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Bayesian two-sample prediction problem for the Rayleigh distribution under progressively Type-II censoring with random removals

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Abstract. In this paper, we study the prediction problem in the two-sample case for predicting future progressively Type-II censored order statistics based on observed progressively Type-II censored order statistics with random removals from the Rayleigh distribution. We consider two important distributions for random removals, binomial and discrete uniform distributions. In both cases, Bayesian point and interval predictors are obtained. In the following, through a simulation study, the results are compared to each other. Finally, a real data set is given to illustrate the output results.

Keywords: Bayesian point predictor, Mean squared prediction error, Rayleigh distribution, Random censoring scheme.

2000 Mathematics subject classification: 62F15, 62N01.

1. Introduction

The Rayleigh distribution was introduced by Rayleigh [20]. This model has an extensive range of applications in many statistical fields, including life-testing and reliability theory, as its failure rate is a linear function of time. The origin and other aspects of this distribution can be

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found in Siddiqui [21] and Miller and Sackrowitz [19]. Inference and prediction problems for the Rayleigh distribution have been discussed by several authors. Fernández [16] considered a Bayesian approach to inference in reliability studies based on Type-II doubly censored data from a Rayleigh distribution. Bayesian prediction of progressively Type-II censored data from the Rayleigh distribution was considered by Ali Mousa and Al-Sagheer [2]. Asgharzadeh and Azizpour [3] based on a hybrid censored sample from a Rayleigh distribution obtained Bayes estimators and highest posterior density credible intervals. Asgharzadeh et al. [4] obtained exact confidence intervals and regions for the location and scale parameters of the Rayleigh distribution. Basiri [10] studied the problem of finding the optimal number of failures in Type-II censoring by considering two criteria, total cost of experiment and mean squared prediction error in the Rayleigh distribution.

The scheme of progressive Type-II censoring is an important method of obtaining data in lifetime studies. It allows the experimenter to remove units from a life test at various stages during the experiment. Under the progressively Type-II censoring scheme, n units are placed on a lifetime test. At the first failure time, r_1 surviving items are randomly withdrawn from the test. At the second failure time, r_2 surviving items are selected at random and taken out of the experiment, and so on. Finally, at the time of the mth failure, the remaining $r_m = n-m-\sum_{i=1}^{m-1} r_i$ objects are removed. If the failure times are based on an absolutely continuous cumulative distribution function (cdf) $F(\cdot)$ and probability density function (pdf) $f(\cdot)$, and denote the ith failure time by $X_{i:m:n}^{\tilde{r}}$ then random variables $X_{1:m:n}^{\tilde{r}}, \ldots, X_{m:m:n}^{\tilde{r}}$ are called *progressively Type-II censored order statistics* (PCOs) based on censoring scheme $\tilde{r} = (r_1, \ldots, r_m)$, where $n = m + \sum_{j=1}^{m} r_j$. For notational simplicity, hereafter we use $X_{i:m:n}$ instead of $X_{i:m:n}^{\tilde{r}}$. For a detailed discussion of progressive censoring, we refer the reader to Balakrishnan and Aggarwala [7] Balakrishnan [6], Balakrishnan and Cramer [8] and the references contained therein.

One of the issues that the experimenter always pays attention to in lifetime tests is to predict future events and future failure times. So far, several researchers have studied the problem of prediction and both the classical and Bayesian approaches have been utilized. For example, two methods for obtaining prediction intervals for the times to failure of units censored in multiple stages in a progressively censored sample from proportional hazard rate models have been presented by Asgharzadeh and Valiollahi [5]. Ahmadi et al. [1] derived Bayesian prediction intervals for k-record values from a future sequence based on observed progressively

Type-II censored data from an exponential distribution. Basiri and Ahmadi [11] obtained non-parametric prediction intervals for progressively Type-II censored order statistics in terms of upper and lower k-records. Basiri and Ahmadi [12] studied the problem of non-parametric predicting future generalized order statistics, by assuming the future sample size is a random variable. They also considered predicting future progressively Type-II censored order statistics with random sample size as a special case.

All these mentioned works assumed that the censoring scheme, or values of $r_1, \ldots, r_m$ are all prefixed. But, for example in a clinical test, the number of patients that dropped out at each stage is random and cannot be fixed. In such cases, the pattern of removal at each failure is random. Soliman et al. [22] studied the problem of predicting future records and order statistics (two-sample prediction) based on progressive Type-II censored with binomial removals. Meshkat and Dehqani [18] obtained some different point predictors such as maximum likelihood predictors, best unbiased predictors and conditional median predictors, for failure times of units censored in a progressively censored sample from proportional hazard rate models, where the number of units removed at each failure time follows a binomial distribution. The problem of finding optimal censoring scheme in progressively Type-II censoring with binomial removals, has been considered by Basiri and Beigi [13].

Motivated by these mentioned articles, the aim of this article is to study the prediction problem in the two-sample case for predicting future progressively Type-II censored order statistics based on observed progressively Type-II censored order statistics, when the removals are random variables. Two important distributions, namely binomial and discrete uniform distributions, are considered for random removals.

The rest of the paper is organized as follows. At the beginning of Section 2, we introduce the notations used throughout the paper. Then, we construct the Bayesian point and interval predictors for future progressively Type-II censored order statistics based on observed progressively Type-II censored order statistics with random removal. Two distributions namely binomial and discrete uniform distributions are considered for censoring scheme. The results are compared to each other through a simulation study in Section 3. Finally, our results are applied to one real data set in Section 4.

2. Main results

In this article, we consider one parameter Rayleigh distribution, denoted by $Ray(\theta)$, with the following probability density function (pdf) and cumulative distribution function (cdf)

$$f(x|\theta) = 2\theta x e^{-\theta x^2}, \quad \text{and} \quad F(x|\theta) = 1 - e^{-\theta x^2}, \quad x > 0,\ \theta > 0, \quad (2.1)$$

where $\sqrt{\theta}$ is the scale parameter.

Let $\tilde{x} = (x_{1:m_1:n_1}, \cdots, x_{m_1:m_1:n_1})$ be an observed progressively Type-II right censored sample from a life test of size m_1 from a sample of size n_1 with independent and identically distributed (iid) continuous random variables and censoring scheme $\tilde{R} = (R_1, R_2, \cdots, R_{m_1})$, where R_i, $(i = 1, \cdots, m_1)$, are random variables independent on $\tilde{X}$, such that $\sum_{i=1}^{m_1} R_i = n_1 - m_1$. Moreover, lifetimes have the one parameter Rayleigh distribution with pdf and cdf as given by (2.1). With a pre-determined number of removal of units from the test, say $R_1 = r_1, R_2 = r_2, \cdots, R_{m_1} = r_{m_1}$, the conditional likelihood function, takes the form (see, for example, Balakrishnan and Aggarwala [7])

$$\begin{aligned} L(\theta, \tilde{x}|\tilde{R} = \tilde{r}) &= C \prod_{i=1}^{m_1} f(x_i|\theta)(\bar{F}(x_i|\theta))^{r_i} \\ &= C^* \theta^{m_1} \left(\prod_{i=1}^{m_1} x_i \right) \exp(-\theta T), \end{aligned} \quad (2.2)$$

where $\bar{F}(x|\theta) = 1 - F(x|\theta)$ is the reliability function of the X-sample, $T = \sum_{i=1}^{m_1} (1 + r_i) x_i^2$, $C^* = 2^{m_1} C$ and $C = \prod_{j=1}^{m_1} (n_1 - \sum_{i=1}^{j-1} r_i - j + 1)$.

Independently, let $Y_{s:m_2:n_2}$ be the sth future progressively Type-II right censored order statistic from a life test of size m_2 from a sample of size n_2 from the same distribution and censoring scheme $\tilde{R}' = (R'_1, R'_2, \cdots, R'_{m_2})$, where R'_i, $(i = 1, \cdots, m_2)$, are random variables independent on $\tilde{Y}$, such that $\sum_{i=1}^{m_2} R'_i = n_2 - m_2$. Then, given $R'_1 = r'_1, \cdots, R'_{s-1} = r'_{s-1}$ (with $r'_0 = 0$), the marginal pdf of $Y_{s:m_2:n_2}$, $(1 \le s \le m_2)$ from the one parameter Rayleigh distribution, is given by (see, for example, Balakrishnan and Aggarwala [7])

$$\begin{aligned} f_{Y_{s:m_2:n_2}|R'_1 = r'_1, \cdots, R'_{s-1} = r'_{s-1}}(y) &= c'_{s-1} \sum_{i=1}^{s} a'_{i,s} (\bar{F}(y|\theta))^{\gamma'_i - 1} f(y|\theta) \\ &= 2\theta y c'_{s-1} \sum_{i=1}^{s} a'_{i,s} \exp\{-\theta \gamma'_i y^2\}, \quad y > 0, \end{aligned} \quad (2.3)$$

where $\gamma'_i = n_2 - i + 1 - \sum_{j=1}^{i-1} r'_j$, $c'_{s-1} = \prod_{j=1}^{s} \gamma'_j$ and $a'_{i,s} = \displaystyle\prod_{j=1, j\neq i}^{s} \frac{1}{\gamma'_j - \gamma'_i}$, $1 \le i \le s \le m_2$. So, the marginal pdf of $Y_{s:m_2:n_2}$, $(1 \le s \le m_2)$ can be evaluated by taking expectation on both sides (2.3) with respect to $\tilde{R}' = (R'_1, R'_2, \cdots, R'_{s-1})$. That is

$$\begin{aligned} f_{Y_{s:m_2:n_2}}(y) &= \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} f_{Y_{s:m_2:n_2}|R'_1=r'_1,\cdots,R'_{s-1}=r'_{s-1}}(y) \\ &\quad \times P(R'_1 = r'_1, \cdots, R'_{s-1} = r'_{s-1}) \\ &= \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} \sum_{i=1}^{s} 2\theta y c'_{s-1} a'_{i,s} e^{-\theta \gamma'_i y^2} \\ &\quad \times P(R'_1 = r'_1, \cdots, R'_{s-1} = r'_{s-1}), \end{aligned} \tag{2.4}$$

where $g(r'_i) = n_2 - m_2 - \sum_{j=0}^{i-1} r'_j$, $i = 1, \cdots, s-1$.

In this paper, we want to predict the value of $Y_{s:m_2:n_2}$ based on $\tilde{x} = (x_{1:m_1:n_1}, \cdots, x_{m_1:m_1:n_1})$. To this end, in what follows, we consider two cases for removing units from the test for both observed and future sample, namely binomial and discrete uniform distributions.

2.1. **Binomial distribution.** Presume that an individual unit being removed from the life test is independent of the others but with the same probability p, $(0 < p < 1)$, the number of units removed at each failure time follows a binomial distribution such that

$$P(R_1 = r_1) = \binom{n_1 - m_1}{r_1} p^{r_1} (1-p)^{n_1 - m_1 - r_1}, \quad r_1 = 0, \cdots, n_1 - m_1, \tag{2.5}$$

and

$$\begin{aligned} &P(R_i = r_i | R_1 = r_1, \cdots, R_{i-1} = r_{i-1}) \\ &= \binom{n_1 - m_1 - \sum_{k=1}^{i-1} r_k}{r_i} p^{r_i} (1-p)^{n_1 - m_1 - \sum_{k=1}^{i} r_k}, \end{aligned} \tag{2.6}$$

for $r_i = 0, \cdots, n_1 - m_1 - \sum_{k=1}^{i-1} r_k$, $i = 2, \cdots, m_1 - 1$ and all the remaining items, if there are some, are all removed from the test at the m_1-th failure with probability one. So, from (2.5) and (2.6), it can be written that

$$\begin{aligned} P(\tilde{R} = \tilde{r}) &= P(R_1 = r_1) P(R_2 = r_2 | R_1 = r_1) \cdots \\ &\quad \times P(R_{m_1-1} = r_{m_1-1} | R_1 = r_1, \cdots, R_{m_1-2} = r_{m_1-2}) \\ &= A(\tilde{R}, m_1, n_1, m_1) p^{\mu_1(\tilde{R}, m_1)} (1-p)^{\mu_2(\tilde{R}, m_1, n_1, m_1)}, \end{aligned} \tag{2.7}$$

where

$$A(\tilde{R}, m, n, j) = \frac{(n-m)!}{\prod_{i=1}^{j-1} r_i!(n-m-\sum_{i=1}^{j-1} r_i)!}, \tag{2.8}$$

$$\mu_1(\tilde{R}, j) = \sum_{i=1}^{j-1} r_i, \quad \text{and} \quad \mu_2(\tilde{R}, m, n, j) = (j-1)(n-m) - \sum_{i=1}^{j-1}(j-i)r_i. \tag{2.9}$$

Therefore, using (2.2) and (2.7), the joint likelihood function of $\tilde{X} = (X_{1:m_1:n_1}, \cdots, X_{m_1:m_1:n_1})$ and $\tilde{R} = (R_1, \cdots, R_m)$ can be expressed as

$$\begin{aligned} L(\theta, p; \tilde{x}, \tilde{r}) &= L(\theta, \tilde{x}|\tilde{R} = \tilde{r})P(\tilde{R} = \tilde{r}) \\ &= AL_1(\theta)L_2(p), \end{aligned} \tag{2.10}$$

where $A = C^* \left(\prod_{i=1}^{m_1} x_i\right) A(\tilde{R}, m_1, n_1, m_1)$ does not depend on the parameters θ and p and

$$L_1(\theta) = \theta^{m_1} \exp\{-\theta T\}, \tag{2.11}$$

$$L_2(p) = p^{\mu_1(\tilde{R}, m_1)}(1-p)^{\mu_2(\tilde{R}, m_1, n_1, m_1)}. \tag{2.12}$$

The conjugate prior distribution for θ and p are considered as follows

$$\pi_1(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, \quad \theta > 0, \quad a, b > 0, \tag{2.13}$$

and

$$\pi_2(p) = \frac{1}{\beta(c,d)}p^{c-1}(1-p)^{d-1}, \quad 0 < p < 1, \quad c, d > 0, \tag{2.14}$$

respectively, where $\Gamma(\cdot)$ and $\beta(\cdot, \cdot)$ are the complete Gamma and Beta functions, respectively. Moreover, θ and p are assumed to be independent. Thus, the joint prior distribution for θ and p is given by

$$\pi(\theta, p) = \frac{b^a}{\Gamma(a)\beta(c,d)}\theta^{a-1}e^{-b\theta}p^{c-1}(1-p)^{d-1}, \quad \theta > 0, \quad 0 < p < 1, \tag{2.15}$$

for $a, b, c, d > 0$. Therefore, the joint posterior distribution of θ and p will be obtained as

$$\begin{aligned} &\pi(\theta, p|\tilde{x}, \tilde{r}) \\ &= \frac{\theta^{a+m_1-1}e^{-\theta(b+T)}p^{c+\mu_1(\tilde{R},m_1)-1}(1-p)^{d+\mu_2(\tilde{R},m_1,n_1,m_1)-1}}{\int_0^\infty \int_0^1 \theta^{a+m_1-1}e^{-\theta(b+T)}p^{c+\mu_1(\tilde{R},m_1)-1}(1-p)^{d+\mu_2(\tilde{R},m_1,n_1,m_1)-1}dpd\theta} \\ &= \frac{(b+T)^{a+m_1}\theta^{a+m_1-1}e^{-\theta(b+T)}p^{c+\mu_1(\tilde{R},m_1)-1}(1-p)^{d+\mu_2(\tilde{R},m_1,n_1,m_1)-1}}{\Gamma(a+m_1)\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}, \end{aligned} \tag{2.16}$$

where $\mu_1(\cdot, \cdot)$ and $\mu_2(\cdot, \cdot, \cdot, \cdot)$ are defined in (2.9).

On the other hand, the marginal pdf of $Y_{s:m_2:n_2}$, $(1 \leq s \leq m_2)$ from (2.4) is obtained as

$$\begin{aligned} f_{Y_{s:m_2:n_2}}(y) &= \sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')}\sum_{i=1}^{s} 2A(\tilde{R}', m_2, n_2, s)\theta y c_{s-1}' a_{i,s}' e^{-\theta\gamma_i' y^2} \\ &\quad \times p^{\mu_1(\tilde{R}',s)}(1-p)^{\mu_2(\tilde{R}',m_2,n_2,s)}, \end{aligned} \tag{2.17}$$

where $A(\cdot,\cdot,\cdot,\cdot)$, $\mu_1(\cdot,\cdot)$ and $\mu_2(\cdot,\cdot,\cdot,\cdot)$ are defined in (2.8) and (2.9), respectively. So, the predictive density function for $Y_{s:m_2:n_2}$ is given by (see, for example, Dunsmore [15])

$$\begin{aligned} & f^*_{Y_{s:m_2:n_2}}(y|\tilde{x},\tilde{r}) \\ &= \int_0^1\int_0^\infty f_{Y_{s:m_2:n_2}}(y)\pi(\theta,p|\tilde{x},\tilde{r})\mathrm{d}\theta\mathrm{d}p \\ &= \sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')}\sum_{i=1}^{s} \frac{2A(\tilde{R}', m_2, n_2, s) y c_{s-1}' a_{i,s}'(b+T)^{a+m_1}}{\Gamma(a+m_1)\beta\Big(c+\mu_1(\tilde{R},m_1), d+\mu_2(\tilde{R},m_1,n_1,m_1)\Big)} \\ &\times \int_0^\infty \theta^{a+m_1} e^{-\theta(b+T+\gamma_i' y^2)}\mathrm{d}\theta \\ &\times \int_0^1 p^{\mu_1(c+\mu_1(\tilde{R}',s)+\mu_1(\tilde{R},m_1)-1}(1-p)^{d+\mu_2(\tilde{R}',m_2,n_2,s)+\mu_2(\tilde{R},m_1,n_1,m_1)-1}\mathrm{d}p \\ &= 2y\sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')} A(\tilde{R}', m_2, n_2, s)c_{s-1}'(a+m_1)(b+T)^{a+m_1} \\ &\times \frac{\beta\Big(c+\mu_1(\tilde{R}',s)+\mu_1(\tilde{R},m_1), d+\mu_2(\tilde{R}',m_2,n_2,s)+\mu_2(\tilde{R},m_1,n_1,m_1)\Big)}{\beta\Big(c+\mu_1(\tilde{R},m_1), d+\mu_2(\tilde{R},m_1,n_1,m_1)\Big)} \\ &\times \sum_{i=1}^{s}\frac{a_{i,s}'}{(b+T+\gamma_i' y^2)^{a+m_1+1}}. \end{aligned}$$

Moreover, the predictive posterior survival function of $Y_{s:m_2:n_2}$ can be written as

$$\begin{aligned}
&\bar{F}^*_{Y_{s:m_2:n_2}}(y|\tilde{x},\tilde{r}) \\
&= \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} A(\tilde{R}', m_2, n_2, s) c'_{s-1} (b+T)^{a+m_1} \\
&\times \frac{\beta\Big(c+\mu_1(\tilde{R}', s)+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}', m_2, n_2, s)+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}{\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)} \\
&\times \sum_{i=1}^{s} \frac{a'_{i,s}}{\gamma'_i (b+T+\gamma'_i y^2)^{a+m_1}}.
\end{aligned}$$

Interval Predictors. The Bayesian predictive bounds of a two-sided equi-tailed $100(1-\alpha)\%$ prediction interval for $Y_{s:m_2:n_2}$, $1 \le s \le m_2$, can be obtained through solving the following equations:

$$\bar{F}^*_{Y_{s:m_2:n_2}}(L|\tilde{x},\tilde{r}) = 1-\frac{\alpha}{2}, \qquad \text{and} \qquad \bar{F}^*_{Y_{s:m_2:n_2}}(U|\tilde{x},\tilde{r}) = \frac{\alpha}{2}, \tag{2.18}$$

where L and U are the lower and upper bounds, respectively. Suppose that $\zeta_{Y_{s:m_2:n_2},\alpha}(\boldsymbol{x})$ is the upper quantile of the predictive density, i.e.

$$\bar{F}^*_{Y_{s:m_2:n_2}}\left(\zeta_{Y_{s:m_2:n_2},\alpha}(\tilde{x})|\tilde{x},\tilde{r}\right) = \alpha,$$

then clearly we have

$$L = \zeta_{Y_{s:m_2:n_2},1-\alpha/2}(\tilde{x}), \qquad \text{and} \qquad U = \zeta_{Y_{s:m_2:n_2},\alpha/2}(\tilde{x}).$$

In general, we do not have a closed-form expression for quantiles but they can numerically be calculated using statistical software packages. For the special case, $s=1$, the minimum of the future sample, we have

$$\zeta_{Y_{1:m_2:n_2},\alpha}(\tilde{x}) = \sqrt{\frac{(b+T)}{n_2}\left\{\left(\frac{1}{\alpha}\right)^{\frac{1}{a+m_1}} - 1\right\}}.$$

The predictive density function $f^*_{Y_{s:m_2:n_2}}(y|\tilde{x},\tilde{r})$ is unimodal. So, the $100(1-\alpha)\%$ highest posterior density prediction interval (HPD PI) of the form (w_1, w_2), when $s \ge 2$, can be derived by solving the following equations simultaneously

$$\bar{F}^*_{Y_{s:m_2:n_2}}(w_1|\tilde{x},\tilde{r}) - \bar{F}^*_{Y_{s:m_2:n_2}}(w_2|\tilde{x},\tilde{r}) = 1-\alpha \tag{2.19}$$

and

$$f^*_{Y_{s:m_2:n_2}}(w_1|\tilde{x},\tilde{r}) = f^*_{Y_{s:m_2:n_2}}(w_2|\tilde{x},\tilde{r}). \tag{2.20}$$

Since the predictive density of $Y_{1:m_2:n_2}$ is decreasing with respect to y, the $100(1-\alpha)\%$ HPD prediction interval for $Y_{1:m_2:n_2}$ takes the following form

$$\left(0, \sqrt{\frac{(b+T)}{n_2}\left\{\left(\frac{1}{\alpha}\right)^{\frac{1}{a+m_1}}-1\right\}}\right).$$

Point Predictors. First, we recall that if $Z_1, Z_2, \ldots, Z_m$ are iid random variables from the standard exponential distribution, then (see for example, Balakrishnan and Aggarwala [7], p. 19)

$$W_{s:m:n} \stackrel{\mathrm{d}}{=} \sum_{l=1}^{s} \frac{Z_l}{n-\sum_{k=0}^{l-1} r_k - l + 1}, \tag{2.21}$$

where $W_{s:m:n}$ denotes the sth progressively Type-II right censored order statistic from the standard exponential distribution and $\stackrel{\mathrm{d}}{=}$ stands for identical in distribution. So, from (2.21), we can write

$$E(W_{s:m:n}) = g(s,m,n), \quad \text{where} \quad g(s,m,n) = \sum_{l=1}^{s} \frac{1}{n-\sum_{k=0}^{l-1} r_k - l + 1}. \tag{2.22}$$

The point predictor for $Y_{s:m_2:n_2}$, $1 \leq s \leq m_2$, under squared error loss (SEL) function is (details are given in the appendix)

$$\begin{aligned}
\widehat{Y}_{s:m_2:n_2} &= (b+T)^{\frac{1}{2}} \beta\left(a+m_1-\frac{1}{2}, \frac{1}{2}\right) \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} A(\tilde{R}', m_2, n_2, s) \\
&\quad \times \frac{\beta\Big(c+\mu_1(\tilde{R}', s)+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}', m_2, n_2, s)+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}{\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)} \\
&\quad \times k(s, m_2, n_2),
\end{aligned} \tag{2.23}$$

where $A(\cdot,\cdot,\cdot,\cdot)$, $\mu_1(\cdot,\cdot)$ and $\mu_2(\cdot,\cdot,\cdot,\cdot)$ are defined in (2.8) and (2.9) and

$$k(s, m_2, n_2) = \frac{1}{2} c'_{s-1} \sum_{i=1}^{s} \frac{a'_{i,s}}{\gamma_i'^{\frac{3}{2}}}. \tag{2.24}$$

Since in many real applications, no prior knowledge is available about the distribution of the parameters, we may take $a = b = c = d \approx 0$, i.e. the Jeffrey's non-information prior for θ. So, we have

$$\widehat{Y}_{s:m_2:n_2} = T^{\frac{1}{2}} Q, \tag{2.25}$$

where

$$Q = \beta\left(m_1 - \frac{1}{2}, \frac{1}{2}\right) \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} A(\tilde{R}', m_2, n_2, s)$$
$$\times \frac{\beta\Big(\mu_1(\tilde{R}', s) + \mu_1(\tilde{R}, m_1), \mu_2(\tilde{R}', m_2, n_2, s) + \mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}{\beta\Big(\mu_1(\tilde{R}, m_1), \mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}$$
$$\times k(s, m_2, n_2), \tag{2.26}$$

and $A(\cdot,\cdot,\cdot,\cdot)$, $\mu_1(\cdot,\cdot)$ and $\mu_2(\cdot,\cdot,\cdot,\cdot)$ are defined in (2.8) and (2.9).

It is easy to show that $X^2_{i:m_1:n_1}$, $i = 1, \cdots, m_1$, are the corresponding progressively Type-II censored order statistics from the exponential distribution with parameter θ. So, conditioned on $\{\tilde{R} = \tilde{r}\}$, T has the gamma distribution with parameters m_1 and θ, i.e. (see, for example, Balakrishnan and Aggarwala [7], p. 17)

$$T | \tilde{R} = \tilde{r} \sim \Gamma(m_1, \theta).$$

Therefore

$$E(\widehat{Y}_{s:m_2:n_2}) = \frac{Q}{\theta^{1/2}} \frac{\Gamma(m_1 + 1/2)}{\Gamma(m_1)}, \quad \text{and} \quad E((\widehat{Y}_{s:m_2:n_2})^2) = \frac{m_1 Q^2}{\theta},$$

and consequently

$$Var(\widehat{Y}_{s:m_2:n_2}) = \frac{1}{\theta} Q^2 \left\{ m_1 - \left(\frac{\Gamma(m_1 + 1/2)}{\Gamma(m_1)} \right)^2 \right\}.$$

Moreover, from (2.17) we can write

$$E(Y_{s:m_2:n_2}) = \frac{1}{\theta^{1/2}} \Psi_1, \quad \text{and} \quad E((Y_{s:m_2:n_2})^2) = \frac{1}{\theta} \Psi_2,$$

where

$$\Psi_1 = \Gamma(\frac{1}{2}) \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} A(\tilde{R}', m_2, n_2, s) p^{\mu_1(\tilde{R}', s)} (1-p)^{\mu_2(\tilde{R}', m_2, n_2, s)}$$
$$\times k(s, m_2, n_2), \tag{2.27}$$

$$\Psi_2 = \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} A(\tilde{R}', m_2, n_2, s) p^{\mu_1(\tilde{R}', s)} (1-p)^{\mu_2(\tilde{R}', m_2, n_2, s)}$$
$$\times g(s, m_2, n_2), \tag{2.28}$$

and $k(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ are defined as in (2.24) and (2.22). So, we find that

$$V(Y_{s:m_2:n_2}) = \frac{1}{\theta} \left\{ \Psi_2 - \Psi_1^2 \right\}.$$

Thus, we can consider mean squared prediction error (MSPE) of the obtained point predictor $\widehat{Y}_{s:m_2:n_2}$ which is computed to be

$$\begin{aligned} MSPE(\widehat{Y}_{s:m_2:n_2}) &= E\Big(\widehat{Y}_{s:m_2:n_2} - Y_{s:m_2:n_2}\Big)^2 \\ &= \frac{1}{\theta}\left\{ m_1 Q^2 + \Psi_2 - 2Q\Psi_1 \frac{\Gamma(m_1+\frac{1}{2})}{\Gamma(m_1)} \right\}. \end{aligned}$$

2.2. **Uniform distribution.** Now, suppose that the number of units removed at each failure time follows a discrete uniform distribution such that

$$P(R_1 = r_1) = \frac{1}{n_1 - m_1 + 1}, \quad r_1 = 0, \cdots, n_1 - m_1, \tag{2.29}$$

and

$$P(R_i = r_i | R_1 = r_1, \cdots, R_{i-1} = r_{i-1}) = \frac{1}{n_1 - m_1 - \sum_{j=1}^{i-1} r_j + 1}, \tag{2.30}$$

for $r_i = 0, \cdots, n_1 - m_1 - \sum_{k=1}^{i-1} r_k$, $i = 2, \cdots, m_1 - 1$, and all the remaining items, if there are some, are all removed from the test at the m_1-th failure with probability one. So, relations (2.29) and (2.30) provide

$$P(\tilde{R} = \tilde{r}) = B(\tilde{R}, m_1, n_1, m_1), \tag{2.31}$$

where

$$B(\tilde{R}, m, n, j) = \prod_{i=1}^{j-1} \frac{1}{n - m - \sum_{k=1}^{i-1} r_k + 1}. \tag{2.32}$$

Now using (2.2) and (2.31), we can write the full likelihood function as

$$L(\theta; \tilde{x}, \tilde{r}) = BL_1(\theta), \tag{2.33}$$

where $B = C^* \left(\prod_{i=1}^{m_1} x_i\right) B(\tilde{R}, m_1, n_1, m_1)$ does not depend on the parameter θ and $L_1(\theta)$ is defined as in (2.11). Based on the conjugate prior distribution for θ considered as (2.13), the posterior distribution of θ will be obtained as

$$\pi(\theta | \tilde{x}, \tilde{r}) = \frac{(b+T)^{a+m_1} \theta^{a+m_1-1} e^{-\theta(b+T)}}{\Gamma(a+m_1)}.$$

On the other hand, the marginal pdf of $Y_{s:m_2:n_2}$, $(1 \le s \le m_2)$ from (2.4) can be expressed as

$$f_{Y_{s:m_2:n_2}}(y) = \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} \sum_{i=1}^{s} 2B(\tilde{R}', m_2, n_2, s)\theta y c'_{s-1} a'_{i,s} e^{-\theta\gamma'_i y^2},$$

where $B(\cdot,\cdot,\cdot,\cdot)$ is defined in (2.32). Then, the predictive density function for $Y_{s:m_2:n_2}$ can be obtained by

$$\begin{aligned} f^*_{Y_{s:m_2:n_2}}(y|\tilde{x},\tilde{r}) &= \sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')}\sum_{i=1}^{s}\frac{2B(\tilde{R}',m_2,n_2,s)yc'_{s-1}a'_{i,s}(b+T)^{a+m_1}}{\Gamma(a+m_1)} \\ &\quad\times\int_0^\infty \theta^{a+m_1}e^{-\theta(b+T+\gamma_i'y^2)}\mathrm{d}\theta \\ &= 2y\sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')}B(\tilde{R}',m_2,n_2,s)c'_{s-1}(a+m_1)(b+T)^{a+m_1} \\ &\quad\times\sum_{i=1}^{s}\frac{a'_{i,s}}{(b+T+\gamma_i'y^2)^{a+m_1+1}}. \end{aligned}$$

Finally, the predictive posterior survival function of $Y_{s:m_2:n_2}$ can be written as

$$\begin{aligned} \bar{F}^*_{Y_{s:m_2:n_2}}(y|\tilde{x},\tilde{r}) &= \sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')}B(\tilde{R}',m_2,n_2,s)c'_{s-1}(b+T)^{a+m_1} \\ &\quad\times\sum_{i=1}^{s}\frac{a'_{i,s}}{\gamma_i'(b+T+\gamma_i'y^2)^{a+m_1}}. \end{aligned} \tag{2.34}$$

Upon substituting (2.34) into (2.18), (2.19) and (2.20), the Bayesian predictive bounds of a two-sided equi-tailed $100(1-\alpha)\%$ interval and the $100(1-\alpha)\%$ HPD PI for $Y_{s:m_2:n_2}$, $1 \le s \le m_2$, can be obtained.

It is important to note that for the special case, $s = 1$, the result based on discrete uniform distribution is the same as the one for binomial distribution.

The point predictor for $Y_{s:m_2:n_2}$, $1 \le s \le m_2$, under SEL function is given by

$$\begin{aligned} \widehat{Y}_{s:m_2:n_2} &= (b+T)^{\frac{1}{2}}\beta\left(a+m_1-\frac{1}{2},\frac{1}{2}\right)\sum_{r_1'=0}^{g(r_1')}\sum_{r_2'=0}^{g(r_2')}\cdots\sum_{r_{s-1}'=0}^{g(r_{s-1}')}B(\tilde{R}',m_2,n_2,s) \\ &\quad\times k(s,m_2,n_2), \end{aligned}$$

where $B(\cdot,\cdot,\cdot,\cdot)$ and $k(\cdot,\cdot,\cdot)$ are defined in (2.32) and (2.24), respectively.

Assuming $a = b \approx 0$, we find

$$\widehat{Y}_{s:m_2:n_2} = T^{\frac{1}{2}}Q', \tag{2.35}$$

where

$$Q' = \beta\left(m_1 - \frac{1}{2}, \frac{1}{2}\right) \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} B(\tilde{R}', m_2, n_2, s) k(s, m_2, n_2), \quad (2.36)$$

and $B(\cdot,\cdot,\cdot,\cdot)$ is defined in (2.32). Also, we can write

$$Var(\widehat{Y}_{s:m_2:n_2}) = \frac{1}{\theta} Q'^2 \left\{ m_1 - \left(\frac{\Gamma(m_1 + 1/2)}{\Gamma(m_1)} \right)^2 \right\}.$$

Moreover, from (2.34) we can write

$$V(Y_{s:m_2:n_2}) = \frac{1}{\theta} \left\{ \Psi'_2 - \Psi'^2_1 \right\},$$

where

$$\Psi'_1 = \Gamma(\frac{1}{2}) \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} B(\tilde{R}', m_2, n_2, s) k(s, m_2, n_2), \quad (2.37)$$

$$\Psi'_2 = \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} B(\tilde{R}', m_2, n_2, s) g(s, m_2, n_2), \quad (2.38)$$

and $k(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ are defined as in (2.24) and (2.22).

Therefore, the MSPE of the obtained point predictor $\widehat{Y}_{s:m_2:n_2}$ can be computed as

$$MSPE(\widehat{Y}_{s:m_2:n_2}) = \frac{1}{\theta} \left\{ m_1 Q'^2 + \Psi'_2 - 2Q'\Psi'_1 \frac{\Gamma(m_1 + \frac{1}{2})}{\Gamma(m_1)} \right\}. \quad (2.39)$$

3. Simulation study

In this section, a simulation study is carried out in order to assess the performances of the results in Section 2. To do this, first, we assume binomial distribution for the censoring scheme. Based on the algorithm proposed by Balakrishnan and Sandhu [9], we have used the following algorithm. In all cases we have taken $a = b = c = d \approx 0$.

Algorithm 3.1. Take $\theta = 1$ and suppose $(1-\alpha)$, s, m_1, m_2, n_1, n_2 and p are all given. Then:

(1) Generate values of r_i, $(i = 1, \cdots, m_1)$ and r'_i, $(i = 1, \cdots, m_2)$ from

$$r_i \sim Bin(n_1 - m_1 - \sum_{j=1}^{i-1} r_j, p),\ r_{m_1} = n_1 - m_1 - \sum_{j=1}^{m_1-1} r_j,\ i = 1, 2, \cdots, m_1 - 1,$$

$$r'_i \sim Bin(n_2 - m_2 - \sum_{j=1}^{i-1} r'_j, p),\ r_{m_2} = n_2 - m_2 - \sum_{j=1}^{m_2-1} r'_j,\ i = 1, 2, \cdots, m_2 - 1.$$

(2) Generate m_1 and m_2 independent Uniform $(0,1)$ random variables $W_1, \ldots, W_{m_1}$ and $W'_1, \ldots, W'_{m_2}$.
(3) Set $V_i = W_i^{\frac{1}{i+\sum_{j=m_1-i+1}^{m_1} r_j}}$ for $i = 1, \cdots, m_1$ and $V'_i = W_i'^{\frac{1}{i+\sum_{j=m_2-i+1}^{m_2} r'_j}}$ for $i = 1, \cdots, m_2$.
(4) Take $U_i = 1 - \prod_{j=m_1-i+1}^{m_1} V_j$ for $i = 1, \cdots, m_1$ and $U'_i = 1 - \prod_{j=m_2-i+1}^{m_2} V'_j$ for $i = 1, \cdots, m_2$.
(5) Set $X_{i:m_1:n_1} = F^{-1}(U_i)$ for $i = 1, \cdots, m_1$ and $Y_{i:m_2:n_2} = F^{-1}(U'_i)$ for $i = 1, \cdots, m_2$, where $F^{-1}(\cdot)$ is the inverse cumulative distribution function of the Rayleigh distribution.
(6) Obtain the $100(1-\alpha)\%$ equi-tailed PI, the $100(1-\alpha)\%$ HPD PI and the point predictor for $Y_{s:m_2:n_2}$ based on $X_{1:m_1:n_1}, \ldots, X_{m_1:m_1:n_1}$, by using (2.18), (2.19), (2.20) and (2.25), respectively.
(7) Repeat the Steps 1-6 for $K = 10000$ times and let $\widehat{Y}_{s:m_2:n_2}(i)$, $L(i)$ and $U(i)$ be the point predictor, the lower bound and the upper bound of the PIs obtained from Step 6 in the ith iteration, $i = 1, \ldots, K$. Also, let $Y_{s:m_2:n_2}(i)$ be the sth progressive order statistic of a sample of size m_2 and $X_{1:m_1:n_1}(i), \ldots, X_{m_1:m_1:n_1}(i)$ be the sample of size m_1 generated in Step 5 in the ith iteration. Then, calculate the mean point predictors (MPPs), the estimated MSPEs (EMSPEs), the average widths (AWs) of the PIs, and the coverage probabilities (CPs) of the PIs by using the relations $\widetilde{Y}_{s:m_2,n_2} = \dfrac{1}{K}\displaystyle\sum_{i=1}^{K} \widehat{Y}_{s:m_2:n_2}(i)$, $EMSPE(\widehat{Y}_{s:m_2:n_2}) = \frac{1}{K}\sum_{i=1}^{K}\left(\widehat{Y}_{s:m_2:n_2}(i) - Y_{s:m_2:n_2}(i)\right)^2$, $AW = \dfrac{1}{K}\displaystyle\sum_{i=1}^{K}(U(i) - L(i))$, and $CP = \dfrac{1}{K}\displaystyle\sum_{i=1}^{K} I_{(U(i)-L(i))}(Y_{s:m_2:n_2}(i))$, respectively, where $I_A(\cdot)$ is the indicator function, i.e. $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$, otherwise.

Now, assume the censoring scheme follows a discrete uniform distribution. The steps of Algorithm 3.1 can be used except Step 1 that should be modified as:

(1) Generate values of r_i, $(i = 1, \cdots, m_1)$ and r'_i, $(i = 1, \cdots, m_2)$ from

$$r_i \sim DU\left\{0, \cdots, n_1 - m_1 - \sum_{j=1}^{i-1} r_j\right\},\ r_{m_1} = n_1 - m_1 - \sum_{j=1}^{m_1-1} r_j,\ i = 1, 2, \cdots, m_1 - 1,$$

$$r'_i \sim DU\left\{0, \cdots, n_2 - m_2 - \sum_{j=1}^{i-1} r'_j\right\},\ r_{m_2} = n_2 - m_2 - \sum_{j=1}^{m_2-1} r'_j,\ i = 1, 2, \cdots, m_2 - 1.$$

Based on Algorithm 3.1, we have computed the values of MPPs, EMSPEs, AWs and CPs for different values of s and p when, $1 - \alpha = 0.95$, $m_1 = m_2 = 5$, and $n_1 = n_2 = 10$. In order to achieve computational results, the steps in Algorithm 3.1 have been implemented in MATLAB software. The results are tabulated in Table 1. From Table 1 we can see that by increasing values of s the values of MPPs, i.e. $\widehat{Y}_{s:m_2:n_2}$, increase, as we expect. Also, the values of AWs and EMSPEs are increasing with respect to s, when the other components are held fixed. As we would expect, the AWs of HPD PIs are smaller than the corresponding AWs of the equi-tailed PIs. Moreover, the coverage probabilities, CPs, are near the nominal confidence level 0.95 for both HPD and equi-tailed PIs. The results for binomial distribution when $p = 0.5$ and discrete uniform distribution are near for most cases. Generally, we observe that the type of distribution of random removals does not have a very effect on the results.

TABLE 1. Values of MPPs, EMSPEs, AWs and CPs of %95 equi-tailed (ET) and HPD PIs for different values of s and p when $n_1 = n_2 = 10$ and $m_1 = m_2 = 5$.

					HPD PIs		ET PIs	
Distribution	p	s	MPP	$EMSPE$	AW	CP	AW	CP
Binomial	0.1	1	0.2950	0.0256	0.6221	0.9490	0.6685	0.9530
		3	0.6175	0.0520	0.9086	0.9510	0.9595	0.9450
		5	0.9339	0.1082	1.2713	0.9430	1.3622	0.9420
	0.5	1	0.2959	0.0259	0.6241	0.9530	0.6706	0.9490
		3	0.7321	0.0773	1.1103	0.9500	1.1797	0.9560
		5	1.3222	0.2510	1.9066	0.9360	2.0428	0.9450
	0.9	1	0.2953	0.0247	0.6229	0.9490	0.6693	0.9540
		3	0.8157	0.0982	1.2310	0.9440	1.3050	0.9460
		5	1.4835	0.2792	2.0441	0.9535	2.1745	0.9500
Uniform		1	0.2971	0.0265	0.6265	0.9505	0.6732	0.9510
		3	0.7261	0.0797	1.1006	0.9500	1.1692	0.9560
		5	1.3352	0.2363	1.9191	0.9535	2.0556	0.9510

TABLE 2. The censoring schemes obtained by Dey and Dey [14].

number	p	scheme
[1]	0.2	(3,3,0,1,0,1,0,0,0,0,0,0,0,0,0)
[2]	0.5	(4,2,2,0,0,0,0,0,0,0,0,0,0,0,0)
[3]	0.7	(7,0,1,0,0,0,0,0,0,0,0,0,0,0,0)
[4]	0.9	(5,2,1,0,0,0,0,0,0,0,0,0,0,0,0)

4. Real example

In this section, the theoretical results of the paper are illustrated with an example.

Example 1. In this example, we consider the data which represent the number of million revolutions before failure for each of 23 ball bearings in a life test originally reported by Leiblein and Zelen [17]. These failure times are

$$0.1788,\ 0.2892,\ 0.3300,\ 0.4152,\ 0.4212,\ 0.4560,\ 0.4848,\ 0.5184,$$
$$0.5196,\ 0.5412,\ 0.5556,\ 0.6780,\ 0.6864,\ 0.6864,\ 0.6888,\ 0.8412,$$
$$0.9312,\ 0.9864,\ 1.0512,\ 1.0584,\ 1.2792,\ 1.2804,\ 1.7340.$$

Dey and Dey [14] used these data set to provide progressive Type-II censoring from the Rayleigh distribution using binomial removal scheme. They considered different values for p to generate several removal schemes for $m = 15$. The censoring schemes obtained by Dey and Dey [14] are shown in Table 2.

TABLE 3. Values of $\widehat{Y}_{s:m_2:n_2}$, MSPEs and %95 equi-tailed (ET) and HPD PIs with their length for different values of s when $n_2 = 10$, $m_2 = 5$ and $\tilde{R}' = (2, 2, 1, 0, 0)$.

				HPD PIs		ET PIs	
Scheme	s	$\widehat{Y}_{s:m_2:n_2}$	$MSPE$	PIs	Length	PIs	Length
1	1	0.2839	0.0228	(0, 0.5686)	0.5686	(0.0497, 0.6386)	0.5889
	3	0.6460	0.4279	(0.2447, 1.0919)	0.8472	(0.2832, 1.1547)	0.8715
	5	1.1433	1.3404	(0.4776, 1.9436)	1.4660	(0.5541, 2.0908)	1.5367
2	1	0.2829	0.2280	(0, 0.5667)	0.5667	(0.0495, 0.6364)	0.5869
	3	0.6653	0.4454	(0.2493, 1.1299)	0.8806	(0.2898, 1.1966)	0.9068
	5	1.2144	1.4820	(0.5058, 2.0532)	1.5474	(0.5832, 2.1962)	1.6130
3	1	0.2782	0.0228	(0, 0.5572)	0.5572	(0.0487, 0.6257)	0.5770
	3	0.6605	0.4015	(0.2469, 1.1228)	0.8759	(0.2873, 1.1894)	0.9021
	5	1.2145	1.3457	(0.5071, 2.0468)	1.5397	(0.5833, 2.1858)	1.6025
4	1	0.2818	0.0228	(0, 0.5645)	0.5645	(0.0493, 0.6339)	0.5846
	3	0.6681	0.2388	(0.2498, 1.1356)	0.8858	(0.2906, 1.2030)	0.9124
	5	1.2263	0.7644	(0.5114, 2.0685)	1.5571	(0.5886, 2.2098)	1.6212

For the purpose of illustrating the methods discussed in this paper, we use the censoring schemes in Table 2 to construct an observed progressively Type-II censored sample. Moreover, for the future sample we assume $n_2 = 10$, $m_2 = 5$ and $\tilde{R}' = (2, 2, 1, 0, 0)$ that has been generated from the binomial distribution. Values of $\widehat{Y}_{s:m_2:n_2}$, MSPEs and %95 equi-tailed (ET) and HPD PIs with their length for different values of s when $n_2 = 10$, $m_2 = 5$ and $\tilde{R}' = (2, 2, 1, 0, 0)$ have been computed and are presented in Table 3. Both HPD and equi-tailed PIs contain the point predictor $\widehat{Y}_{s:m_2:n_2}$. For all cases, the HPDs are of the shortest length than that of the equi-tailed prediction intervals. Again we can observe that the values of MSPE and the length of both HPD and equi-tailed prediction intervals are increasing in s.

5. Appendix

From (2.18), the point predictor for $Y_{s:m_2:n_2}$, $1 \le s \le m_2$, under SEL function is given by

$$
\begin{aligned}
\widehat{Y}_{s:m_2:n_2} &= \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} \sum_{i=1}^{s} \frac{A(\tilde{R}', m_2, n_2, s) c'_{s-1} a'_{i,s} (b+T)^{a+m_1}}{\Gamma(a+m_1)\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)} \\
&\quad \times \int_0^\infty \int_0^\infty 2y^2 \theta^{a+m_1} e^{-\theta(b+T+\gamma'_i y^2)} \mathrm{d}y \mathrm{d}\theta \\
&\quad \times \int_0^1 p^{\mu_1(c+\mu_1(\tilde{R}', s)+\mu_1(\tilde{R}, m_1)-1} (1-p)^{d+\mu_2(\tilde{R}', m_2, n_2, s)+\mu_2(\tilde{R}, m_1, n_1, m_1)-1} \mathrm{d}p \\
&= \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} \sum_{i=1}^{s} A(\tilde{R}', m_2, n_2, s) c'_{s-1} a'_{i,s} \frac{(b+T)^{a+m_1}}{\Gamma(a+m_1)} \\
&\quad \times \frac{\beta\Big(c+\mu_1(\tilde{R}', s)+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}', m_2, n_2, s)+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}{\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)} \frac{\Gamma(\frac{3}{2})}{\gamma_i'^{3/2}} \\
&\quad \times \int_0^\infty \theta^{a+m_1-3/2} e^{-\theta(b+T)} \mathrm{d}\theta \\
&= \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} \sum_{i=1}^{s} A(\tilde{R}', m_2, n_2, s) c'_{s-1} a'_{i,s} (b+T)^{1/2} \\
&\quad \times \frac{\beta\Big(c+\mu_1(\tilde{R}', s)+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}', m_2, n_2, s)+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}{\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)} \\
&\quad \times \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(a+m_1-\frac{1}{2})}{\Gamma(a+m_1)\gamma_i'^{3/2}} \\
&= (b+T)^{\frac{1}{2}} \beta\left(a+m_1-\frac{1}{2}, \frac{1}{2}\right) \sum_{r'_1=0}^{g(r'_1)} \sum_{r'_2=0}^{g(r'_2)} \cdots \sum_{r'_{s-1}=0}^{g(r'_{s-1})} A(\tilde{R}', m_2, n_2, s) \\
&\quad \times \frac{\beta\Big(c+\mu_1(\tilde{R}', s)+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}', m_2, n_2, s)+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)}{\beta\Big(c+\mu_1(\tilde{R}, m_1), d+\mu_2(\tilde{R}, m_1, n_1, m_1)\Big)} \\
&\quad \times \frac{1}{2} c'_{s-1} \sum_{i=1}^{s} \frac{a'_{i,s}}{\gamma_i'^{\frac{3}{2}}}.
\end{aligned}
$$

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