} \frac{|\psi(z)|w(z)^{\frac{1}{p}}}{(1-|\varphi(z)|^2)^{\frac{N+1}{p}} v(\varphi(z))^{\frac{1}{p}}} < \infty, \tag{4.1}$$

then the operator $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ is bounded.

Theorem 4.3. *Let u and μ be two analytic functions on B_N such that $u(z) \neq 0$ on B_N. Put $v(z) = |u(z)|$ and $w(z) = |\mu(z)|$, $z \in B_N$. If $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ be a bounded operator, then*

$$\sup_{z\in B_N} \frac{w(z)^{\frac{1}{p}}|\psi(z)|(1-|z|^2)^{\frac{N+1}{p}}}{(1-|\varphi(z)|^2)^{\frac{N+1}{p}} v(\varphi(z))^{\frac{1}{p}}} < \infty.$$

Proof. Suppose that $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ be bounded. Fix $a \in B_N$ and define

$$f_a(z) = \frac{1}{u(z)^{\frac{1}{p}}} \left(\frac{1 - |\varphi(a)|^2}{(1 - \langle z, \varphi(a) \rangle)^2} \right)^{\frac{N+1}{p}}.$$

Then $f_a \in A_{v,p}(B_N)$ and $||f_a||_{v,p} = 1$. Since $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ is bounded, there exists a positive constant M such that

$$M \geq ||\psi C_\varphi(f_a)||^p_{w,p} = \int_{B_N} w(z)|\psi(z)|^p |f_a(\varphi(z))|^p \, dV(z).$$

Theorem 2.1 of [15] implies that

$$\begin{aligned} \frac{w(a)|\psi(a)|^p}{(1-|\varphi(a)|^2)^{N+1} v(\varphi(a))} =& |\mu(a)||\psi(a)|^p |f_a(\varphi(a))|^p \\ \leq& \frac{1}{(1-|a|^2)^{N+1}} \int_{B_N} |\mu(z)||\psi(z)|^p |f_a(\varphi(z))|^p \, dV(z) \\ \leq& M \frac{1}{(1-|a|^2)^{N+1}}. \end{aligned}$$

Hence, we have

$$\frac{w(a)|\psi(a)|^p (1-|a|^2)^{N+1}}{(1-|\varphi(a)|^2)^{N+1} v(\varphi(a))} \leq M.$$

Now the proof is complete. □

In the same way as in Theorems 3.4 and 3.5, we can prove the similar result for the compactness.

Theorem 4.4. *Let w be a weight and v be a weight as in Lemma 4.1, and $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ be a bounded operator. If*

$$\lim_{|\varphi(z)| \to 1} \frac{w(z)^{\frac{1}{p}} |\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{N+1}{p}} v(\varphi(z))^{\frac{1}{p}}} = 0, \tag{4.2}$$

then $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ is compact.

Theorem 4.5. *Let u and μ be two analytic functions on B_N such that $u(z) \neq 0$ on B_N. Put $v(z) = |u(z)|$ and $w(z) = |\mu(z)|$, $z \in B_N$. If $\psi C_\varphi : A_{v,p}(B_N) \longrightarrow A_{w,p}(B_N)$ be a compact operator, then*

$$\lim_{|\varphi(z)| \to 1} \frac{w(z)^{\frac{1}{p}} |\psi(z)| (1-|z|^2)^{\frac{N+1}{p}}}{(1-|\varphi(z)|^2)^{\frac{N+1}{p}} v(\varphi(z))^{\frac{1}{p}}} = 0. \tag{4.3}$$

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