Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260

CJMS. **9**(1)(2020), 86-99

Normal Fermi-Walker Derivative in E_1^3

Özgür Keskin [1] and Yusuf Yayli [2]
[1,2] Department of Mathematics, Faculty of Science, University of Ankara, TURKEY

Abstract. In this paper, firstly, in E_1^3, we defined normal Fermi-Walker derivative and applied for the adapted frame. Normal Fermi-Walker parallelism, normal non-rotating frame, and Darboux vector expressions of normal Fermi-Walker derivative by normal Fermi-Walker derivative are given for adapted frame. Being conditions of normal Fermi-Walker derivative and normal non-rotating frame are examined for frames throughout spacelike, timelike, lightlike curves. It is shown that the vector field which takes part in [17] is normal Fermi-Walker parallel by the normal Fermi-Walker derivative throughout the spacelike, timelike, and lightlike general helix. Also, we show that the Frenet frame is a normal non-rotating frame using the normal Fermi-Walker derivative. Afterward, we testified that the adapted frame is a normal non-rotating frame throughout the spacelike, timelike, and lightlike general helix.

Keywords: Frenet frame, Darboux frame, Normal Fermi-Walker derivative, Normal non-rotating frame, Spacelike curve, Timelike curve, Lightlike curve.

2000 Mathematics subject classification: 53A04, 53A05, 53Z05, 57R25; 53B30.

1. Introduction

Fermi-Walker transport is a process used to define a coordinate system or reference frame in general relativity. All the curvatures in the

[1]Corresponding author: ozgur.keskin@ankara.edu.tr
Received: 10 August 2016
Revised: 8 January 2020
Accepted: 14 January 2020

reference frame are due to the presence of mass-energy density. These curvatures are not arbitrary spin or rotation of the frame. Fermi-Walker derivative, Fermi-Walker parallelism, and non-rotating frame expresses are identified according to this derivative for Bishop and Frenet frames. Then, the obtained notions are applied for Lie groups in $\mathbb{E}^4$ and are expressed on any hypersurface in $\mathbb{E}^{n+1}$ [9, 10, 12].

In Minkowski 3-space, the Fermi-Walker derivative, Fermi-Walker parallelism, Fermi-Walker Darboux vector, and being condition of the non-rotating frame are shown. These expresses are investigated on the spacelike and the timelike surfaces and are explained with a spacelike or timelike principal normal using the spherical indicatrix of a spacelike curve. The Fermi-Walker derivative is obtained throughout the tangent vector of a spacelike curve [11, 13, 14].

Many researchers have attracted the attention of the Fermi-Walker derivative from time to time, and it has been used in numerous studies. Some of these studies are as follows. The unit vector fields according to Fermi-Walker transported are proved along Rytov-Legendre curves. Slant and Legendre curves are defined in 3-dimensional warped products and are parametrized via the scalar products between the normal of these curves and the vertical vector field [5, 4]. The constant precession curve is expressed as a unit-speed curve that its central returns approximate a stable axis with stationary angle and stationary speed. The arc-length parametrized closed-shape solution of the native equations is obtained according to this constant precession curves with the help of direct geometric analysis [16]. Spherical images of a curve are investigated and are called C-slant helices. Furthermore, some new characterizations for the C-slant helices are given and showed that the C-constant precession curves are C-slant helices [17]. In recent years, some application is made by using the normal of slant helix. In light of these opinions, we defined derivative according to the normal of the curve.

Fermi-Walker derivative is given with the help of the tangent vector of the curve. We defined a new normal Fermi-Walker derivative by using the normal vector of any curve according to the Frenet frame in E_1^3. In our study, normal Fermi-Walker derivative, normal Fermi-Walker parallelism, normal non-rotating frame, and Darboux vector expressions of normal Fermi-Walker derivative are given for the adapted frame in Minkowski 3-Space E_1^3. Also, we show if the Frenet frame is non-rotating frame, and $\{N, C, W\}$ frame is non-rotating frame according to the normal Fermi-Walker derivative according to spacelike, timelike or lightlike curves. In this wise, the Fermi-Walker definitions can be defined by the first vector of other frames.

2. Preliminaries

Firstly, we give some basic notions about in E_1^3. Afterward, we define a new derivative according to this space. The Minkowski 3-space E_1^3 is an affine 3-space. An indefinable inner product of this space is given as

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is coordinates of E_1^3. If $t = (t_1, t_2, t_3)$ and $k = (k_1, k_2, k_3)$ are arbitrary vectors in E_1^3. The Minkowski vector product of t and k is defined as follows:

$$t \times k = \begin{vmatrix} -i & j & k \\ t_1 & t_2 & t_3 \\ k_1 & k_2 & k_3 \end{vmatrix}$$

Since $\langle , \rangle$ is an indefinite metric, three cases are possible for $k \epsilon E_1^3$: it can be spacelike if $\langle k, k \rangle > 0$ or k=0, timelike if $\langle k, k \rangle < 0$ and lightlike if $\langle k, k \rangle = 0$ and $k \neq 0$. $\|k\| = \sqrt{\langle k, k \rangle}$ is the norm of k vector. If $\langle k, t \rangle = 0$, k and t vectors are orthogonal [15].

Let $\beta : M \subset \mathbb{R} \to E_1^3$ is a unit speed curve. Denote by

$$\{N, C, N \wedge C = W\}$$

the moving adapted frame throughout the curve $\beta(s)$ in the space E_1^3. The adapted frame equations are

$$\begin{bmatrix} N'(s) \\ C'(s) \\ W'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -\epsilon_0 \epsilon_1 f(s) & 0 & g(s) \\ 0 & -\epsilon_1 \epsilon_2 g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix}$$

and the Lorentzian vector products of adapted vectors are given as

$$\begin{aligned} N \wedge C &= W, \\ C \wedge W &= -\epsilon_1 N, \\ W \wedge N &= -\epsilon_0 C, \end{aligned}$$

where $\langle N, N \rangle = \epsilon_0 = \pm 1$, $\langle C, C \rangle = \epsilon_1 = \pm 1$ and $\langle W, W \rangle = \epsilon_2 = \pm 1$ [14].

Fermi-Walker derivative is given with the help of the tangent vector of curve. In this study, for the first time, we will give the following definition by using the normal vector of curve in Minkowski 3-Space. We will say **normal Fermi-Walker derivative** name of this definition in Minkowski 3-Space.

3. Adapted Frame and Normal Fermi-Walker Derivative in E_1^3

Firstly, we express a different definition of the normal Fermi-Walker derivative with the help of the adapted frame in E_1^3. Afterward, we describe some theorems and results by being conditions spacelike curve, timelike curve, or lightlike curve.

Definition 3.1. Let $\beta : M \subset \mathbb{R} \to E_1^3$ be a unit-speed curve and X be a vector field throughout the $\beta(s)$ curve in E_1^3.

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} - \epsilon\langle N, X\rangle N' + \epsilon\langle N', X\rangle N.$$

Here N is any normal vector of Frenet frame and $\langle N, N\rangle = \epsilon = \pm 1$.

Lemma 3.2. *Let $\beta : M \subset \mathbb{R} \to E_1^3$ be a curve in E_1^3 and X be a vector field throughout the $\beta(s)$ space curve. Afterwards, normal Fermi-Walker derivative can be expressed of the*

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \epsilon f(W \wedge X).$$

Proof.

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} - \epsilon\langle N, X\rangle \frac{dN}{ds} + \epsilon\langle \frac{dN}{ds}, X\rangle N,$$

If $\frac{dN}{ds}=fC$ substitutes and the necessary actions are done,

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \epsilon f(W \wedge X),$$

is procured. □

Now, we will investigate the conditions that a vector field is normal Fermi-Walker parallel according to the normal Fermi-Walker derivative in E_1^3.

Theorem 3.3. *Let $\beta(s)$ be a unit-speed spacelike curve. N, C and W are the timelike principal normal vector, the spacelike vector and the spacelike Darboux vector in E_1^3, respectively. $X = \lambda_1 N + \lambda_2 C + \lambda_3 W$ be a vector field throughout $\beta(s)$. This X vector field is normal Fermi-Walker parallel in accordance with the normal Fermi-Walker derivative in E_1^3 if and only if:*

$$\begin{aligned} \lambda_1(s) &= constant, \\ \lambda_2(s) &= c_1 cos(\int_1^s g(s)ds) + c_2 sin(\int_1^s g(s)ds), \\ \lambda_3(s) &= c_2 cos(\int_1^s g(s)ds) - c_1 sin(\int_1^s g(s)ds). \end{aligned}$$

Here λ_1, λ_2, λ_3 real parameters are continuous differentiable functions.

Proof. $\Rightarrow$: Let X be normal Fermi-Walker parallel according to the normal Fermi-Walker derivative throughout $\beta(s)$ in Minkowski 3-Space E_1^3. Since N is the timelike principal normal vector, C is the spacelike vector and W is the spacelike Darboux vector;

$$\begin{aligned} N'(s) &= f(s)C(s), \\ C'(s) &= f(s)N(s) + g(s)W(s), \\ W'(s) &= -g(s)C(s). \end{aligned}$$

and

$$\begin{aligned} N \wedge C &= W, \\ C \wedge W &= -N, \\ W \wedge N &= C. \end{aligned}$$

Also,

$$\langle N, N \rangle = \epsilon = -1.$$

$$\begin{aligned} \frac{\tilde{D}X}{\tilde{D}s} &= \frac{dX}{ds} + \epsilon f(W \wedge X), \\ \frac{\tilde{D}X}{\tilde{D}s} = (\frac{d\lambda_1}{ds})N + (\frac{d\lambda_2}{ds} - g(s)\lambda_3)C &+ (\frac{d\lambda_3}{ds} + g(s)\lambda_2)W, \end{aligned}$$

is procured. X vector field is normal Fermi-Walker parallel according to the normal Fermi-Walker derivative in E_1^3 and $\frac{\tilde{D}X}{\tilde{D}s}$=0 so,

$$\begin{aligned} \frac{d\lambda_1}{ds} &= 0, \\ \frac{d\lambda_2}{ds} - g(s)\lambda_3 &= 0, \\ \frac{d\lambda_3}{ds} + g(s)\lambda_2 &= 0, \end{aligned}$$

is procured. If the equation system solve,

$$\lambda_1(s) = constant,$$
$$\lambda_2(s) = c_1 cos(\int_1^s g(s)ds) + c_2 sin(\int_1^s g(s)ds),$$
$$\lambda_3(s) = c_2 cos(\int_1^s g(s)ds) - c_1 sin(\int_1^s g(s)ds),$$

is obtained.

$\Longleftarrow$: Let $X = \lambda_1 N + \lambda_2 C + \lambda_3 W$ be a vector field and $\theta = (\int_1^s g(s)ds)$ in Minkowski 3-Space E_1^3,

$$\lambda_1(s) = constant,$$
$$\lambda_2(s) = c_1 cos\theta + c_2 sin\theta,$$
$$\lambda_3(s) = c_2 cos\theta - c_1 sin\theta,$$

and

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \epsilon f(W \wedge X),$$
$$\frac{\tilde{D}X}{\tilde{D}s} = (\frac{d\lambda_1}{ds})N + (\frac{d\lambda_2}{ds} - g(s)\lambda_3)C + (\frac{d\lambda_3}{ds} + g(s)\lambda_2)W,$$
$$\frac{\tilde{D}X}{\tilde{D}s} = 0,$$

is obtained. λ_i parameters are constant. □

Theorem 3.4. *Let $\beta(s)$ be a unit-speed timelike curve. (N, C and W are the spacelike principal normal vector, the spacelike vector and the timelike Darboux vector in E_1^3, respectively.) $X = \lambda_1 N + \lambda_2 C + \lambda_3 W$ be a vector field throughout $\beta(s)$. X vector field is normal Fermi-Walker parallel according to the normal Fermi-Walker derivative in Minkowski 3-Space E_1^3 if and only if:*

$$\lambda_1(s) = constant,$$
$$\lambda_2(s) = c_1 cosh(\int_1^s g(s)ds) + c_2 sinh(\int_1^s g(s)ds),$$
$$\lambda_3(s) = -c_1 sinh(\int_1^s g(s)ds) - c_2 cosh(\int_1^s g(s)ds).$$

Here λ_1, λ_2, λ_3 real parameters are continuous differentiable functions.

Theorem 3.5. *Let $\beta(s)$ be a unit-speed lightlike curve. (N, C and W are the spacelike principal normal vector, the lightlike vector and the lightlike Darboux vector in E_1^3,respectively.) $X = \lambda_1 N + \lambda_2 C + \lambda_3 W$ be a vector field throughout $\beta(s)$. X vector field is normal Fermi-Walker parallel in accordance with the normal Fermi-Walker derivative in Minkowski 3-Space E_1^3 if and only if:*

$$\lambda_1(s) = -c_1 g sinh\sqrt{g}s - c_2 g cosh\sqrt{g}s + c_3,$$
$$\lambda_2(s) = \sqrt{g}(c_1 cosh\sqrt{g}s + c_2 sinh\sqrt{g}s),$$
$$\lambda_3(s) = c_1 sinh\sqrt{g}s + c_2 cosh\sqrt{g}s.$$

Here λ_1, λ_2, λ_3 real parameters are continuous differentiable functions.

Now, we can give a new theorem as follows.

Theorem 3.6. *Let $\beta(s)$ be a spacelike general helix in E_1^3. Then $X = \lambda_1 N + \lambda_2 C + \lambda_3 W$ vector field is normal Fermi-Walker parallel according to the normal Fermi-Walker derivative along $\beta(s)$. Here λ_1, λ_2, λ_3 are constant.*

Proof. In E_1^3, let $\beta(s)$ be any spacelike curve, N be the timelike principal normal vector, C be the spacelike vector, and W is the spacelike Darboux vector. If obtained adapted frame equations and the Lorentzian vector products of the adapted frame are put back, then

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \epsilon f(W \wedge X),$$
$$\frac{\tilde{D}X}{\tilde{D}s} = \lambda_1' N + (\lambda_2' - g\lambda_3)C + (g\lambda_2 + \lambda_3')W,$$
$$= g(\lambda_2 W - \lambda_3 C),$$

is obtained. Since $\beta(s)$ is any spacelike general helix, then $g = 0$. So, $\frac{\tilde{D}X}{\tilde{D}s} = 0$ is procured. □

Moreover, in Minkowski 3-Space E_1^3, if $\beta(s)$ is any timelike general helix, N is the spacelike principal normal vector, C is the spacelike vector and W is the timelike Darboux vector; the same vector field is a normal Fermi-Walker parallel in accordance with the normal Fermi-Walker derivative along $\beta(s)$.

Theorem 3.7. *Let $\beta(s)$ be lightlike general helix in Minkowski 3-Space E_1^3. Then $X = \lambda_1 N + \lambda_2 C + \lambda_3 W$ vector field is not normal Fermi-Walker parallel in accordance with the normal Fermi-Walker derivative along $\beta(s)$. Here λ_1, λ_2, λ_3 are constant.*

Proof. In E_1^3, let $\beta(s)$ be any lightlike curve, N be the spacelike principal normal vector, C be the lightlike vector, and W is the lightlike Darboux vector. If obtained adapted frame equations and the Lorentzian vector products of the adapted frame are put back in Lemma (3.2), then

$$\frac{\tilde{D}X}{\tilde{D}s} = g(\lambda_2 N - \lambda_3 C) - \lambda_2 W$$

is obtained. Even if $\beta(s)$ is a lightlike general helix, that is $g = 0$, $\frac{\tilde{D}X}{\tilde{D}s} \neq 0$. □

Example 3.8. $\{N, C, W\}$ vectors are normal Fermi-Walker parallel using the normal Fermi-Walker derivative along the timelike and spacelike general helix in E_1^3.

Now, in three dimensional Minkowski Space E_1^3, we described a relationship between the normal Fermi-Walker parallelism and Euclid parallelism as follows.

Corollary 3.9. *Let X be a vector field throughout the $\beta(s)$ space curve in E_1^3. Normal Fermi-Walker derivative along the space curve of X coincides with Euclidean derivative of X iff*

$$X = \lambda W.$$

λ is constant.

Proof. Due to Lemma (3.2),

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds},$$

iff

$$X = \lambda W$$

is obtained. □

As a result, if we pay attention to the Corollary (3.9), it is seen that the Corollary (3.9) holds for the timelike, the spacelike and the lightlike curves.

$\{T, N, B\}$ frame is not non-rotating in accordance with Fermi-Walker derivative. But now, we will analyze to see if $\{T, N, B\}$ frame is normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative along the timelike, the spacelike or the lightlike curves in E_1^3.

Corollary 3.10. *Let $\beta(s)$ be a unit-speed spacelike curve and $\{T, N, B\}$ be the Frenet frame of $\beta(s)$. The $\{T, N, B\}$ is normal Fermi-Walker non-rotating frame throughout the spacelike curves according to normal Fermi-Walker derivative in E_1^3.*

Proof. In E_1^3, the normal Fermi-Walker derivative is given Definition (3.1) and Lemma (3.2).

Here $\beta(s)$ is a unit-speed spacelike curve, T is the spacelike vector, N is the timelike principal normal vector and B is the spacelike vector. From Definition (3.1), if necessary calculations are done,

$$\frac{\tilde{D}T}{\tilde{D}s} = T' - \epsilon\langle N, T\rangle N' + \epsilon\langle N', T\rangle N.$$

Since $\langle N, T\rangle = 0$, $T' = \kappa N$, $N' = \kappa T + \tau B$ and $\langle N, N\rangle = \epsilon = -1$,

$$\frac{\tilde{D}T}{\tilde{D}s} = 0.$$

Likewise, if you do others,

$$\frac{\tilde{D}N}{\tilde{D}s} = 0,$$
$$\frac{\tilde{D}B}{\tilde{D}s} = 0.$$

□

Although $\beta(s)$ is a unit-speed spacelike curve, the $\{T, N, B\}$ is not normal Fermi-Walker non-rotating frame throughout the spacelike curves according to normal Fermi-Walker derivative in E_1^3. The reason for this is that T is the spacelike vector, N is the spacelike principal normal vector and B is the timelike vector.

Corollary 3.11. *Let $\beta(s)$ be a unit-speed timelike curve and $\{T, N, B\}$ be the Frenet frame of $\beta(s)$. The $\{T, N, B\}$ is normal Fermi-Walker non-rotating frame throughout the timelike curves according to normal Fermi-Walker derivative in three dimensional Minkowski Space E_1^3. Here T is the timelike vector, N is the spacelike principal normal vector, and B is the spacelike vector.*

Also, if $\beta(s)$ is a unit-speed lightlike curve, the $\{T, N, B\}$ is not normal Fermi-Walker non-rotating frame along $\beta(s)$ according to normal Fermi-Walker derivative in three dimensional Minkowski Space E_1^3.

Corollary 3.12. *Let $\beta(s)$ be any spacelike curve. $\{N, C, W\}$ is the adapted frame of $\beta(s)$. The $\{N, C, W\}$ is normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative iff $g = 0$ in three dimensional Minkowski Space E_1^3.*

Proof. Since $\beta(s)$ is any spacelike curve, N is the spacelike principal normal vector, C is the timelike vector, and W is the spacelike Darboux vector, obtained adapted frame equations and the Lorentzian vector products of the adapted frame are put back in Definition (3.1). We will compute the following equations to show that it is a non-rotating frame of $\{N, C, W\}$ frame according to normal Fermi-Walker derivative in E_1^3. If necessary calculations are done,

$$\frac{\tilde{D}N}{\tilde{D}s} = fC - fC + \langle fC, N\rangle N,$$
$$\frac{\tilde{D}N}{\tilde{D}s} = 0.$$

Similarly, if you do others,

$$\frac{\tilde{D}C}{\tilde{D}s} = gW,$$
$$\frac{\tilde{D}W}{\tilde{D}s} = gC.$$

Here, there should be $g = 0$ to be normal Fermi-Walker parallel according to normal Fermi-Walker derivative. Thus, $\beta(s)$ must be a spacelike general helix. Theorem (3.6) is obtained again from the result. □

But, if N is the timelike principal normal vector, C is the spacelike vector and W is the spacelike Darboux vector, the $\{N, C, W\}$ is not normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative in Minkowski 3-Space E_1^3.
Because,

$$N'(s) = f(s)C(s),$$
$$C'(s) = f(s)N(s) + g(s)W(s),$$
$$W'(s) = -g(s)C(s).$$

and

$$\langle N, N \rangle = \epsilon = -1,$$

and

$$\frac{\tilde{D}N}{\tilde{D}s} = 2fC.$$

Similarly, if you do others,

$$\frac{\tilde{D}C}{\tilde{D}s} = gW,$$
$$\frac{\tilde{D}W}{\tilde{D}s} = -gC.$$

Hence, even if $g = 0$, the $\{N, C, W\}$ cannot normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative in E_1^3.

Corollary 3.13. *Let $\beta(s)$ be timelike curve. $\{N, C, W\}$ is the adapted frame of $\beta(s)$. The $\{N, C, W\}$ is normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative iff $g = 0$ in three dimensional Minkowski Space E_1^3. Here N is the spacelike principal normal vector, C is the spacelike vector, and W is the timelike vector.*

On the other hand, in three dimensional Minkowski Space E_1^3, if N is the spacelike principal normal vector, C is the timelike vector, and W is the spacelike vector, the $\{N, C, W\}$ is normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative again. On the contrary, if $\beta(s)$ is a unit-speed lightlike curve, the $\{N, C, W\}$ is not normal Fermi-Walker non-rotating frame according to normal Fermi-Walker derivative.

Corollary 3.14. *Let $\{N, C, W\}$ be the adapted frame of $\beta(s)$ spacelike curve. In accordance with the adapted frame, Darboux vector of the normal Fermi-Walker derivative is $\omega_* = gN$ in Minkowski 3-Space E_1^3.*

Proof. Since $\beta(s)$ is any spacelike curve, N is the spacelike principal normal vector, C is the timelike vector, and W is the spacelike Darboux vector, obtained adapted frame equations and the Lorentzian vector products of the adapted frame put back in Definition (3.1). If the calculations are done,

$$\frac{\tilde{D}N}{\tilde{D}s} = \omega_* \wedge N,$$
$$\frac{\tilde{D}C}{\tilde{D}s} = \omega_* \wedge C,$$
$$\frac{\tilde{D}W}{\tilde{D}s} = \omega_* \wedge W.$$

But, if N is the timelike principal normal vector, C is the spacelike vector and W is the spacelike Darboux vector, these equations don't satisfy. □

Corollary 3.15. *Let $\{N, C, W\}$ be the adapted frame of $\beta(s)$ timelike curve. In accordance with the adapted frame, the Darboux vector of the normal Fermi-Walker derivative is $\omega_* = gN$ in three dimensional Minkowski Space E_1^3. Here N is the spacelike principal normal vector, C is the spacelike vector, and W is the timelike vector.*

Additionally, if N is the spacelike principal normal vector, C is the timelike vector, and W is the spacelike vector, Darboux vector of the normal Fermi-Walker derivative is $\omega_* = gN$ throughout $\beta(s)$ timelike curve using the adapted frame in E_1^3.

Theorem 3.16. *Let $\beta(s)$ be a unit-speed spacelike curve. ω_*=gN. Darboux vector of the normal Fermi-Walker derivative is normal Fermi-Walker parallel according to the normal Fermi- Walker derivative iff g is constant in three dimensional Minkowski Space E_1^3.*

Proof.

$$\frac{\tilde{D}\omega^*}{\tilde{D}s} = \frac{d\omega^*}{ds} + \epsilon f(W \wedge \omega^*),$$
$$\frac{\tilde{D}\omega^*}{\tilde{D}s} = \frac{\tilde{D}(gN)}{\tilde{D}s} = \frac{d(gN)}{ds} = \frac{d(g)}{ds}N + \frac{d(N)}{ds}g,$$
$$\frac{\tilde{D}\omega^*}{\tilde{D}s} = \frac{d(g)}{ds}N + N'g,$$

Since N is the spacelike principal normal vector, C is the timelike vector and W is the spacelike Darboux vector, obtained adapted frame equations and the Lorentzian vector products of adapted frame put back in Definition(3.1). If the calculations are done,

$$f(W \wedge \omega^*) = f(W \wedge gN) = -fgC,$$

and

$$\frac{\tilde{D}\omega^*}{\tilde{D}s} = \frac{d(g)}{ds}N.$$

Then, $\frac{\tilde{D}\omega^*}{\tilde{D}s} = 0$ iff g is constant. When N is the timelike principal normal vector, C is the spacelike vector and W is the spacelike Darboux vector, $\omega_* \neq gN$. Hence, Theorem (3.16) is not satisfied. □

Theorem 3.17. *Let $\beta(s)$ be a unit-speed timelike curve. ω_*=gN. Darboux vector of the normal Fermi-Walker derivative is normal Fermi-Walker parallel by the normal Fermi- Walker derivative iff g is constant in three dimensional Minkowski Space E_1^3. Here N is the spacelike principal normal vector, C is the spacelike vector, and W is the timelike vector.*

Besides, if N is the spacelike principal normal vector, C is the timelike vector, and W is the spacelike vector, Theorem (3.17) is satisfied.

Example 3.18. If $\beta(s)$ is a spacelike and timelike constant precession curve, ω^* is normal Fermi-Walker parallel using the normal Fermi-Walker derivative. Infact, if $\beta(s)$ is a constant precession curve, $f = constant$, $\sigma = constant$. Hence, g is constant.

Conclusion

In our study, we clarified normal Fermi-Walker derivative, normal Fermi-Walker parallelism, normal non-rotating frame, normal Fermi-Walker derivative Darboux vector concepts by the adapted frame in E_1^3.

The Frenet frame is not non-rotating according to Fermi-Walker derivative. Thus, we described the new normal Fermi-Walker derivative to be non-rotating of the Frenet frame in Minkowski 3-Space E_1^3. Then, we showed if the Frenet frame is a normal non-rotating frame using the normal Fermi-Walker derivative in E_1^3. Otherwise, the adapted frame is not non-rotating frame by this normal Fermi-Walker derivative in E_1^3. However, if a curve is a spacelike, timelike or lightlike general helix, the adapted frame can be non-rotating frame using this normal Fermi-Walker derivative in E_1^3.

In E_1^3, the Fermi-Walker derivative can be defined by the first vector of other frames. In this way, various Fermi-Walker derivatives can be redefined for different frames.

Acknowledgement

The first author would like to thank Tubitak-Bidep for their financial supports during her PhD studies.

References

[1] R. Balakrishnan, Space curves, anholonomy and nonlinearity, *Prama J. Phys.* **64(4)***(2005), 607-615.*

[2] I. M. Benn, and R. W. Tucker, Wave mechanics and inertial guidance, *Phys. Rev. D.* **39(6)***(1989), 1594-1601.*

[3] M.V. Berry, Quantal phase factors accompanying adiabatic changes, *Proc. Roy. Soc. London A. 1960, 392.*

[4] C. Calin, and M. Crasmareanu, Slant Curves and Particles in three-dimensional Warped Products and their Lancret invariants, *Bulletin of the Australian Mathematical Society.* **88(1)***(2013), 128-142.*

[5] M. Crasmareanu, and C. Frigioiu, Unitary vector fields are Fermi-Walker transported along Rytov-Legendre curves, *Int. Journal of Geometric Methods in Modern Physics.* **12***(2015), 1550111.*

[6] R. Dandolof, Berry's phase and Fermi-Walker parallel transport, *Phys. Lett. A.* **139 (1,2)***(1989), 19-20.*

[7] E. Fermi, Sopra i fenomeni che avvengono in vicinanza di una linea oraria, *Atti Accad. Naz. Lincei Cl. Sci. Fiz. Mat. Nat.* **31***(1922), 184–306.*

[8] S. W. Hawking, and G. F. R. Ellis, The Large Scale Structure of Spacetime, *Cambridge University Press, 1973.*

[9] F. Karakuş, and Y. Yaylı, On the Fermi-Walker derivative and non-rotating frame, *Int. Journal of Geometric Methods in Modern Physics.* **(9,8)***(2012), 1250066.*

[10] F. Karakuş, and Y. Yaylı, The Fermi- Walker derivative in Lie groups, *Int. Journal of Geometric Methods in Modern Physics.* **10(7)***(2013), Article ID 1320011:10p.*

[11] F. Karakuş, and Y. Yaylı, The Fermi-Walker derivative in Minkowski 3-Space E_1^3, *2nd International Eurasian Conference On Mathematics Sciences And Applications, Proceedings, 2013.*

[12] F. Karakuş, and Y. Yaylı, The Fermi derivative in the hypersurfaces, *Int. Journal of Geometric Methods in Modern Physics.* **12(1)***(2015), Article ID 1550002:12p.*

[13] F. Karakuş, and Y. Yaylı, On the Surface the Fermi- Walker derivative in Minkowski 3-Space E_1^3, *Advances in Applied Clifford Algebras. Springer International Publishing. (2015), 1-12.*

[14] F. Karakuş, and Y. Yaylı, The Fermi-Walker derivative on the Spherical Indicatrix of Spacelike curve in Minkowski 3-Space E_1^3, *Adv. Appl. Clifford Algebras. Springer International Publishing. (2016), Article DOI 10.1007/s00006-015-0635-9.*

[15] R. López, Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, *Int. Electron. J. Geom.* **7(1)***(2014), 44-107.*

[16] P. D. Scofield, Curves of Constant Precession, *The American Mathematical Monthly.* **(102)6***(1995), 531-537.*

[17] B. Uzunoğlu, I. Gök, and Y. Yaylı, A new approach on curves of constant precession, *Applied Mathematics and Computation.* **275***(2016), 317–323.*