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# Translation surfaces according to a new frame

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Abstract. In this paper we studied the translation surfaces according to a new frame called q-frame in three dimensional Euclidean space. The curvatures of the translation surface are obtained in terms of q-frame curvatures. Finally some special cases are investigated for these surfaces.

Keywords: Translation surfaces, Gauss curvature, q-frame.

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## 1. Introduction

The theory of translation surfaces is usually one of the interesting topics in the spaces. Translation surfaces have been investigated from the various viewpoints by many differential geometers. L. Verstraelen, J. Walrave and S. Yaprak have investigated minimal translation surfaces in n-dimensional Euclidean space [16]. H. Liu has given the classification of the translation surfaces with constant mean curvature or constant Gauss curvature in 3- dimensional Euclidean space $E^3$ and 3-dimensional Minkowski space $E_1^3$ [12]. M. I. Munteanu and A. I. Nistor have studied the second fundamental form of translation surfaces in $E^3$ [14]. Ali et

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al. have given a classification of some special points on translation surfaces in $E^3$ [1]. Çetin et al. have investigated the translation surfaces in 3-dimensional Euclidean space by using the non-planar space curves and they gave the differential geometric properties for both translation surfaces and minimal translation surfaces [6]. Moreover, Çetin et al. have contributed the translation surface with Frenet frames and Darboux frame and some differential geometric properties of the translation surfaces in [7].

A surface that can be generated from two space curves by translating either one of them parallel to itself is called a translation surface. The properties along a three-dimensional space curve, such as curvature and torsion, would be an advantage for mathematicians. The parallel transport frame [3], Serret-Frenet frame [4] and Darboux frame [8] provide such properties, however Serret-Frenet frame is not defined for all points along every space curve. Also, the rotation to a tangent of a general spine curve with Serret-Frenet frame can usually not be defined as the point at which undesirable twist occurs in the tubular surface modeling. Due to its minimal twist, Bishop frame is a frame which is needed for a kind of mathematical analysis that is frequently used in computer graphics. But, it is not easy to compute [17]. On the other hand, inspired by the 3D offset curve application of the quasi-normal vector $\mathbf{n}_q$ [5], Dede et al. have defined the q-frame along a space curve [9] and for D-tubular surface modeling [10]. The q-frame is more useful by virtue of disadvantages of Serret-Frenet frame [10].

In this paper, we have researched the translation surfaces according to q-frame in 3-dimensional Euclidean space. The curvatures of the translation surface are obtained in terms of q-frame curvatures. Finally, some special cases are investigated and some examples are given for these surfaces.

## 2. Preliminaries

In order to construct the q-fame, firstly, Coquillart [5] has introduced the quasi-normal vector $\mathbf{n}_q$ of a space curve. The quasi-normal vector is defined for each point of the curve and lies in the plane which is perpendicular to the tangent of the curve at this point [15]. Later, as an alternative to the Frenet frame we define a new adapted frame along a space curve, called as the q-frame. Given a regular space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors which are the unit tangent vector $\mathbf{t}$, the quasi-normal $\mathbf{n}_q$ and the quasi-binormal vector $\mathbf{b}_q$. The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ along $\alpha(t)$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q. \tag{2.1}$$

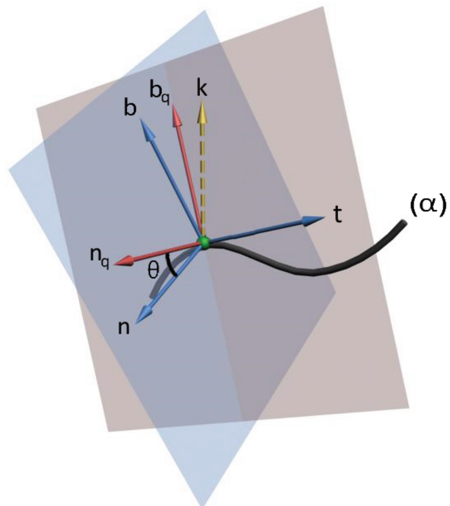


FIGURE 1. The q-frame and Frenet frame.

where **k** is the projection vector [9, 10]. The q-frame along a space curve is shown in Figure 1.

We can define Euclidean angle $\theta$ between the principal normal $\mathbf{n}$ and quasi-normal $\mathbf{n}_q$ vectors. Then, the relation matrix may be expressed as [9],

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2.2}$$

and

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{2.3}$$

Let $\alpha(s)$ be a curve that is parameterized by arc length $s$. The derivative formulas which correspond to q-frame are in the following form [9],

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \| \alpha' \| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{2.4}$$

where the q-curvatures are

$$k_1 = \frac{\langle \mathbf{t}', \mathbf{n}_q \rangle}{\|\alpha'\|}, \ k_2 = \frac{\langle \mathbf{t}', \mathbf{b}_q \rangle}{\|\alpha'\|}, \ k_3 = \frac{\langle \mathbf{n}_q', \mathbf{b}_q \rangle}{\|\alpha'\|}. \tag{2.5}$$

A surface that can be generated from two space curves by translating either one of them parallel to itself in such a way that each of its points describes a curve that is a translation of the other curve. The resulting surface can be considered as a translation surface. Consequently, the class of translation surfaces is restricted to those that can be parameterized as the sum of two space curves. So, it can be parameterized by a patch

$$M(u,v) = \alpha(u) + \beta(v),$$

where $u$ and $v$ are the parameters of the arc lengths of the curves $\alpha$ and $\beta$, respectively [11].

Let $\{\mathbf{t}_\alpha, \mathbf{n}_\alpha, \mathbf{b}_\alpha\}$ be the Frenet frame field of $\alpha$ with curvature $\kappa_\alpha$ and torsion $\tau_\alpha$. Also, let $\{\mathbf{t}_\beta, \mathbf{n}_\beta, \mathbf{b}_\beta\}$ be the Frenet frame field of $\beta$ with curvature $\kappa_\beta$ and torsion $\tau_\beta$. Let $M(u,v)$ be a translation surface in 3-dimensional Euclidean space.

The unit normal of the translation surface can be defined by

$$U(u,v) = \frac{1}{\sin\varphi}(\mathbf{t}_\alpha \wedge \mathbf{t}_\beta) \ \ ,\sin\varphi \neq 0 \tag{2.6}$$

where $\varphi(u)$ is the angle between tangent vectors of $\alpha(u)$ and $\beta(v)$ [6]. The first fundamental form I of the surface $M(u,v)$ is

$$I = Edu^2 + 2Fdudv + Gdv^2 \tag{2.7}$$

with coefficients $E = \langle M_u, M_u\rangle$, $F = \langle M_u, M_v\rangle$ and $G = \langle M_v, M_v\rangle$. Also, the second fundamental form of the surface $M(u,v)$ is given by

$$II = Ldu^2 + 2Mdudv + Ndv^2 \tag{2.8}$$

where $L = \langle M_{uu}, U\rangle$, $M = \langle M_{uv}, U\rangle$ and $N = \langle M_{vv}, U\rangle$. Gauss and mean curvatures have the classical expressions

$$K = \frac{LN - M^2}{EG - F^2} \text{ and } 2H = \frac{LG - 2MF + NE}{EG - F^2}, \tag{2.9}$$

respectively [13]. If the surface $M(u,v)$ is a translation surface, the first and second fundamental form of the surface $M(u,v)$ are

$$I = du^2 + 2cos\varphi dudv + dv^2 \tag{2.10}$$

and

$$II = \kappa_\alpha \cos\theta_\alpha du^2 + \kappa_\beta \cos\theta_\beta dv^2, \tag{2.11}$$

respectively, where $\theta_\alpha$ and $\theta_\beta$ are the angles between $U$ and $\mathbf{n}_\alpha$, $\mathbf{n}_\beta$, respectively. Also, Gauss curvature $K$ and mean curvature $H$ are

$$\begin{aligned} K &= \frac{\kappa_\alpha \kappa_\beta \cos\theta_\alpha \cos\theta_\beta}{\sin^2\varphi} \\ H &= \frac{\kappa_\alpha \cos\theta_\alpha + \kappa_\beta \cos\theta_\beta}{2\sin^2\varphi}, \end{aligned} \tag{2.12}$$

respectively [6].

## 3. On The Translation Surfaces with q-Frame

Let $M(u,v)$ be a translation surface in 3-dimensional Euclidean space. Then $M(u,v)$ is parameterized by the translation surface can be given as

$$M(u,v) = \alpha(u) + \beta(v), \tag{3.1}$$

where $\alpha$ and $\beta$ are unit-speed space curves of the arclength parameters $u$ and $v$, respectively.

Let $M(u,v)$ be a translation surface in 3-dimensional Euclidean space. Then, we have

$$M(u,v) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3) \tag{3.2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$. The partial derivatives of $M(u,v)$, with respect to $u$ and $v$, are determined by

$$M_u = \mathbf{t}_\alpha \text{ and } M_v = \mathbf{t}_\beta. \tag{3.3}$$

Let $\{\mathbf{t}_\alpha, \mathbf{n}_{q\alpha}, \mathbf{b}_{q\alpha}\}$ be the q-frame field of $\alpha$ with q-curvatures $k_1^\alpha$, $k_2^\alpha$ and $k_3^\alpha$ and also let $\{\mathbf{t}_\beta, \mathbf{n}_{q\beta}, \mathbf{b}_{q\beta}\}$ be the q-frame field of $\beta$ with q-curvatures $k_1^\beta$, $k_2^\beta$ and $k_3^\beta$. By introducing the notion of the q-frame, the derivative q-frame formulas along the curves $\alpha$ and $\beta$ are given as

$$\begin{bmatrix} \mathbf{t}'_\alpha \\ \mathbf{n}'_{q\alpha} \\ \mathbf{b}'_{q\alpha} \end{bmatrix} = \begin{bmatrix} 0 & k_1^\alpha & k_2^\alpha \\ -k_1^\alpha & 0 & k_3^\alpha \\ -k_2^\alpha & -k_3^\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_\alpha \\ \mathbf{n}_{q\alpha} \\ \mathbf{b}_{q\alpha} \end{bmatrix}, \tag{3.4}$$

and

$$\begin{bmatrix} \mathbf{t}'_\beta \\ \mathbf{n}'_{q\beta} \\ \mathbf{b}'_{q\beta} \end{bmatrix} = \begin{bmatrix} 0 & k_1^\beta & k_2^\beta \\ -k_1^\beta & 0 & k_3^\beta \\ -k_2^\beta & -k_3^\beta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_\beta \\ \mathbf{n}_{q\beta} \\ \mathbf{b}_{q\beta} \end{bmatrix}$$

respectively. By using equations (2.6) and (3.3), the unit normal of the translation surface can be defined by

$$U(u,v) = \frac{1}{\sin\varphi} (\mathbf{t}_\alpha \wedge \mathbf{t}_\beta), \ \ \sin\varphi \neq 0 \tag{3.5}$$

where $\varphi(u)$ is the angle between the tangent vectors of $\alpha(u)$ and $\beta(v)$.

**Corollary 3.1.** The point $M(u_0, v_0)$ of the translation surface $M(u,v)$ is a singular point if and only if $\sin\varphi = 0$.

**Corollary 3.2.** If the point $M(u_0, v_0)$ of the translation surface $M(u,v)$ is a singular point, then

i) The angle between tangent vectors of $\alpha(u)$ and $\beta(v)$ is $n\pi$, $n \in Z$.

ii) Owing to $k_1^\alpha = k_2^\alpha = k_1^\beta = k_2^\beta = 0$, the generated curves $\alpha(u)$ and $\beta(v)$ are degenerate curves.

iii) The point $M(u_0, v_0)$ of the translation surface $M(u,v)$ is an umbilical point.

**Theorem 3.1.** Gauss and mean curvatures of the translation surface are given by

$$K = \frac{(k_1^\alpha \cos\phi_\alpha - k_2^\alpha \sin\phi_\alpha)(k_1^\beta \cos\phi_\beta - k_2^\beta \sin\phi_\beta)}{\sin^2\varphi} \tag{3.6}$$

and

$$H = \frac{k_1^\beta \cos\phi_\beta - k_2^\beta \sin\phi_\beta + k_1^\alpha \cos\phi_\alpha - k_2^\alpha \sin\phi_\alpha}{2\sin^2\varphi}, \tag{3.7}$$

respectively.

**Proof:** From (2.7) and (3.3), the first fundamental form $I$ of the translation surface $M(u,v)$ is

$$I = du^2 + 2\cos\varphi du dv + dv^2. \tag{3.8}$$

Similarly by (2.8), the second fundamental form $I$ of the translation surface $M(u,v)$ is

$$II = (k_1^\alpha \cos\phi_\alpha - k_2^\alpha \sin\phi_\alpha)du^2 + (k_1^\beta \cos\phi_\beta - k_2^\beta \sin\phi_\beta)dv^2 \tag{3.9}$$

where $\phi_\alpha$ is the angle between vectors $U$ and $\mathbf{n}_{q\alpha}$, $\phi_\beta$ is the angle between vectors $U$ and $\mathbf{n}_{q\beta}$.

By substituting components of the equations (3.8) and (3.9) into the equation (2.9), Gauss and mean curvatures of the translation surface are obtained by

$$K = \frac{(k_1^\alpha \cos\phi_\alpha - k_2^\alpha \sin\phi_\alpha)(k_1^\beta \cos\phi_\beta - k_2^\beta \sin\phi_\beta)}{\sin^2\varphi}$$

and

$$H = \frac{k_1^\beta \cos\phi_\beta - k_2^\beta \sin\phi_\beta + k_1^\alpha \cos\phi_\alpha - k_2^\alpha \sin\phi_\alpha}{2\sin^2\varphi},$$

respectively.

Note that if a curve is a plane curve then the q-frame coincides with the Frenet and Bishop frames. In this case, the curvature $k_2$ of a plane curve vanishes. Therefore we state the following corollary.

**Corollary 3.3**. Let $M(u,v)$ be a translation surface in three dimensional Euclidean space $E^3$ given with the parametrization (3.1). If the curves $\alpha$ and $\beta$ are planar curves, then Gauss and mean curvatures of $M(u,v)$ are obtained as

$$K = \frac{k_1^\alpha k_1^\beta \cos\phi_\alpha \cos\phi_\beta}{\sin^2\varphi} \tag{3.10}$$

and

$$H = \frac{k_1^\alpha \cos\phi_\alpha + k_1^\beta \cos\phi_\beta}{2\sin^2\varphi}, \tag{3.11}$$

respectively.

**Corollary 3.4.** Let $M(u,v)$ be a translation surface in three dimensional Euclidean space $E^3$ given with the parametrization (3.1). The point is a planar in the translation surface $M(u,v)$ for the following cases:

i) $k_1^\alpha = k_2^\alpha = k_1^\beta = k_2^\beta = 0$, which implies to Gauss and mean curvatures vanish and the space curves $\alpha$ and $\beta$ are the straight lines.

ii) $k_1^\alpha = k_1^\beta = 0$ and $\phi_\alpha = \phi_\beta = n\pi$, $n \in Z$, which implies to Gauss and mean curvatures vanish.

iii) $k_2^\alpha = k_2^\beta = 0$ and $\phi_\alpha = \phi_\beta = (2n+1)\,\pi$, $n \in Z$, which implies to Gauss and mean curvatures vanish.

Hence, the translation surface $M(u,v)$ is both minimal and flat surface.

**Theorem 3.2.** Gauss curvature of a translation surface generated by the curves $\alpha$ and $\beta$ vanish if and only if

$$\frac{k_1^\alpha}{k_2^\alpha} = \tan\phi_\alpha \text{ or } \frac{k_1^\beta}{k_2^\beta} = \tan\phi_\beta \tag{3.12}$$

where $k_i^\alpha$, $k_i^\beta$ $(i = 1,2)$ are q-curvatures of $\alpha$ and $\beta$, respectively.

**Proof:** Let Gauss curvature be zero. Using the equation (3.6), we get

$$\frac{k_1^\alpha}{k_2^\alpha} = \tan\phi_\alpha \text{ or } \frac{k_1^\beta}{k_2^\beta} = \tan\phi_\beta.$$

Conversely, if we have

$$\frac{k_1^\alpha}{k_2^\alpha} = \tan\phi_\alpha \text{ or } \frac{k_1^\beta}{k_2^\beta} = \tan\phi_\beta$$

where $\phi_\alpha$ is the angle between vectors $U$ and $\mathbf{n}_{q\alpha}$, $\phi_\beta$ is the angle between vectors $U$ and $\mathbf{n}_{q\beta}$, then we obtain

$$k_1^\alpha \cos\phi_\alpha - k_2^\alpha \sin\phi_\alpha = 0 \text{ or } k_1^\beta \cos\phi_\beta - k_2^\beta \sin\phi_\beta = 0. \tag{3.13}$$

By substituting the equations (3.13) into the equation (3.6), it is easy to see that $K = 0$.

**Definition 3.1.** Let $\alpha(s)$ be a curve that is parameterized by arc length $s$ with q-frame $\{\mathbf{t}_\alpha, \mathbf{n}_{q\alpha}, \mathbf{b}_{q\alpha}\}$ on a surface $M$. If

$$\langle U, \mathbf{n}_{q\alpha} \rangle = 0$$

where the vector $U$ is the unit normal of the surface $M$, then the curve $\alpha$ is called $\mathbf{n}_{q\alpha}$-line.

**Theorem 3.3.** If the curve $\alpha$ is a $\mathbf{n}_{q\alpha}$-line of the translation surface, then we have $\dfrac{k_2^\alpha}{k_3^\alpha} = -\tan\varphi$.

**Proof** By using $\langle U, \mathbf{n}_{q\alpha} \rangle = \cos\phi_\alpha$ and (3.5), we have

$$\cos\phi_\alpha = -\frac{1}{\sin\varphi}\langle \mathbf{b}_{q\alpha}, \mathbf{t}_\beta \rangle. \tag{3.14}$$

Differentiating (3.14) with respect to $u$ and using (3.4) and $\langle \mathbf{b}_{q\alpha}, \mathbf{t}'_\beta \rangle = 0$ gives

$$-\phi'_\alpha \sin\phi_\alpha = \cot\varphi(-\varphi'\cos\phi_\alpha + k_2^\alpha) + k_3^\alpha. \tag{3.15}$$

Since $\alpha$ is a $\mathbf{n}_{q\alpha}$-line, we have $\cos\phi_\alpha = 0$, $\sin\phi_\alpha = \pm 1$ and $\phi'_\alpha = 0$. Therefore from (3.15), we get

$$\frac{k_2^\alpha}{k_3^\alpha} = -\tan\varphi. \tag{3.16}$$

**Corollary** Let $\beta$ be not a geodesic of the translation surface $M$, then $\beta$ is a $\mathbf{n}_{q\beta}$-line if and only if the angle $\phi_\beta$ is a constant.

## 4. EXAMPLES

**Example 4.1.** Let $M(u,v)$ be a translation surface given by

$$M(u,v) = \left(u + v^3, u^2 + v^2, 10u^3 + v\right)$$

with the generator curves

$$\alpha(u) = (u, u^2, 10u^3) \text{ and } \beta(v) = (v^3, v, v^2).$$

The tangent, the quasi-normal and the quasi-binormal vectors of $\alpha$ are

$$\mathbf{t}_\alpha = \frac{1}{\sqrt{1 + 4u^2 + 900u^4}}\left(1, 2u, 30u^2\right)$$

$$\mathbf{n}_{q\alpha} = \frac{1}{\sqrt{1 + 4u^2}}\left(2u, -1, 0\right)$$

and

$$\mathbf{b}_{q\alpha} = \frac{1}{\sqrt{(1 + 4u^2 + 900u^4)(1 + 4u^2)}}\left(30u^2, 60u^3, -1 - 4u^2\right),$$

respectively, where the projection vector $\mathbf{k}$ is $(0,0,1)$. Similarly, the tangent, the quasi-normal and the quasi-binormal vectors of $\beta$ are

$$\mathbf{t}_\beta = \frac{1}{\sqrt{1 + 4v^2 + 9v^4}}\left(3v^2, 1, 2v\right)$$

$$\mathbf{n}_{q\beta} = \frac{1}{\sqrt{1 + 9v^4}}\left(1, -3v^2, 0\right)$$

and

$$\mathbf{b}_{q\beta} = \frac{1}{\sqrt{(1 + 4v^2 + 9v^4)(1 + 9v^4)}}\left(6v^3, 2v, -9v^4 - 1\right),$$

respectively, where the projection vector $\mathbf{k}$ is $(0,0,1)$. Gauss and mean curvatures of the translation surface are obtained as

$$\begin{aligned} K &= \frac{4(1-60uv+90u^2v^2)(1-6uv+90u^2v^2)}{(1+4u^2+900u^4)^2(1+4v^2+9v^4)^2} \\ H &= \frac{4(1-33uv+90u^2v^2)}{(1+4u^2+900u^4)^{3/2}(1+4v^2+9v^4)^{1/2}}, \end{aligned}$$

respectively. Then the translation surface is shown in Figure 2.

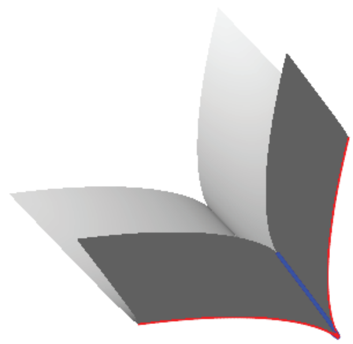

FIGURE 2.

**Example 4.2.** Let $M(u,v)$ be a translation surface given by

$$M(u,v) = \left(\sqrt{3}\sin\frac{u}{2} + 2\cos\frac{v}{3} - 2, u + 2\sin\frac{v}{3}, \frac{3u+2\sqrt{5}v}{6}\right)$$

with the generator curves

$$\alpha(u) = \left(\sqrt{3}\sin\frac{u}{2}, u, \frac{u}{2}\right) \text{ and } \beta(v) = (2\cos\frac{v}{3} - 2, 2\sin\frac{v}{3}, \frac{\sqrt{5}v}{3}).$$

The tangent, the quasi-normal and the quasi-binormal vectors of $\alpha$ are calculated as

$$\mathbf{t}_\alpha = \frac{1}{\sqrt{3\cos^2\frac{u}{2}+5}}(\sqrt{3}\cos\frac{u}{2}, 2, 1)$$

$$\mathbf{n}_{q\alpha} = \frac{1}{\sqrt{3\cos^2\frac{u}{2}+4}\sqrt{3\cos^2\frac{u}{2}+5}}(2, -\sqrt{3}\cos\frac{u}{2}, 0)$$

and

$$\mathbf{b}_{q\alpha} = \frac{1}{\sqrt{3\cos^2\frac{u}{2}+4}\sqrt{3\cos^2\frac{u}{2}+5}}(\sqrt{3}\cos\frac{u}{2}, 2, -3\cos^2\frac{u}{2}-4),$$

respectively, where the projection vector $\mathbf{k}$ is $(0, 0, 1)$.

Similarly, the tangent, the quasi-normal and the quasi-binormal vector of $\beta$ are computed as

$$\mathbf{t}_{\beta} = \left(-\frac{2}{3}\sin\frac{v}{3}, \frac{2}{3}\cos\frac{v}{3}, \frac{\sqrt{5}}{3}\right)$$

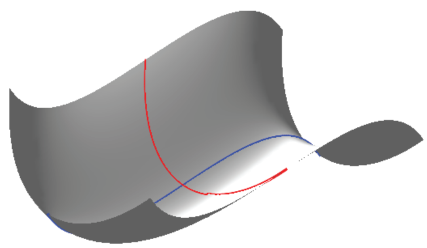

FIGURE 3.

$$\mathbf{n}_{q\beta} = \left(\cos\frac{v}{3}, \sin\frac{v}{3}, 0\right)$$

and

$$\mathbf{b}_{q\beta} = \left(-\frac{\sqrt{5}}{3}\sin\frac{v}{3}, \frac{\sqrt{5}}{3}\cos\frac{v}{3}, -\frac{2}{3}\right),$$

respectively, where the projection vector $\mathbf{k}$ is $(0, 0, 1)$. Then the translation surface is shown in Figure 3.

## 5. ACKNOWLEDGEMENTS

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