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On the D-concircular curvature tensor Of a generalized Sasakian-space-form

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Abstract. The object of this paper is to study of D-concircular curvature tensor on generalized Sasakian-space-forms. Actually we consider generalized Sasakian-space-forms when it is, respectively: D-concircularly flat; D-concircular-pseudosymmetric; D-concircularly Ricci-semisymmetric; D-concircularly symmetric; $V(\xi, X) . R = 0$. Most of the main results obtained in this paper are in the form of necessary and sufficient conditions.

Keywords: Generalized Sasakian-space-form, D-concircular curvature tensor, Kenmotsu-space-form, Einstein manifold.

2000 Mathematics subject classification: 53C25,53D15.

1. Introduction

In [1], the notion of a generalized Sasakian-space-form is introduced as follow: If (ϕ, ξ, η, g) is an almost contact metric structure on a manifold M, R the curvature tensor and there exist three differential functions f_1, f_2 and f_3 such that

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad +g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned}$$

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for X, Y and Z vector fields on M, then M is said to be a generalized Sasakian-space-form. This generalizes the concept of Sasakian space form as well as generalized complex space form did with complex space form. Moreover, the authors have given some examples of generalized Sasakian-space-forms in terms of warped-product spaces. U. C. De and A. Sarkar [9], studied some conditions regarding the projective curvature tensor of a generalized Sasakian-space-form. Derivation conditions $\tilde{C}(X,Y)Z = 0$, $\nabla S = 0$ and $R(X,Y).S = 0$ were studied where $\tilde{C}$ is the quasi-conformal curvature tensor, in [15]. In [16] ϕ-recurrent generalized Sasakian-space-forms was studied. In [2], it is shown that in dimensions ≥ 5 a contact metric generalized Sasakian-space-form is a Sasakian-space-form ($f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}, c$ being the constant ϕ-sectional curvature). In the 3-dimensional case non-Sasakian contact metric generalized Sasakian-space- forms exist and the curvature tensor is also studied in [2]. For more details about generalized Sasakian-space-forms see also [3, 7, 11, 12, 14, 17]. This paper is organized as follows:
In section 2, some preliminaries results of generalized Sasakian-space-forms are given. In section 3, we study D-concircularly at generalized Sasakian-space-form and obtain necessary and sufficient conditions for a generalized Sasakian-space-form to be D-concircularly at. In next, D-concircular-pseudosymmetric generalized Sasakian-space-form, D-concircularly Ricci-semisymmetric generalized Sasakian-space-form and D-concircularly symmetric generalized Sasakian-space-form were studied. Finally, the end of this section contains generalized Sasakian-space-forms satisfying $V(\xi, X)\ .\ R = 0$.

2. Generalized Sasakian-space-forms

In an n-dimensional ($n = 2m+1$) almost contact Riemannian manifold with (ϕ, η, ξ, g) almost contact metric structure, where ϕ is a (1,1)-tensor filed, η is a 1-form, ξ is the associated vector field and g is the Riemannian metric we have following conditions [4, 5, 18]

$$\phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ \eta(\xi) = 1, \tag{2.2}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0. \tag{2.5}$$

for all vector fields $X, Y \in \chi(M^n)$.
For an n-dimensional generalized Sasakian-space-form we have [1].

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{2.6}$$

$$\begin{aligned} S(X,Y) &= \{(n-1)f_1 + 3f_2 - f_3\}g(X,Y) \\ &- \{3f_2 + (n-2)f_3\}\eta(X)\eta(Y) \end{aligned} \tag{2.7}$$

$$QX = \{(n-1)f_1 + 3f_2 - f_3\}X - \{3f_2 + (n-2)f_3\}\eta(X)\xi \tag{2.8}$$

$$r = n(n-1)f_1 + 3(n-1)f_2 - 2(n-1)f_3 \tag{2.9}$$

$$\nabla_X \xi = -(f_1 - f_3)\phi X, \quad (\nabla_X \eta)Y = g(\nabla_X \xi, Y) \tag{2.10}$$

where r and Q are the scalar curvature and the Ricci operator of generalized Sasakian-space-form $M(f_1, f_2, f_3)$, respectively. From (2.2), (2.6) and (2.7) we have

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{2.11}$$

$$R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} \tag{2.12}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{2.13}$$

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X) \tag{2.14}$$

$$S(\xi,\xi) = (n-1)(f_1 - f_3) \tag{2.15}$$

for all vector fields $X, Y \in \chi(M^n)$.

Since $S(X,Y) = g(QX,Y)$ and $Q\phi = \phi Q$, from Eq. (2.4) and (2.14), it follows that

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(f_1 - f_3)\eta(X)\eta(Y). \tag{2.16}$$

Theorem 2.1. *([9]) An n-dimensional generalized Sasakian-space-form is projectively flat if and only if* $f_3 = \frac{3f_2}{2-n}$.

3. The D-concircular curvature tensor of a generalized Sasakian-space-form

The D-Concircular curvature tensor V on a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension n is defined by [6]

$$\begin{aligned} V(X,Y)Z &= R(X,Y)Z \\ &+ \frac{r+2(n-1)}{(n-1)(n-2)}\{g(X,Z)Y - g(Y,Z)X\} \\ &- \frac{r+n(n-1)}{(n-1)(n-2)}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \end{aligned} \tag{3.1}$$

where R is the curvature tensor and r is the scalar curvature.

Proposition 3.1. *In an n-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$, the D-Concircular curvature tensor V satisfies*

$$V(X,Y)\xi = \big(f_1 - f_3 + 1\big)\{\eta(Y)X - \eta(X)Y\}, \tag{3.2}$$

$$\eta(V(X,Y)Z) = \big(f_1 - f_3 + 1\big)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{3.3}$$

$$V(\xi,X)Y = \big(f_1 - f_3 + 1\big)\{g(X,Y)\xi - \eta(Y)X\} \tag{3.4}$$

for all vector fields X, Y, Z on $M(f_1 f_2, f_3)$.

Proof. From (2.9)-(2.14) and (3.1) the Eqs. (3.2)-(3.4) follow easily. □

For an n-dimensional D-concirculary flat at generalized Sasakian-space-form from (3.1), we have

$$\begin{aligned} R(X,Y)Z = &-\frac{r+2(n-1)}{(n-1)(n-2)}\{g(X,Z)Y - g(Y,Z)X\} \\ &+ \frac{r+n(n-1)}{(n-1)(n-2)}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \end{aligned} \tag{3.5}$$

In view of (2.6), (2.7) and (2.14) in (3.5) we get

$$\begin{aligned} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ &= -\frac{r+2(n-1)}{(n-1)(n-2)}\{g(X,Z)Y - g(Y,Z)X\} \\ &+ \frac{r+n(n-1)}{(n-1)(n-2)}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \end{aligned} \tag{3.6}$$

In (3.6) putting $X=\xi$, from (2.1) and (2.3) we obtain

$$(f_1-f_3+1)\{g(Y,Z)\xi-\eta(Z)Y\}=0. \tag{3.7}$$

Replacing Z by ϕZ, from (3.7) and in view of (2.9), the above equation reduce to

$$(f_1-f_3+1)g(Y,\phi Z)\xi=0. \tag{3.8}$$

Since $g(Y,\phi Z)\xi\neq 0$ equation (3.8) implies that

$$f_1-f_3=-1. \tag{3.9}$$

Conversely suppose that $f_1-f_3=-1$. Applying (2.6), (2.7) and (2.9) to (3.1), we get

$$\begin{aligned} V(X,Y,Z,W) &= f_1\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ &+ f_2\{g(X,\phi Z)g(\phi Y,W)-g(Y,\phi Z)g(\phi X,W) \\ &+ 2g(X,\phi Y)g(\phi Z,W)\}+f_3\{\eta(X)\eta(Z)g(Y,W) \\ &- \eta(Y)\eta(Z)g(X,W)+g(X,Z)\eta(Y)\eta(W) \\ &- g(Y,Z)\eta(X)\eta(W)\} \\ &+ \frac{r+2(n-1)}{(n-1)(n-2)}\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\} \\ &- \frac{r+n(n-1)}{(n-1)(n-2)}\{g(X,Z)\eta(Y)\eta(W) \\ &- g(Y,Z)\eta(X)\eta(W)+\eta(X)\eta(Z)g(Y,W) \\ &- \eta(Y)\eta(Z)g(X,W)\}. \end{aligned} \tag{3.10}$$

In (3.10), replacing Z by ϕZ and W by ϕW, we have

$$\begin{aligned} V(X,Y,\phi Z,\phi W) &= f_1\{g(Y,\phi Z)g(X,\phi W)-g(X,\phi Z)g(Y,\phi W)\} \\ &+ f_2\{g(X,\phi^2 Z)g(\phi Y,\phi W)-g(Y,\phi^2 Z)g(\phi X,\phi W) \\ &+ 2g(X,\phi Y)g(\phi^2 Z,\phi W)\} \\ &+ \frac{r+2(n-1)}{(n-1)(n-2)}\{g(X,\phi Z)g(Y,\phi W) \\ &- g(Y,\phi Z)g(X,\phi W)\}. \end{aligned} \tag{3.11}$$

In Eq. (3.11) putting $Y=W=e_i$, where $\{e_i : i=1,2,\cdots,n\}$, is an orthonormal basis of the tangent space at any point of manifold, and taking summation over i and in view of (2.4) and (2.5), we get

$$\sum_{i=1}^{n} V(X,e_i,\phi Z,\phi e_i) = \big[-f_1-nf_2+\frac{r+2(n-1)}{(n-1)(n-2)}\big]g(\phi X,\phi Z).$$

From the above equation by a contraction, we get

$$(n-1)\big[-f_1-nf_2+\frac{r+2(n-1)}{(n-1)(n-2)}\big]=0. \tag{3.12}$$

In view of (2.9) and (3.9) in (3.12), we have

$$f_2=0. \tag{3.13}$$

Equations (3.9), (3.12) and (3.13) implies that

$$\frac{r+2(n-1)}{(n-1)(n-2)}=f_1 \tag{3.14}$$

and

$$\frac{r+n(n-1)}{(n-1)(n-2)}=f_3 \tag{3.15}$$

Applying eqs. (3.13)-(3.15) to (3.1), we have $V(X,Y)Z=0$.

Hence we have the following:

Theorem 3.2. *An n-dimensional generalized Sasakian-space-form is D-concircularly flat if and only if* $f_1-f_3=-1$.

In [1], as example of a generalized Sasakian-space-form, it is shown that, a Kenmotsu-space-form, i.e., a Kenmotsu manifold with constant ϕ-sectional curvature c, is a generalized Sasakian-space-form with $f_1=\frac{c-3}{4}$ and $f_2=f_3=\frac{c+1}{4}$, hence $f_1-f_3=-1$. Thus we have the following corollary:

Corollary 3.3. *An n-dimensional Kenmotsu-space-form is D-concircularly flat.*

Since in a Sasakian-space-form $f_1=\frac{c+3}{4}$ and $f_3=\frac{c-1}{4}$, so $f_1-f_3\neq-1$. Thus we may state the following:

Corollary 3.4. *There is no D-concircularly flat Sasakian-space-form.*

A Riemannian manifold (M^n,g) is said to be pseudosymmetric [12] if

$$(R(X,Y).R)(Z,U)W \quad = \quad L_R[((X\wedge Y).R)(Z,U)W]$$

holds on $U_R=\{x\in M|R-\frac{r}{n(n-1)}G\neq 0 \ at \ x\}$, where G is the (0,4)-tensor defined by $G(X,Y,Z,W)=g((X\wedge Y)Z,W)$, L_R is some smooth function on U_R,

$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y. \tag{3.16}$$

and

$$\begin{aligned}(R(X,Y).R)(Z,U)W &= R(X,Y)R(Z,U)W \\ &- R(R(X,Y)Z,U)W \\ &- R(Z,R(X,Y)U)W-R(Z,U)R(X,Y)W.\end{aligned} \tag{3.17}$$

A Riemannian manifold (M^n, g) is said to be D-concircular-pseudosymmetric if

$$(R(X,Y).V)(Z,U)W = L_V[((X \wedge Y).V)(Z,U)W] \quad (3.18)$$

holds on $U_V = \{x \in M | V \neq 0 \ at \ x\}$, where L_V is some function on U_V and V is the D-concircular curvature tensor. Every pseudosymmetric manifold is D-concircular-pseudosymmertic, but the converse is not true. If $R.V = 0$ then (M^n, g) is called D-concircular-semisymmetric.
Let $M(f_1, f_2, f_3)$ be a D-concircular-pseudosymmetric generalized Sasakian-space form. Then from (3.20), we have

$$(R(\xi,Y).V)(Z,U)W = L_V\big[((\xi \wedge Y).V)(Z,U)W\big]. \quad (3.19)$$

In view of (3.16) and (3.17) in (3.19), we have

$$\begin{aligned} & R(\xi,Y)V(Z,U)W - V(R(\xi,Y)Z,U)W \\ - \ & V(Z,R(\xi,Y)U)W - V(Z,U)R(\xi,Y)W \\ = \ & L_V\big[(\xi \wedge Y)V(Z,U)W - V((\xi \wedge Y)Z,U)W \\ - \ & V(Z,(\xi \wedge Y)U)W - V(Z,U)(\xi \wedge Y)W\big]. \end{aligned}$$

Using (2.12) in the above equation, we can see

$$\begin{aligned} 0 = \ & \big[L_V - (f_1 - f_3)\big]\big\{V(Z,U,W,Y)\xi - \eta(V(Z,U)W)Y \\ - \ & g(Y,Z)V(\xi,U)W + \eta(Z)V(Y,U)W - g(Y,U)V(Z,\xi)W \\ + \ & \eta(U)V(Z,Y)W - g(Y,W)V(Z,U)\xi + \eta(W)V(Z,U)Y\big\}, \end{aligned}$$

which implies that either $L_V = f_1 - f_3$ or

$$\begin{aligned} 0 = \ & V(Z,U,W,Y)\xi - \eta(V(Z,Y)W)Y - g(Y,Z)V(\xi,U)W \quad (3.20) \\ + \ & \eta(Z)V(Y,U)W - g(Y,U)V(Z,\xi)W + \eta(U)V(Z,Y)W \\ - \ & g(Y,W)V(Z,U)\xi + \eta(W)V(Z,U)Y. \end{aligned}$$

Assume that $L_V \neq f_1 - f_3$. Taking the inner product of (3.20) with ξ we obtain

$$\begin{aligned} 0 = \ & V(Z,U,W,Y) - \eta(Y)\eta(V(Z,U)W) - g(Y,Z)\eta(V(\xi,U)W) \quad (3.21) \\ + \ & \eta(Z)\eta(V(Y,U)W) - g(Y,U)\eta(V(Z,\xi)W) + \eta(U)\eta(V(Z,Y)W) \\ - \ & g(Y,W)\eta(V(Z,U)\xi) + \eta(W)\eta(V(Z,U)Y). \end{aligned}$$

Putting $Y = Z$ in (3.21), we find

$$\begin{aligned} 0 = \ & V(Y,U,W,Y) - g(Y,Y)\eta(V(\xi,U)W) \quad (3.22) \\ - \ & g(Y,U)\eta(V(Y,\xi)W) + \eta(W)\eta(V(Y,U)Y). \end{aligned}$$

In Eq. (3.22) putting $Y = e_i$, where $\{e_i : i = 1, 2, \cdots, n\}$, is an orthonormal basis of the tangent space at any point of manifold, and taking summation over i, and using (3.3) and (3.4) we get

$$\begin{aligned} S(U,W) &= [f_1 + 3f_2 + (n-3)f_3 + 2 - n]g(U,W) \\ &- [nf_1 + 3f_2 - 2f_3 + n]\eta(U)\eta(W). \end{aligned} \tag{3.23}$$

From (2.7) and (3.23) we can get $f_1 - f_3 = -1$. according to the theorem 3.2, this means that $M(f_1, f_2, f_3)$ is D-concircularly flat.
Hence we can state the following theorem:

Theorem 3.5. *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. If $M(f_1, f_2, f_3)$ is D-concircular-pseudosymmetric then $M(f_1, f_2, f_3)$ is either $L_V = f_1 - f_3$ or $M(f_1, f_2, f_3)$ is D-concircularly flat.*

From the above theorem, it can be seen that:

Corollary 3.6. *Every D-concircular-pseudosymmetric generalized Sasakian-space-form is of the form*

$$(R(X,Y)\ .\ V)(Z,U)W = (f_1 - f_3)\left[((X \wedge Y).V)(Z,U)W\right].$$

Since a Sasakian space form $f_1 - f_3 = 1$ and due to the corollary 3.4, it can be seen that

Corollary 3.7. *For every Sasakian-space-form, we have*

$$(R(X,Y)\ .\ V)(Z,U)W = \left[((X \wedge Y).V)(Z,U)W\right].$$

A Riemannian manifold is said to be D-concircularly Ricci-semisymmetric if the relation $V(X,Y)\ .\ S = 0$ holds.
Now we prove the following theorem:

Theorem 3.8. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space-form. $M(f_1, f_2, f_3)$ is D-concircularly Ricci-semisymmetric if and only if $f_3 = \frac{3f_2}{2-n}$ or $M(f_1, f_2, f_3)$ is D-concircularly flat.*

Proof. Assume that $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space-form. The condition $V(X,Y)\ .\ S = 0$, implies that

$$S(V(X,Y)Z, W) + S(W, V(X,Y)Z) = 0. \tag{3.24}$$

In view of 2.7 in 3.24 we get

$$0 = [3f_2 + (n-2)f_3]\left[\eta(V(X,Y)Z)\eta(W) + \eta(Z)\eta(V(X,Y)W)\right]. \tag{3.25}$$

The equation 3.25, implies that either

$$f_3 = \frac{3f_2}{2-n}, \tag{3.26}$$

or

$$0 = \eta(V(X,Y)Z)\eta(W) + \eta(Z)\eta(V(X,Y)W). \tag{3.27}$$

From 3.3 in (3.27) we have

$$\begin{aligned} 0 &= (f_1 - f_3 + 1)\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ &+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)\}. \end{aligned} \tag{3.28}$$

Taking $Z = \xi$ in (3.28), by a contraction, we get

$$f_1 - f_3 = -1. \tag{3.29}$$

According to the theorem 3.2 and the equation (3.29), $M(f_1, f_2, f_3)$ is D-concircularly flat.
Conversely suppose that

$$f_3 = \frac{3f_2}{2-n}, \tag{3.30}$$

Applying (3.30) in (2.7), we can see that $M(f_1, f_2, f_3)$ is an Einstein manifold. Therefore it's easily visible that $M(f_1, f_2, f_3)$ is D-concircularly Ricci-semisymmetric. Also if $M(f_1, f_2, f_3)$ be D-concircularly flat it's trivial $M(f_1, f_2, f_3)$ is D-concircularly Ricci-semisymmetric. □

From the theorem 2.1, we have the following:

Corollary 3.9. *A non D-concircularly at generalized Sasakian-space-form is D-concircularly Ricci- semisymmetric if and only if it is Ricci-semisymmetric.*

Now, we prove the following:

Theorem 3.10. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space-form. Then $M(f_1, f_2, f_3)$ satisfies the condition $R(\xi, X).V = 0$ if and only if $f1 = f3$ or $M(f_1, f_2, f_3)$ is D-concircularly flat.*

Proof. Assume that $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space-form and satisfies the condition $R(\xi, X).V = 0$, we can write

$$\begin{aligned} 0 &= R(\xi,X)V(Y,Z)W - V(R(\xi,X)Y,Z)W \\ &- V(Y,R(\xi,X)Z)W - V(Y,Z)R(\xi,X)W, \end{aligned} \tag{3.31}$$

for all vector fields X, Y, Z, W. Using (2.12), in (3.31) we find

$$\begin{aligned} 0 &= (f_1 - f_3)\Big\{V(Y,Z,W,X)\xi - \eta(V(Y,Z)W)X \\ &- g(X,Y)V(\xi,Z)W + V(X,Z)W\eta(Y), \\ &- g(X,Z)V(Y,\xi)W + V(Y,X)W\eta(Z) \\ &- g(X,W)V(Y,Z)\xi + V(Y,Z)X\eta(W)\Big\}, \end{aligned} \tag{3.32}$$

which implies that either $f_1 - f_3 = 0$ or

$$\begin{aligned} 0 &= V(Y,Z,W,X)\xi - \eta(V(Y,Z)W)X \\ &- g(X,Y)V(\xi,Z)W + V(X,Z)W\eta(Y), \\ &- g(X,Z)V(Y,\xi)W + V(Y,X)W\eta(Z) \\ &- g(X,W)V(Y,Z)\xi + V(Y,Z)X\eta(W). \end{aligned} \tag{3.33}$$

Assume that $f_1 \neq f_3$. Taking the inner product of (3.33) with ξ we obtain

$$\begin{aligned} 0 &= V(Y,Z,W,X) - \eta(V(Y,Z)W)\eta(X) \\ &- g(X,Y)\eta(V(\xi,Z)W) + \eta(V(X,Z)W)\eta(Y), \\ &- g(X,Z)\eta(V(Y,\xi)W) + \eta(V(Y,X)W)\eta(Z) \\ &- g(X,W)\eta(V(Y,Z)\xi) + \eta(V(Y,Z)X)\eta(W). \end{aligned} \tag{3.34}$$

Hence in view of (3.2)-(3.4) the Eq. (3.34) is reduced to

$$\begin{aligned} V(Y,Z,W,X) &= [f_1 - f_3 + 1] \\ &\quad \times \{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\}. \end{aligned} \tag{3.35}$$

So by a suitable contraction of (3.35) we get

$$\begin{aligned} S(Z,W) &= (n-1)\left[\frac{r+(n-1)}{n-1} + (n-1)(f_1 - f_3 + 1)\right] \\ &\quad \times g(Z,W) - \left[\frac{r+(n-1)}{n-1}\right]\eta(Z)\eta(W). \end{aligned} \tag{3.36}$$

From (2.7), (2.9) and (3.36) we have

$$f_1 - f_3 = -1 \tag{3.37}$$

According to the theorem 3.2, the above equation implies that $M(f_1, f_2, f_3)$ is a D-concircularly flat manifold.
Conversely, if $f_1 = f_3$ then from (2.12), we can see that $R(\xi, X) = 0$. Obviously $R(\xi, X).V = 0$. Also if $M(f_1, f_2, f_3)$ be D-concircularly flat, in view of (3.31), this shows that $R(\xi, X).V = 0$. This completes the proof of the theorem. □

A Riemannian manifold is said to be D-concircularly symmetric if it satisfies $\nabla V = 0$ and is said to be D-concircularly semisymmetric if it satisfies $R(X,Y).V = 0$. It can be seen easily that if $M(f_1, f_2, f_3)$ is a D-concircularly symmetric, then it is D-concircularly semisymmetric. Thus we have the following two corollaries:

Corollary 3.11. *If $M(f_1, f_2, f_3)$ be an n-dimensional D-concircularly symmetric generalized Sasakian- space-form, then $f_1 = f_3$ or $M(f_1, f_2, f_3)$ is D-concircularly flat.*

Corollary 3.12. *If $M(f_1, f_2, f_3)$ be an n-dimensional D-concircularly semisymmetric generalized Sasakian- space-form, then $f_1 = f_3$ or $M(f_1, f_2, f_3)$ is D-concircularly flat.*

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