Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **9**(2)(2020), 159-169

On Character Projectivity Of Banach Modules

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Abstract. Let A be a Banach algebra, $\Omega(A)$ be the character space of A and $\alpha \in \Omega(A)$. In this paper, we examine the characteristics of α-projective (injective) A-modules and demonstrate that these character-based A-modules also satisfy well-known classical homological properties on Banach A-modules.

Keywords: Banach Algebra, A- Module, Banach A-Module, Projective, Injective Modules.

2000 Mathematics subject classification: 46M10, 46H25.

1. Introduction

This paper has been devoted to the study of some homological properties of Banach algebras and Banach modules. Many mathematicians work on the Lifting Problem (projectivity, injectivity and flatness) of Banach modules and Banach algebras, to name but a few, we may mention [1, 3, 4, 5, 7, 8, 11, 13, 14, 15]. The underlying concepts of these meanings were originally introduced by Helemskii, [3, 4, 5]. Selivanov [14] characterized biprojective banach algebras. Subsequently, Selivanov, Helemskii, Pirkovski [5, 12] and many other mathematicians have provided research works with abound results on homological properties of Banach algebras, C^*-algeras, Banach modules and Frechet algebras,

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Received: 12 October 2019
Revised: 28 March 2020
Accepted: 04 April 2020

e.g., see [1, 3, 4, 5, 7, 8, 11, 13, 14, 15].
Kaniuth et al. [9], for each character on a Banach algebra introduced the concept of character amenability and character contractibility.

On the other hand, R. Nasr and S. Soltani Renani [10] have developed the concepts of character projective and injective Banach modules and further demonstrated that these notions admit by character amenability of Kaniuth et al [9] [10, for more details].

In this paper, we study some intrinsic homological properties of Banach modules and Banach algebras in character homological properties.

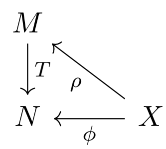
Let A be a Banach algebra. We denote by A_+ the Banach algebra obtained by adjoining identity e^+ to A. Closely akin to [4, 7, 13], the category of left Banach A-modules, right Banach A-modules and A-bimodules will be denoted by A-**mod** **mod**-A and A-**mod**-A, respectively.
For each $M, N \in A$-**mod** (correspondingly **mod**-A and A-**mod**-A), the space ${}_A\mathcal{H}(M,N)$ (correspondingly $\mathcal{H}_A(M,N)$ and ${}_A\mathcal{H}_A(M,N)$) is defined as the collection of all left A-module (correspondingly right A-module and bi-A-module) morphisms from M to N. The morphism $T \in {}_A\mathcal{H}(M,N)$ is called an admissible epimorphism (monomorphism) if T is epimorphism(monomorphism) and has a right (left) inverse as a morphism between two locally convex spaces M and N.

The A-module X is called projective (injective) if for every admissible epimorphism (monomorphism) $T \in {}_A\mathcal{H}(M,N)$ and further each $\phi \in {}_A\mathcal{H}(X,N)$ $(\mathcal{H}_A(M,X))$, there exists $\psi \in {}_A\mathcal{H}(X,M)$ $(\mathcal{H}_A(N,X))$ such that $T \circ \psi = \phi$ $(\psi \circ T = \phi)$. Note that for the right and two-sided modules, projectivity and injectivity can be defined in a parallel manner noting however that, with regard to a two-sided module, the module X is called biprojective. Each Banach algebra A is biprojective as a Banach A-bimodule if and only if the admissiblle epimorphism $\pi_A : A\widehat{\bigotimes}A \longrightarrow A$ defined by $\pi_A(a \otimes b) = a \cdot b$ for each $a, b \in A$, is a retraction (has a right inverse in A-**mod**-A) [4, 11, 14].
Consequently, if A is biprojective and I is a closed bi-ideal of A such that for some closed bi-ideal J of A, $A = I \oplus J$, then J is biprojective [4, 11, 12]. In this paper, we further investigate this property as well as some other properteis for charecter projectivity.

2. Main result

Let A be a Banach algebra and let $\Omega(A)$ be the character spase of A. For each $X \in A$-**mod**-A, as in [10], $\mathcal{I}(\alpha, X)$ denotes the span of $\{a \cdot x - \alpha(a)x : a \in A, x \in X\}$ in X. Immediately, $\mathcal{I}(\alpha, X) = 0$ if and only if module multiplication on X is of the form $a \cdot x$ for every $a \in A$ and $x \in X$.

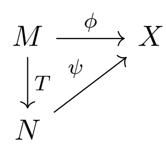
Definition 2.1. Let X be a banach A-module and $\alpha \in \Omega(A)$. The space X is called α-projective A-module whenever for each admissible epimorphism $T \in {}_A\mathcal{H}(M, N)$, with $\mathcal{I}(\alpha, \ker T) = 0$ and each morphism $\phi \in {}_A\mathcal{H}(X, N)$, there exists a morphism $\rho \in {}_A\mathcal{H}(X, N)$ such that the following diagram is commutative:



It is obvious that every projective A-module is an α-projective A-module. Moreover, if $X \in A$-**mod** is α-projective and $Y \in A$-**mod** is a retraction of X – i.e., there exists morphism $\theta : X \longrightarrow Y$ which has a right inverse – then Y is α-projective; indeed, if $T \in {}_A\mathcal{H}(M, N)$ such that $\mathcal{I}(\alpha, \ker T) = 0$ and $\phi \in {}_A\mathcal{H}(X, N)$, $\phi\theta$ is a morhism from X to N. therefore, there is $\tau \in {}_A\mathcal{H}(X, M)$ such that $T \circ \tau = \phi \circ \theta$. Now, if ρ is an inverse for θ, we set $\psi = \theta \circ \rho$, then

$$T \circ \psi = T \circ \tau \circ \rho = \phi \circ \theta \circ \rho = \phi.$$

Definition 2.2. Let A be a Banach algera and $\alpha \in \Omega(A)$. The Banach left A-module X is called α-injective if for each admissible monomorphism $T \in {}_A\mathcal{H}(M, N)$ with $\mathcal{I}(\alpha, N) \subseteq Im(T)$ and any morphism $\phi \in {}_A\mathcal{H}(M, X)$, there exists morphism $\psi \in {}_A\mathcal{H}(N, X)$ such that the following diagram is commutative:

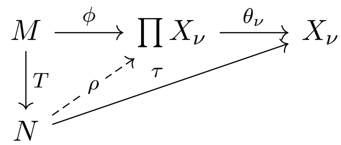


It is easy to observe that every retract of α-injective A-module is α-injective [10].
We recall that the categories A-**mod**, **mod**-A and A-**mod**-A admit product and coproduct. Consider the familly of Banach A-modules $\{X_\nu; \nu \in \Lambda\}$. The product and coproduct of this familly are denoted by $\prod X_\nu$ and $\coprod X_\nu$, respectively. For a more detailed account on these concepts, we refer the reader to [6, 7].

Theorem 2.3. *Suppose that $\prod\{X_\nu; \nu \in \Lambda\}$ is the product of familly of Banach left A-modules $\{X_\nu\}_{\nu\in\Lambda}$ and $\alpha \in \Omega(A)$. Then, $\prod X_\nu$ is α-injective if and only if each X_ν be α-injective.*

Proof. First, we suppose that all X_νs are α-injective. Let $T \in {}_A\mathcal{H}(M, N)$ be an admissible monomorphism with $\mathcal{I}(\alpha, N) \subseteq \Im(T)$ and $\phi \in {}_A\mathcal{H}(M, \prod X_\nu)$. If θ_ν be a projection from $\prod X_\nu$ to $\{X_\nu\}$, then

$\theta_\nu \circ \phi$ is a morphism from M to $\{X_\nu\}$, thereby there exist morphism $\tau \in {}_A\mathcal{H}(N, X_\nu)$ such that the diagram



is commutative. On the other hand, from universal property of product, there is $\rho \in {}_A\mathcal{H}(N, \prod X_\nu)$ such that $\theta_\nu \circ \rho = \tau$. Now we have $\tau \circ T = \theta_\nu \circ \phi$ which means $\theta_\nu \circ \rho \circ T = \theta_\nu \circ \phi$, and so $\rho \circ T = \phi$.
The converse is deduced from the fact that retract of α-injective A-module is α-injective. □

Theorem 2.4. *Suppose that $\{X_\nu; \nu \in \Lambda\}$ is a familly of Banach A-modules. The following statements are equivalent:*
(i) *All the objects of X_ν are α-projective.*
(ii) *The coproduct of X_ν's, $\coprod X_\nu$, is α-projective.*

Proof. The proof is analogous to the proof of Theorem 2.3. □

It is obvious that if α be a character of Banach algebra A, then $\alpha \otimes \alpha$ is ancharacter of $A \widehat{\bigotimes} A$.

Proposition 2.5. *Let A be a Banach algebra and $\alpha \in \Omega(A)$, and let I be a closed two-sided ideal of A. If A is $\alpha\otimes\alpha$-biprojetive Banach algebra, then the Banach A-bimodule $A/A \cdot I$ is $\alpha \otimes \alpha$-biprojective.*

Proof. Suppose that $T \in {}_A\mathcal{H}_A(M, N)$ is an admissible epimorphism with $\mathcal{I}(\alpha \otimes \alpha, \ker T) = 0$. If $\phi \in {}_A\mathcal{H}_A(A/A \cdot I, N)$ and ψ are canonical projection from A to $A/A \cdot I$, then $\psi \circ \phi \in {}_A\mathcal{H}_A(A, N)$ and thus $\alpha \otimes \alpha$-biprojetivity of A follows from the fact that there exist A-bimodule morphism ρ_0 from A to M such that $T \circ \rho_0 = \phi \circ \psi$. Now we have

$$\rho_0(a \cdot d) = \phi \circ \psi(a \cdot d) = 0,$$

for each $a \in A$ and $d \in I$. Therefore, $\rho_0(A \cdot I) = 0$ which means that there exist a morphism $\rho \in {}_A\mathcal{H}_A(A/A \cdot I, M)$ defined by the formula

$$\rho\big(\psi(a)\big)(a) = \rho_0(a), \qquad a \in A.$$

Observe that for any $a \in A$ we have

$$\begin{aligned} T \circ \rho(a + A.I) &= T \circ \rho \circ \psi(a) \\ &= T \circ \rho_0(a) \\ &= \phi \circ \psi(a) \\ &= \phi(a + A \cdot I), \end{aligned}$$

and it concludes that the Banach A-bimodule $A/A{\cdot}I$ is $\alpha{\otimes}\alpha$-biprojective.
□

Corollary 2.6. *Let A be a Banach algebra and $\alpha \in \Omega(A)$, and let I be a two-sided ideal of A such that $A = I \oplus J$ for some closed essential closed two-sided ideal J of A. If A is $\alpha \otimes \alpha$-biprojetive Banach algebra, then I is an A-bimodule $\alpha \otimes \alpha$-biprojetive.*

Proof. It follows from the previous proposition and the fact that

$$I \cong A/J = A/\overline{A \cdot J}.$$

□

Let A be a Banach algebra, $\Omega(A)$ its character space and $X \in A$-mod. We show the canonical projection from $A_+\widehat{\bigotimes}X$ to X defined by $a{\otimes}x \longmapsto a.x$ on elementary members by π_X^+. Further, we use this notion for morphism $A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+ \longrightarrow X$ and $X\widehat{\bigotimes}A_+ \longrightarrow X$ when X is an object in A-mod-A and mod-A, respectively.
Consider $X \in$A-**mod**, $\alpha \in \Omega(A)$ and the morphism

$${}_\alpha\Upsilon_X : A_+\widehat{\bigotimes}X/\overline{\mathcal{I}(\alpha, \ker \pi_X^+)} \longrightarrow X$$

defined by the fomula

$${}_\alpha\Upsilon_X(a \otimes x + \overline{\mathcal{I}(\alpha, \ker \pi_X^+)}) = a \cdot x,$$

for each $a \in A_+$ and $x \in X$. If we denote the space $A_+\widehat{\bigotimes}X/\overline{\mathcal{I}(\alpha, \ker \pi_X^+)}$ as in [10] by ${}_\alpha A_+\widehat{\bigotimes}X$, then the morphism ${}_\alpha\Upsilon_X$ is an element in ${}_A\mathcal{H}({}_\alpha A_+\widehat{\bigotimes}X, X)$ with $\mathcal{I}(\alpha, \ker {}_\alpha\Upsilon_X) = 0$.

The following theorem is taken from [10].

Theorem 2.7. *Let A be a Banach algebra and let $\alpha \in \Omega(A)$. For $X \in A$-**mod**, the following statements are equivalent.*

(i) X is α-projective.

(ii) The left A-module morphism ${}_\alpha\Upsilon_X \in {}_A\mathcal{H}({}_\alpha A_+\widehat{\bigotimes}X, X)$ is a retraction; there exist morphism ${}_\alpha\rho_X \in {}_A\mathcal{H}(X, {}_\alpha A_+\widehat{\bigotimes}X)$ such that it is a right inverse for ${}_\alpha\Upsilon_X$.

It is clear that if $M, N \in$ A-**mod**-A and $T \in {}_A\mathcal{H}_A(M, N)$, then $T \in \mathcal{H}_A(M, N)$ and $T \in {}_A\mathcal{H}(M, N)$.

Theorem 2.8. *Let A be a Banach algebra and let $\alpha \in \Omega(A)$. Then $\alpha \otimes \alpha$-projectivity of $X \in A$-**mod**-A concludes that X is α-projective in both A-**mod** and **mod**-A.*

Proof. Suppose that ${}_{\alpha\otimes\alpha}\rho$ is a right inverse of the mapping ${}_{\alpha\otimes\alpha}\Upsilon_X$ that come out in Theorem 2.7 in A-**mod**-A category. Let $\theta : A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+ \longrightarrow$

$A_+\widehat{\bigotimes}X$ be the morphism defined by $\theta(u\otimes b) = u\cdot b$ for each $u \in A_+\widehat{\bigotimes}X$ and $b \in A_+$. Now we show that $\theta\big(\overline{\mathcal{I}(\alpha\otimes\alpha, \ker{}_{A-A}\pi_X)}\big) \subseteq \overline{\mathcal{I}(\alpha, \ker{}_A\pi_X)}$. Let $a,b,c\in A$ and $u\in A_+\widehat{\bigotimes}X$, we have

$$a\cdot u\cdot cb - \alpha(a)u\cdot c\alpha(b) = \big(a-\alpha(a)\big)\cdot u\cdot cb + \big(\alpha(a)u\cdot c\big)\big(b-\alpha(b)\big).$$

It is immediate that if $u\otimes c\in \ker{}_{A-A}\pi_X$, then $u.c\in\ker{}_A\pi_X$ as same as $u\cdot cb$. Therefore, $\theta\big(\mathcal{I}(\alpha\otimes\alpha,\ker{}_{A-A}\pi^+{}_X)\big)\subseteq\mathcal{I}(\alpha,\ker{}_A\pi^+{}_X)$ and by the continuity of θ,

$$\theta\big(\overline{\mathcal{I}(\alpha\otimes\alpha,\ker{}_{A-A}\pi^+{}_X)}\big)\subseteq\overline{\mathcal{I}(\alpha,\ker{}_A\pi^+{}_X)}.$$

Now consider A-bimodule morphism

$$\Theta : \frac{A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+}{\overline{\mathcal{I}(\alpha\otimes\alpha,\ker{}_{A-A}\pi^+{}_X)}} \longrightarrow \frac{A_+\widehat{\bigotimes}X}{\overline{\mathcal{I}(\alpha,\ker{}_A\pi^+{}_X)}}$$

produced by θ. We set ${}_\alpha\rho = \Theta\circ{}_{\alpha\otimes\alpha}\rho$. This is an A-module morphism that is a right inverse for ${}_\alpha\Upsilon_X$. For the case **mod**-A, the proof is similar. $\square$

Theorem 2.9. *Suppose that $\kappa : A\longrightarrow B$ is a morphism of Banach algebras with dense range. Let $\alpha\in\Omega(B)$, and $X\in B$-**mod**. If X_κ be an $\alpha\circ\kappa$-projective Banach A-module, then X is an α-projective Banach B-module.*

Proof. Let $\rho_A : X\longrightarrow \frac{A_+\widehat{\bigotimes}X}{\overline{\mathcal{I}(\alpha\circ\kappa,\ker{}_A\pi^+{}_X)}}$ be a right inverse for ${}_{\alpha\circ\kappa}\Upsilon_{X_\kappa}$ in A-**mod**.

Consider morphism

$$\kappa\widehat{\otimes}Id_X : A_+\widehat{\bigotimes}X\longrightarrow B_+\widehat{\bigotimes}X.$$

For any $a,b\in A$ and each $x\in X$ we have,

$$\begin{aligned}\kappa\widehat{\otimes}Id_X\big(a.(b\otimes x)-\alpha\circ\kappa(a)(b\otimes x)\big) &= \kappa(ab)\otimes\big(\alpha\circ\kappa(a)\big)\big(\kappa(b)\otimes x\big)\\ &= \Big(\kappa(a)-\alpha\big(\kappa(a)\big)\Big)\big(\kappa(b)\otimes x\big).\end{aligned}$$

Therefore,

$$\kappa\widehat{\otimes}Id_X\big(\mathcal{I}(\alpha\circ\kappa, A_+\widehat{\bigotimes}X)\big)\subseteq\mathcal{I}(\alpha, B_+\widehat{\bigotimes}X).$$

On the other hand, it is clear that if $u\in\ker\pi^+_{X_\kappa}$ then $\kappa\widehat{\otimes}Id_X(u)\in\ker\pi^+_X$. This implies that

$$\kappa\widehat{\otimes}Id_X\big(\overline{\mathcal{I}(\alpha\circ\kappa, A_+\widehat{\bigotimes}X)}\big)\subseteq\overline{\mathcal{I}(\alpha, B_+\widehat{\bigotimes}X)},$$

and thus there exists A-module morphism

$$\theta : \frac{A_+\widehat{\bigotimes} X}{\overline{\mathcal{I}(\alpha\circ\kappa, \ker \pi^+_{X_\kappa})}} \longrightarrow \frac{B_+\widehat{\bigotimes} X}{\overline{\mathcal{I}(\alpha, \ker \pi^+_{X})}}.$$

Next we set $\rho = \theta \circ \rho_A$ and, in the two succeeding steps, we show that this morphism is a B-module morphism inverse for ${}_\alpha\Upsilon_X$.

(i) The morphism ρ is a B-module morphism; for this, let $x \in X$ and $b \in B$. Since $\overline{Im(\kappa)} = B$, there is a sequence $\big(a_i\big)_i \subseteq A$ such that $\lim_i \kappa(a_i) = b$. Thus,

$$\begin{aligned}
\rho(b.x) &= \theta \circ \rho_A(b \cdot x)\\
&= \lim_i \theta \circ \rho_A\big(\kappa(a_i) \cdot x\big)\\
&= \lim_i \theta\big(a_i \cdot \rho_A(x)\big)\\
&= \lim_i \kappa(a_i) \cdot \theta \circ \rho_A(x)\\
&= b \cdot \rho(x).
\end{aligned}$$

(ii) The morphism ρ is a right inverse for ${}_\alpha\Upsilon_X$ in B-**mod**. Let $x \in X$ and let $u = \sum_{j=1}^\infty a_j \otimes x_j$ for some $a_j \in A$ and $x_j \in X$ such that

$$\rho_A(x) = u + \overline{\mathcal{I}(\alpha \circ \kappa, \ker \pi^+_{X_\kappa})}.$$

Then,

$$\theta \circ \rho_A(x) = \textstyle\sum_{j=1}^\infty \kappa(a_j) \otimes x_j + \overline{\mathcal{I}(\alpha, \ker \pi^+_X)}$$

and thus

$$\begin{aligned}
{}_\alpha\Upsilon_X \circ \theta \circ \rho_A(x) &= \sum_{j=1}^\infty \kappa(a_j).x_j\\
&= {}_{\alpha\circ\kappa}\Upsilon_{X_\kappa}\big(u + \overline{\mathcal{I}(\alpha \circ \kappa, \ker \pi^+_{X_\kappa})}\big)\\
&= {}_{\alpha\circ\kappa}\Upsilon_{X_\kappa} \circ \rho_A(x)\\
&= x.
\end{aligned}$$

□

Let A be a Banach algebra, $M, N \in A$-**mod**-A and let $T \in {}_A\mathcal{H}_A(M, N)$. The space $\ker T$ is a left, right and two-side submodule of M. We denote the space $\mathcal{I}(\alpha, \ker T)$ by ${}_A\mathcal{I}(\alpha, \ker T)$ and $\mathcal{I}_A(\alpha, \ker T)$ respectively, when $T \in {}_A\mathcal{H}(M, N)$ and $T \in \mathcal{H}_A(M, N)$.

Definition 2.10. Let A be a Banach algebra, $\alpha \in \Omega(A)$ and $X \in A$-**mod**-A. We say that X is left α-biprojective when for each $M, N \in A$-**mod**-A if $T \in {}_A\mathcal{H}_A(M, N)$ is an admissible epimorphism with

${}_A\mathcal{I}(\alpha, \ker T) = 0$ and $\phi \in \mathcal{H}(X, N)$, then there exists $\psi \in {}_A\mathcal{H}_A(X, M)$ such that $T \circ \psi = \phi$.

Let A be a Banach algebra and $X \in A$-**mod**-A. Suppose that

$$_{A-A}\pi_X^+ : A_+ \widehat{\bigotimes} X \widehat{\bigotimes} A_+ \longrightarrow X$$

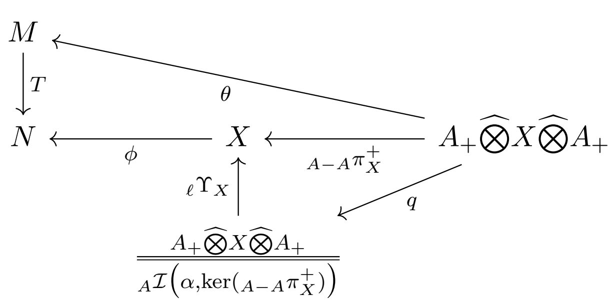
is a canonical morphism. Then $_{A-A}\pi_X^+ : A_+$ is an admissible epimorphism that is a retraction in A-**mod**-A if and only if X be biprojective, see proposition IV.1.1 in [4]. Now we consider the morphism $_\ell\Upsilon_X : \frac{A_+ \widehat{\bigotimes} X \widehat{\bigotimes} A_+}{{}_A\mathcal{I}\left(\alpha, \ker({}_{A-A}\pi_X^+)\right)} \longrightarrow X$ given by

$$_\ell\Upsilon_X\left(a \otimes x \otimes b + \overline{{}_A\mathcal{I}\left(\alpha, \ker({}_{A-A}\pi_X^+)\right)}\right) = a \cdot x \cdot b, \qquad x \in X, a, b \in A.$$

Apparently, $_\ell\Upsilon_X$ is a morphism in A-**mod**-A and ${}_A\mathcal{I}\left(\alpha, \ker({}_\ell\Upsilon_X)\right) = 0$; see [10].

Proposition 2.11. *If A is a Banach algebra, $\alpha \in \Omega(A)$ and $X \in A$-**mod**-A. The following statements are equvalent:*
(i) The Banach A-bimodule X is left α-biprojective.
*(ii) A-bimodule morphism $_\ell\Upsilon_X : \frac{A_+ \widehat{\bigotimes} X \widehat{\bigotimes} A_+}{{}_A\mathcal{I}\left(\alpha, \ker({}_{A-A}\pi_X^+)\right)} \longrightarrow X$ is a retraction in A-**mod**-A.*

Proof. $(i) \Rightarrow (ii)$, The module morphism $_\ell\Upsilon_X$ is an admissible epimorphism with $\mathcal{I}\left(\alpha, \ker({}_\ell\Upsilon_X)\right) = 0$ and so there exist a right morphism for $_\ell\Upsilon_X$.
$(ii) \Rightarrow (i)$, Let $M, N \in A$-**mod**-A, $T \in {}_A\mathcal{H}_A(M, N)$ an admissible epimorphism with ${}_A\mathcal{I}(\alpha, \ker T) = 0$, and $\phi \in {}_A\mathcal{H}_A(X, N)$. We show that there exists morphism $R \in {}_A\mathcal{H}_A(X, M)$ such that $T \circ R = \phi$. For this, the module $A_+ \widehat{\bigotimes} X \widehat{\bigotimes} A_+$ is a biprojective A-bimodule and thus there is $\theta \in {}_A\mathcal{H}_A(A_+ \widehat{\bigotimes} X \widehat{\bigotimes} A_+, M)$ which, if we consider q as the quotient mappig, the up side and down side of the following diagram are commutative.



Now $\Theta : \frac{A_+ \widehat{\bigotimes} X \widehat{\bigotimes} A_+}{{}_A\mathcal{I}\left(\alpha, \ker({}_{A-A}\pi_X^+)\right)} \longrightarrow M$, which is defined by

$$\Theta\Big(\nu+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}\Big)=\theta(\nu),$$

for all $\nu\in A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+$, is well defined. If ρ_α is a right inverse for ${}_\ell\Upsilon_X$, then the morphism $R=\Theta\circ\rho_\alpha$ belongs to ${}_A\mathcal{H}_A(X,\frac{A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+}{\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}})$. Next, it is sufficient to show that $T\circ R=\phi$. Let $x\in X$ and for some $u\in A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+$,

$$\rho_\alpha(x)=u+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}.$$

Therefore,

$$\begin{aligned}T\circ R(x)&=T\Big(\Theta\big(\rho_\alpha(x)\big)\Big)\\&=T\left(\Theta\Big(u+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}\Big)\right)\\&=T\big(\theta(u)\big)\\&=\phi\circ{}_{A-A}\pi_X^{+}(u)\\&=\phi\circ{}_\ell\Upsilon_X\circ q(u)\\&=\phi\circ{}_\ell\Upsilon_X\Big(u+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}\Big)\\&=\phi\circ{}_\ell\Upsilon_X\circ\rho_\alpha(x)\\&=\phi(x),\end{aligned}$$

as required. □

Theorem 2.12. *Let A be a Banach algebra and $\alpha\in\Omega(A)$. If $X\in A$-**mod**-A is left α-biprojective, then X is left α-projective.*

Proof. Consider morphism $\theta:A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+\longrightarrow A_+\widehat{\bigotimes}X$ such that for each $a,b\in A_+$ and $x\in X$, $\theta(a\otimes x\otimes B)=a\otimes x\cdot b$. For each $u=\sum_{i=1}^{+\infty}a_i\otimes x_i\otimes b_i\in\ker{}_A\pi_A$ and $a\in A$, we have

$$\theta\Big(\big(a-\alpha(a)\big)\textstyle\sum_{i=1}^{+\infty}a_i\otimes x_i\otimes b_i\Big)=\big(a-\alpha(a)\big)\textstyle\sum_{i=1}^{+\infty}a_i\otimes x_i\cdot b_i,$$

since $\sum_{i=1}^{+\infty}a_i\otimes x_i\cdot b_i\in\ker\pi_X^{+}$, the right hand side of the above equation belongs to $\mathcal{I}\big(\alpha,\ker\pi_X^{+}\big)$. Thus, there is morphism

$$\Theta:\frac{A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+}{\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}}\longrightarrow\frac{A_+\widehat{\bigotimes}X}{\overline{\mathcal{I}\big(\alpha,\ker(\pi_X^{+})\big)}}$$

such that for every $\nu\in A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+$,

$$\Theta\Big(\nu+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^{+})\big)}\Big)=\theta(\nu)+\overline{\mathcal{I}(\alpha,\ker\pi_X^{+})}.$$

If ρ is a right inverse for ${}_\ell\Upsilon_X$, which was concluded from the previous theorem, we set ${}_\alpha\rho = \Theta\circ\rho$. Now, it is sufficient to show that ${}_\alpha\Upsilon_X\circ\rho_\alpha = id_X$. For this, let $x\in X$ and for some $u\in A_+\widehat{\bigotimes}X\widehat{\bigotimes}A_+$,

$$\rho(x) = u+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^+)\big)} = \textstyle\sum_{i=1}^{+\infty} a_i\otimes x_i\otimes b_i+\overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^+)\big)}$$

that $a_i, b_i\in A_+$ and $x_i\in X$. Therefore,

$$\begin{aligned}
{}_\alpha\Upsilon_X\circ\rho_\alpha(x) &= {}_\alpha\Upsilon_X\circ\Theta\circ\rho(x)\\
&= {}_\alpha\Upsilon_X\circ\Theta\Big(\sum_{i=1}^{+\infty} a_i\otimes x_i\otimes b_i + \overline{{}_A\mathcal{I}\big(\alpha,\ker({}_{A-A}\pi_X^+)\big)}\Big)\\
&= {}_\alpha\Upsilon_X\Big(\theta(\sum_{i=1}^{+\infty} a_i\otimes x_i\otimes b_i) + \overline{\mathcal{I}(\alpha,\ker\pi_X^+)}\Big)\\
&= \sum_{i=1}^{+\infty} a_i\cdot x_i\cdot b_i\\
&= x,
\end{aligned}$$

as required. □

3. Questions

Suppose that A is a Banach algebra and $X\in A$-**mod**-A. Let $\mathcal{LM}(X)$ and $\mathcal{RM}(X)$ be respectively the left and right multipliers of X; in other words $L\in\mathcal{LM}(X) = {}_A\mathcal{H}(A,X)$ and $R\in\mathcal{RM}(X) = \mathcal{H}_A(A,X)$. We recall that the continuous operator $D: A\longrightarrow X$ is a derivation if D satisfies the Leibnitz rule:

$$D(ab) = a\cdot D(b) + D(a)\cdot b,$$

for each $a, b\in A$. In [15] or Theorem 3.4 in [11], Selivanov and Pirkovski showed that A is a biprojective Banach algebra if and only if for each derivation from A to X there exist $R\in\mathcal{RM}(X)$ and $L\in\mathcal{LM}(X)$ such that $D = R-L$. Now, the question is

Question 3.1. Let $\alpha\in\Omega(A)$. If the left module multiplication on X is of the form

$$a\cdot x = \alpha(a)x, \qquad (a\in A, x\in X)$$

then is it true that: X is left α-biprojective if and only if for each derivation D from A to X, there exist $R\in\mathcal{RM}(X)$ and $L\in\mathcal{LM}(X)$ such that $D = R-L$?

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