Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **9** (2)(2020), 305-320

Cohomology of $\mathfrak{aff}(1|1)$ acting on the space of bilinear differential operators on the superspace $\mathbb{R}^{1|1}$

Imed Basdouri[a,1], Ammar Derbali[b] , and Mohamed Elkhames Chraygui[b]
[a] Département de Mathématiques, Faculté des Sciences de Gafsa, Zarroug 2112, Gafsa, Tunisie
[b] Département de Mathématiques, Faculté des Sciences de Sfax, BP 802, Sfax 3038, Tunisie

Abstract. We consider the $\mathfrak{aff}(1)$-module structure on the spaces of bilinear differential operators acting on the spaces of weighted densities. We compute the first differential cohomology of the Lie superalgebra $\mathfrak{aff}(1)$ with coefficients in space $\mathcal{D}_{\lambda,\nu;\mu}$ of bilinear differential operators acting on weighted densities. We study also the super analogue of this problem getting the same results.

Keywords: Contact geometry, differential operators, Lie (super)algebra.

2000 Mathematics subject classification: Primary: 53D55; Secondary: 17B56, 58H15.

1. Introduction

Lie theory has its roots in the work of Sophus Lie, who studied certain transformation groups that are now called Lie groups. His work led to the discovery of Lie algebras. By now, both Lie groups and Lie algebras have become essential to many parts of mathematics and theoretical physics. In the meantime, Lie algebras have become a central object of interest in their own right, not least because of their description by the

[1]Corresponding author: basdourimed@yahoo.fr
Received: 6 February 2020
Revised: 2 April 2020
Accepted: 4 April 2020

Serre relations, whose generalizations have been very important. Now, we focus on the use of Lie algebras in a field that is very unavailable and present today in theoretical physics and in mathematics which was the cohomology.
Generally, Lie algebra cohomology is just the cohomology of a particular kind of algebraic theory. There are analogous cohomology theories for groups, associative algebras, and commutative rings. All these theories can be unified by employing the notion of an injective resolution. To broaden the scope further, we can employ category theory and reconceptualize Lie algebra cohomology as a functor from the category of $\mathfrak{g}$-modules to the category of cochain complexes.
Lie algebra cohomology was first formalized by C.Chevalley and S.Eilenberg in an influential paper, in 1948 [8]. The aim was to calculate the cohomology, in the topological sense, of a compact Lie group by using the finite-dimensional data of the corresponding Lie algebra. In this they were inspired by an even earlier idea of Elie Cartan, who was the first to announce that there was a connection between the topology of a Lie group and the algebraic structure of the underlying Lie algebra [9]. What makes this story particularly interesting is that Homological Algebra, as a subject, was launched by the remarkable 1956 book [10] by Cartan and Eilenberg called, oddly enough "Homological Algebra". However, the Cartan involved this time is not Elie, but Henri, the equally remarkable son of the very remarkable Elie. A survey of history of homological algebra by Charles Weibel is available at the $\mathbb{K}$-theory archive [11].

In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem a goal is to determine if all related deformation obstructions vanish and many good techniques were developed to determine when this is so. Deformations of Lie algebras with base and versal deformations were already considered by Fialowski in 1986 [12]. In 1988, Fialowski [13] further introduced deformations whose base is a complete local algebra Also, in [13], the notion of miniversal (or formal versal) deformation was introduced in general, and it was proved that under some cohomology restrictions, a versal deformation exists. Later, Fialowski and Fuchs, using this framework, gave a construction for a versal deformation.
In this article, we are interesting to the study of some differential cohomological structures $\mathrm{H}^1(\mathfrak{g};\mathrm{M})$ where $\mathfrak{g}$ is a (super) Lie algebras and M is a $\mathfrak{g}$-module. We also study some related questions of deformations of some $\mathfrak{g}$-modules. If $\mathrm{M} = \mathrm{End}(V)$ where V is a $\mathfrak{g}$-module then, according

to Nijenhuis-Richardson, the space $\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))$ classifies the infinitesimal deformations of a $\mathfrak{g}$-module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $\mathrm{H}^2(\mathfrak{g};\mathrm{End}(V))$. More generally, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then the $\mathfrak{h}$-relative cohomology $\mathrm{H}^1(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))$ measures the infinitesimal deformations that become trivial once the action is restricted to $\mathfrak{h}$ ($\mathfrak{h}$-*trivial deformations*), while the obstructions to extension of any $\mathfrak{h}$-trivial infinitesimal deformation to a formal one are related to $\mathrm{H}^2(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))$.
Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathfrak{g}-$modules. It is well-known that nontrivial extensions of $\mathfrak{g}-$modules:

$$0 \to \mathcal{M} \to . \to \mathcal{N} \to 0$$

are classified by the first cohomology group $\mathrm{H}^1(\mathfrak{g},\mathrm{Hom}(N,M))$ (see, e.g., [13]). Any $1-$cocycle Λ generates a new action on $\mathcal{M}\oplus\mathcal{N}$ as follows: for all $g\in\mathfrak{g}$ and for all $(a,b)\in\mathcal{M}\oplus\mathcal{N}$, we define $g*(a,b) := (g*a+\Lambda(b), g*b)$.

The space of weighted densities of weight λ on $\mathbb{R}$ (or λ-densities for short), denoted by:

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in C^\infty(\mathbb{R}) \right\},\ \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(\mathrm{T}^*\mathbb{R})^{\otimes^\lambda}$. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ of vector fields $X_h = h\frac{d}{dx}$, where $h\in C^\infty(\mathbb{R})$, acts by the Lie derivative. Alternatively, this action can be written as follows:

$$X_h.(f dx^\lambda) = L^\lambda_{X_h}(f) dx^\lambda \ \textit{with}\ L^\lambda_{X_h}(f) = hf' + \lambda h' f, \tag{1.1}$$

where f', h' are $\frac{df}{dx}$, $\frac{dh}{dx}$. Each bilinear differential operator A on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_\lambda\otimes\mathcal{F}_\nu$ to $\mathcal{F}_\mu$, for any $\lambda,\nu,\mu\in\mathbb{R}$, by $f dx^\lambda\otimes g dx^\nu \mapsto A(f\otimes g)dx^\mu$. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ acts on the space $\mathrm{Hom}(\mathcal{F}_\lambda\otimes\mathcal{F}_\nu,\mathcal{F}_\mu) = \mathcal{D}_{\lambda,\nu;\mu}$ of these differential operators by:

$$X_h.A = L^\mu_{X_h}\circ A - A\circ L^{(\lambda,\nu)}_{X_h}, \tag{1.2}$$

where, $L^{(\lambda,\nu)}_{X_h}$ is the Lie derivative on $\mathcal{F}_\lambda\otimes\mathcal{F}_\nu$ defined by the Leibnitz rule:

$$L^{(\lambda,\nu)}_{X_h}(f\otimes g) = L^\lambda_{X_h}(f)\otimes g + f\otimes L^\nu_{X_h}(g).$$

For the space of tensor densities of weight λ, $\mathcal{F}_\lambda$, viewed as a module over the Lie algebra of smooth vector fields $\mathrm{Vect}(\mathbb{R})$, the classification of nontrivial extensions

$$0 \to \mathcal{F}_\mu \to . \to \mathcal{F}_\lambda \to 0,$$

leads Feigin and Fuks in [7] to compute the cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathrm{Hom}(\mathcal{F}_\lambda,\mathcal{F}_\mu))$. Later, Ovsienko and Bouarroudj in [5] have computed the corresponding

relative
cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu)).$$

Later, Bouarroudj in [4] has computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathrm{Hom}(\mathcal{F}_\lambda \otimes \mathcal{F}_\nu, \mathcal{F}_\mu)).$$

Moreover, it has computed the cohomology group

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathrm{Hom}(\mathcal{F}_\lambda \otimes \mathcal{F}_\nu, \mathcal{F}_\mu)).$$

If we restrict ourselves to the Lie algebra $\mathfrak{aff}(1)$ which is isomorphic to the Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$ spanned by

$$\{X_1, X_x\}$$

In this paper, we will compute the first differential cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}_{\lambda,\nu;\mu})$ and the analogue super structures. More precisely, we compute the first differential cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\nu;\mu})$ where, $\mathfrak{D}_{\lambda,\nu;\mu}$ is the
superspace of bilinear differential operators from $\mathfrak{F}_\lambda \otimes \mathfrak{F}_\nu$ to $\mathfrak{F}_\mu$.

2. Definitions and Notation

2.1. The Lie superalgebra of contact vector fields on $\mathbb{R}^{1|1}$. We define the superspace $\mathbb{R}^{1|1}$ in terms of its superalgebra of functions, denoted by $C^\infty(\mathbb{R})$ and consisting of elements of the form:

$$F(x, \theta) = f_0(x) + f_1(x)\theta,$$

where, x is the even variable, θ is the odd variable ($\theta^2 = 0$) and $f_0(x), f_1(x) \in C^\infty(\mathbb{R})$. Even elements in $C^\infty(\mathbb{R})$ are the functions $F(x, \theta) = f_0(x)$, the functions $F(x, \theta) = \theta f_1(x)$ are odd elements. The parity of homogenous elements F will be denoted $|F|$. We consider the contact bracket on $C^\infty(\mathbb{R})$ defined on $C^\infty(\mathbb{R})$ by:

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|}\overline{\eta}(F)\overline{\eta}(G),$$

Where, $\eta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ and $\overline{\eta} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}$. The superspace $\mathbb{R}^{1|1}$ is equipped with the standard contact structure given by the following 1-form:

$$\alpha = dx + \theta d\theta.$$

Let $\mathrm{Vect}(\mathbb{R}^{1|1})$ be the superspace of vector fields on $\mathbb{R}^{1|1}$:

$$\mathrm{Vect}(\mathbb{R}^{1|1}) = \left\{F_0\partial_x + F_1\partial_\theta | F_i \in C^\infty(\mathbb{R}^{1|1})\right\}$$

where, ∂_θ stands for $\frac{\partial}{\partial\theta}$ and ∂_x stands for $\frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(1)$ of contact vector fields on $\mathbb{R}^{1|1}$. That is, $\mathcal{K}(1)$ is the Lie superalgebra of conformal vector fields on $\mathbb{R}^{1|1}$ with respect to the 1-form α:

$$\mathcal{K}(1) = \left\{X \in \mathrm{Vect}(\mathbb{R}^{1|1}) \mid \textit{there exists } H \in C^\infty(\mathbb{R}^{1|1}) \textit{ such that}, \ \mathfrak{L}_X(\alpha) = H\alpha\right\},$$

where, $\mathfrak{L}_X$ is the Lie derivative along the vector field X. In particular, we have $\mathcal{K}(0) = \mathrm{Vect}(\mathbb{R})$. Any contact vector field on $\mathbb{R}^{1|1}$ has the following explicit form:

$$X_H = H\partial_x - \frac{1}{2}(-1)^{|H|}\overline{\eta}(H)\overline{\eta}, \ \textit{where } H \in C^\infty(\mathbb{R}^{1|1}).$$

The bracket on $\mathcal{K}(1)$ is given by

$$[X_F, X_G] = X_{\{F,G\}}.$$

2.2. **The superalgebra $\mathfrak{aff}(1|1)$.** The Lie algebra $\mathfrak{aff}(1)$ is realized as superalgebra of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ (see, e.g. [3]):

$$\mathfrak{aff}(1) = Span(X_1, X_x).$$

Similarly, we now consider the affine Lie superalgebra as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{aff}(1|1) = Span(X_1, X_x, X_\theta).$$

The space of even elements is isomorphic to $\mathfrak{aff}(1)$, while the space of odd elements is two dimensional:

$$(\mathfrak{aff}(1|1))_{\overline{1}} = Span(X_\theta).$$

The new commutation relations are

$$[X_1, X_x] = X_1, \qquad [X_x, X_\theta] = -\frac{1}{2}X_\theta,$$

$$[X_1, X_\theta] = 0, \qquad [X_\theta, X_\theta] = \frac{1}{2}X_1.$$

2.3. **The space of weighted densities on $\mathbb{R}^{1|1}$.** We have analogous definition of weighted densities in super setting with dx replaced by α. The elements of these spaces are indeed (weighted) densities since all spaces of generalized tensor fields have just one parameter relative $\mathcal{K}(1)$, the value of X_x on the lowest weight vector (the one annihilated by X_θ). From this point of view the volume element (roughly speaking, $dx\frac{\partial}{\partial\theta}$) is indistinguishable from $\alpha^{\frac{1}{2}}$. We denote by $\mathfrak{F}_\lambda$ the space of all weighted densities on $\mathbb{R}^{1|1}$ of weight λ:

$$\mathfrak{F}_\lambda = \left\{F(x,\theta)\alpha^\lambda \mid F(x,\theta) \in C^\infty(\mathbb{R}^{1|1})\right\}.$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty(\mathbb{R}^{1|1})$, but, the Lie derivative of the density $F\alpha^\lambda$ along the vector field X_H in $\mathcal{K}(1)$ is now:

$$\mathfrak{L}_{X_H}(F\alpha^\lambda) = \mathfrak{L}^\lambda_{X_H}(F)\alpha^\lambda, \ \textit{with } \mathfrak{L}^\lambda_{X_H}(F) = \mathfrak{L}_{X_H}(F) + \lambda H'F.$$

Or, if we put $H(x,\theta)=a(x)+b(x)\theta,\ F(x,\theta)=f_0(x)+f_1(x)\theta,$

$$\mathfrak{L}^{\lambda}_{X_H}(F)=L^{\lambda}_{a\partial_x}(f_0)+\frac{1}{2}bf_1+\big(L^{\lambda+\frac{1}{2}}_{a\partial_x}(f_1)+\lambda f_0b'+\frac{1}{2}f_0'b\big)\theta.$$

Especially, we have

$$\left\{\begin{array}{c}\mathfrak{L}^{\lambda}_{X_a}(f_0)=L^{\lambda}_{a\partial_x}(f_0),\qquad \mathfrak{L}^{\lambda}_{X_a}(f_1\theta)=\theta L^{\lambda+\frac{1}{2}}_{a\partial_x}(f_1),\\ and\\ \mathfrak{L}^{\lambda}_{X_{b\theta}}(f_0)=(\lambda f_0b'+\frac{1}{2}f_0'b)\theta,\qquad \mathfrak{L}^{\lambda}_{X_{b\theta}}(f_1\theta)=\frac{1}{2}bf_1.\end{array}\right. \tag{2.1}$$

Of course, for all λ, $\mathfrak{F}_\lambda$ is a $\mathcal{K}(1)$-module:

$$[\mathfrak{L}^{\lambda}_{X_F},\mathfrak{L}^{\lambda}_{X_G}]=\mathfrak{L}^{\lambda}_{[X_F,X_G]}.$$

We thus obtain a one-parameter family of $\mathcal{K}(1)$-modules on $C^\infty(\mathbb{R}^{1|1})$ still denoted by $\mathfrak{F}_\lambda$.

2.4. **Differential operators on weighted densities.** A differential operator on $\mathbb{R}^{1|1}$ is an operator on $C^\infty(\mathbb{R}^{1|1})$ of the following form (see, e.g. [1, 2]) :

$$A=\sum_{i=0}^{\ell}a_i(x,\theta)\partial_x^i+\sum_{i=0}^{\ell}b_i(x,\theta)\partial_x^i\partial_\theta.$$

Of course, any differential operator defines a linear mapping from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any $\lambda,\mu\in\mathbb{R}$, thus, the space of differential operators becomes a family of $\mathcal{K}(1)$ and $\mathfrak{aff}(1|1)$ modules denoted $\mathfrak{D}_{\lambda,\mu}$ for the natural action:

$$X_H.A=\mathfrak{L}^{\mu}_{X_H}\circ A-(-1)^{|A||H|}A\circ\mathfrak{L}^{\lambda}_{X_H}.$$

Similarly, we consider a family of $\mathcal{K}(1)$-modules on the space $\mathfrak{D}_{\lambda,\nu;\mu}$ of bilinear differential operators: $A\ :\ \mathfrak{F}_\lambda\otimes\mathfrak{F}_\nu\to\mathfrak{F}_\mu$ with the $\mathcal{K}(1)$-action

$$X_H.A=\mathfrak{L}^{\mu}_{X_H}\circ A-(-1)^{|A||H|}A\circ\mathfrak{L}^{(\lambda,\nu)}_{X_H}, \tag{2.2}$$

where, $L^{(\lambda,\nu)}_{X_H}$ is the Lie derivative on $\mathfrak{F}_\lambda\otimes\mathfrak{F}_\nu$ defined by the Leibnitz rule :

$$\mathfrak{L}^{(\lambda,\nu)}_{X_H}(F\otimes G)=\mathfrak{L}^{\lambda}_{X_H}(F)\otimes G+(-1)^{|F||H|}F\otimes\mathfrak{L}^{\nu}_{X_H}(G).$$

2.5. **Cohomology.** Let us first recall some fundamental from cohomology theory [13]. Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a Lie superalgebra acting on a superspace $V=V_{\bar{0}}\oplus V_{\bar{1}}$ and let $\mathfrak{h}$ be a superalgebra of $\mathfrak{g}$.(If $\mathfrak{h}$ is omitted it assumed to be $\{0\}$.) The space of $\mathfrak{h}$-relative $\mathfrak{n}$-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g},\mathfrak{h};V):=\mathrm{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h});V).$$

The coboundary operator $\delta_n:C^n(\mathfrak{g},\mathfrak{h};V)\longrightarrow C^{n+1}(\mathfrak{g},\mathfrak{h};V)$ is a $\mathfrak{g}$-map satisfying $\delta_n\circ\delta_{n-1}=0$. The kernel of δ_n denoted $\mathrm{Z}^n(\mathfrak{g},\mathfrak{h};V)$, is the space $\mathfrak{h}$-relative n-cocycle, among them, the elements in the range on

δ_{n-1} are called $\mathfrak{h}$-relative n-coboundaries. We denote $\mathrm{B}^n(\mathfrak{g},\mathfrak{h};V)$ the space
of n-coboundaries.
By definition, the n^{th} $\mathfrak{h}$-relative cohomology space in the quotient space

$$\mathrm{H}^n(\mathfrak{g},\mathfrak{h};V)=\mathrm{Z}^n(\mathfrak{g},\mathfrak{h};V)/\mathrm{B}^n(\mathfrak{g},\mathfrak{h};V).$$

We will only need the formula of δ_n(which will be simply denoted δ) in degrees 0 and 1: for $v\in C^0(\mathfrak{g},\mathfrak{h};V)=V^{\mathfrak{h}}$, $\delta v(g):=(-1)^{p(g)p(v)}g.v$, where,

$$V^{\mathfrak{h}}=\{v\in V|h.v=0\ for\ all\ h\in\mathfrak{h}\},$$

and for $\Upsilon\in C^0(\mathfrak{g},\mathfrak{h};V)$,

$$\delta(\Upsilon)(g,h):=(-1)^{p(g)p(\Upsilon)}g.\Upsilon(h)-(-1)^{p(h)(p(g)+p(\Upsilon))}h.\Upsilon(g)-\Upsilon([g,h])\ for\ any\ g,h\in\mathfrak{g}.$$

3. Cohomology of $\mathfrak{aff}(1)$ acting on $\mathcal{D}_{\lambda,\nu;\mu}$

For the sake of simplicity, the elements fdx^λ of $\mathcal{F}_\lambda$ will be denoted f. Any 1-cochain $c\in\mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{aff}(1),\mathcal{D}_{\lambda,\nu;\mu})$ should retains the following general form:

$$c(X_h,f,g)=\sum_{i,j}\alpha_{i,j}hf^{(i)}g^{(j)}+\sum_{i,j}\beta_{i,j}h'f^{(i)}g^{(j)}.$$

So, for any integer $k\geq 0$, we define the $(k+1)$-homogeneous component of c by

$$c(X_h,f,g)=\sum_{i+j=k+1}\alpha_{i,j}hf^{(i)}g^{(j)}+\sum_{i+j=k}\beta_{i,j}h'f^{(i)}g^{(j)}.$$

The coboundary map δ is homogeneous, therefore, we easily deduce the following lemma:

Lemma 3.1. *Any* 1*-cochain* $c\in\mathrm{C}^1_{\mathrm{diff}}(\mathfrak{aff}(1),\mathcal{D}_{\lambda,\nu;\mu})$ *is a* 1*-cocycle if and only if each of its homogeneous components is a* 1*-cocycle.*

The following lemma gives the general form of any homogeneous 1-cocycle.

Lemma 3.2. *Up to a coboundary, any* k*-homogeneous* 1*-cocycle* $c\in\mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{aff}(1),\mathcal{D}_{\lambda,\nu;\mu})$ *can be expressed as follows. For all* $f\in\mathcal{F}_\lambda$, $g\in\mathcal{F}_\nu$ *and for all* $X_h\in\mathfrak{aff}(1)$*:*

$$c(X_h,f,g)=\sum_{i+j=k}\beta_{i,j}h'f^{(i)}g^{(j)},\qquad(3.1)$$

where $\beta_{i,j}$ *are constants.*

Proof. Any $(k+1)$-homogeneous 1-cocycle on $\mathfrak{aff}(1)$ should retains the following general form:

$$c(X_h, f, g) = \sum_{i+j=k+1} \alpha_{i,j} h f^{(i)} g^{(j)} + \sum_{i+j=k} \beta_{i,j} h' f^{(i)} g^{(j)}$$

where, $\alpha_{i,j}, \beta_{i,j}$ are, a priori, functions.
Now, consider the 1-cocycle condition:

$$c([X_{h_1}, X_{h_2}], f, g) - L^{\lambda,\nu;\mu}_{X_{h_1}} c(X_{h_2}, f, g) + L^{\lambda,\nu;\mu}_{X_{h_2}} c(X_{h_1}, f, g) = 0.$$

where $f \in \mathcal{F}_\lambda$, $g \in \mathcal{F}_\nu$ and $X_{h_1}, X_{h_2} \in \mathfrak{aff}(1)$.
A direct computation proves that we have

$$\frac{d}{dx}(\beta_i) = 0 \; and \; (\alpha_{i,j} = 0) \; if \; \mu - \lambda - \nu = k.$$

□

Theorem 3.3. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}_{\lambda,\nu;\mu})$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}_{\lambda,\nu;\mu}) \simeq \begin{cases} \mathbb{R}^{k+1} & \text{if } \mu = \lambda + \nu + k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Lemma 3.4. *Let* $c : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \to \mathcal{F}_\mu$ *be a bilinear differential operator defined as follows. For all* $f \in \mathcal{F}_\lambda$ *and for all* $g \in \mathcal{F}_\nu$*:*

$$c(f, g) = \sum_{i+j=k} c_{ij} f^{(i)} g^{(j)}$$

where c_{ij} *are constants. Then, for all* $X_h \in \mathfrak{aff}(1)$*, we have*
$L^{\lambda,\nu;\mu}_{X_h} c(f, g) = h' \sum_{i+j=k} (\mu - \lambda - \nu - i - j) c_{ij} f^{(i)} g^{(j)}$.

Proof. Straightforward computation.By using (2.2). □

Proof. Now we are in position to prove Theorem 3.3. Any 1-cocycle on $\mathfrak{aff}(1)$ should retains the following general form:

$$c(X_h, f, g) = \sum_{i+j=k} \beta_{i,j} h' f^{(i)} g^{(j)}, \tag{3.3}$$

where $\beta_{i,j}$ are constants.
the 1-cocycle condition reads as follows: for all $f \in \mathcal{F}_\lambda$, for all $g \in \mathcal{F}_\nu$ and for all $X_{h_1}, X_{h_2} \in \mathfrak{aff}(1)$, we have

$$c([X_{h_1}, X_{h_2}], f, g) - L^{\lambda,\nu;\mu}_{X_{h_1}} c(X_{h_2}, f, g) + L^{\lambda,\nu;\mu}_{X_{h_2}} c(X_{h_1}, f, g) = 0.$$

A direct computation, and by using Lemma 3.2, proves that the coefficient of the component $f^{(i)} g^{(j)}$ in the 1-cocycle condition above is equal to

$$\mu - \lambda - \nu - k = 0. \tag{3.4}$$

Now we are going to deal with trivial 1-cocycles, and show how the general 1-cocycles can be eventually trivial. Any trivial 1-cocycle should be of the forme

$$L_{X_h}^{\lambda,\nu;\mu}c,$$

where, c is an operator $c : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \to \mathcal{F}_\mu$ defined as $c(f,g) = \sum_{i+j=k} c_{ij} f^{(i)} g^{(j)}$. By using Lemma 3.4, we have

$$L_X^{\lambda,\nu;\mu}c = h' \sum_{i+j=k} (\mu - \lambda - \nu - k) c_{ij} f^{(i)} g^{(j)}. \tag{3.5}$$

If $\mu - \lambda - \nu = k$, the corresponding cohomology space is $(k+1)$-dimensional, generated by the 1-cocycles $\mathfrak{c}_0, \mathfrak{c}_1, \cdots, \mathfrak{c}_k$ defined as follows:

$$\mathfrak{c}_0(X_h, f, g) = \beta_0 h' f g^k, \mathfrak{c}_1(X_h, f, g) = \beta_1 h' f' g^{k-1}, \cdots, \mathfrak{c}_k(X_h, f, g) = \beta_k h' f^k g. \tag{3.6}$$

□

4. Cohomology of $\mathfrak{aff}(1|1)$ acting on $\mathfrak{D}_{\lambda,\nu;\mu}$

In this section, we will compute the "differentiable" cohomology of the Lie algebra $\mathfrak{aff}(1|1)$ with coefficients in the space of bilinear differential operators $\mathfrak{D}_{\lambda,\nu;\mu}$. Namely, we consider only cochains that are given by differentiable maps.

Theorem 4.1. *The space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\nu;\mu})$ has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\nu;\mu}) \simeq \begin{cases} \mathbb{R}^{2k+1} & \text{if } \mu - \lambda - \nu = k, \quad k \in \mathbb{N}, \\ \mathbb{R}^{2k+2} & \text{if } \mu - \lambda - \nu = k - \frac{1}{2}, \quad k \in \mathbb{N}^*. \\ 0 & \text{otherwise.} \end{cases}$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\Upsilon_{i,j} = \textstyle\sum_{i+j=k} a_{i,j} \overline{\eta}_1^2(H) F^{(i)} G^{(j)} + \sum_{i+j=k-1} b_{i,j} \overline{\eta}_1^2(H) \eta_1(F^{(i)}) \overline{\eta}_1(G^{(j)}),$$

$$\Psi_{i,j} = \textstyle\sum_{i+j=k} c_{i,j} \overline{\eta}_1^2(H) \overline{\eta}_1(F^{(i)}) G^{(j)} + \sum_{i+j=k} d_{i,j} \overline{\eta}_1^2(H) (-1)^{|F|} F^{(i)} \overline{\eta}_1(G^{(j)}). \tag{4.1}$$

4.1. Relationship between $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\nu;\mu})$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}_{\lambda,\nu;\mu})$.

Before proving the theorem 4.1, we present here some results illustrating the analogy between the cohomology spaces in super and classical settings.

Proposition 4.2. 1. *As a* $\mathfrak{aff}(1)$*-module, we have*

$$\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \text{ and } \mathfrak{aff}(1|1) \simeq \mathfrak{aff}(1) \oplus \Pi(\mathcal{H}),$$

where, $\mathcal{H}$ *is the subspace of* $\mathcal{F}_{-\frac{1}{2}}$ *spanned by* $\{dx^{-\frac{1}{2}}\}$ *and* Π *is the change of parity.*
2. *As a* $\mathfrak{aff}(1)$*-module, we have for the homogenous components of* $\mathfrak{D}_{\lambda,\nu;\mu}$*:*

$$(\mathfrak{D}_{\lambda,\nu;\mu})_{\bar{0}} \simeq \mathcal{D}_{\lambda,\nu;\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu} \oplus \mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu+\frac{1}{2}} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu+\frac{1}{2}}, \quad (4.2)$$

$$(\mathfrak{D}_{\lambda,\nu;\mu})_{\bar{1}} \simeq \Pi(\mathcal{D}_{\lambda,\nu;\mu+\frac{1}{2}} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu} \oplus \mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu+\frac{1}{2}}). \quad (4.3)$$

Proof. 1. The first statement is immediately deduced from (2.1).
2. It is well known that is $M = M_{\bar{0}} \oplus M_{\bar{1}}$ and $N = N_{\bar{0}} \oplus N_{\bar{1}}$ are two $\mathfrak{g}$-modules, where $\mathfrak{g}$ is a (super)algebra, then $Hom(M, N)$ is a $\mathfrak{g}$-module, where the homogenous components are

$$\mathrm{Hom}(M,N)_{\bar{0}} = \mathrm{Hom}(M_{\bar{0}}, N_{\bar{0}}) \oplus \mathrm{Hom}(M_{\bar{1}}, N_{\bar{1}}) \text{ and } \mathrm{Hom}(M,N)_{\bar{1}} = \mathrm{Hom}(M_{\bar{0}}, N_{\bar{1}}) \oplus \mathrm{Hom}(M_{\bar{1}}, N_{\bar{0}})$$

and the $\mathfrak{g}$-action on $Hom(M, N)$ is given by

$$(X.A)(x) = X.(A(x)) - (1)^{|A||X|}A(X.x).$$

Moreover, if $\varphi_1 : M \to M'$ and $\varphi_2 : N \to N'$ are two $\mathfrak{g}$-isomorphisms, then the map $\Psi : \mathrm{Hom}(M,N) \to \mathrm{Hom}(M', N')$ defined by

$$\Psi(A) = \varphi_2 \circ A \circ \varphi_1^{-1}$$

is a $\mathfrak{g}$-isomorphism. In our situation, as a $\mathfrak{aff}(1)$-module, we have for the homogeneous relative parity components:

$$\begin{cases} (\mathfrak{F}_\lambda \otimes \mathfrak{F}_\nu)_{\bar{0}} \simeq \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \\ (\mathfrak{F}_\lambda \otimes \mathfrak{F}_\nu)_{\bar{1}} \simeq \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \mathcal{F}_\nu \oplus \mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}). \end{cases}$$

So, we deduce the two homogenous relative parity components of $\mathfrak{D}_{\lambda,\nu;\mu}$ as a $\mathfrak{aff}(1)$-module. In fact, we have the following isomorphisms:

$$\begin{cases} \mathrm{Hom}_{\mathrm{diff}}(\Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \mathcal{F}_\mu) & \to & \mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu}, & A \mapsto A \circ (\Pi \otimes \Pi), \\ \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \Pi(\mathcal{F}_{\mu+\frac{1}{2}})) & \to & \mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu+\frac{1}{2}}, & A \mapsto \Pi \circ A \circ (Id \otimes \Pi), \\ \mathrm{Hom}_{\mathrm{diff}}(\Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \mathcal{F}_\nu, \Pi(\mathcal{F}_{\mu+\frac{1}{2}})) & \to & \mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu+\frac{1}{2}}, & A \mapsto \Pi \circ A \circ (\Pi \otimes Id). \end{cases}$$

$$\begin{cases} \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda \otimes \mathcal{F}_\nu, \Pi(\mathcal{F}_{\mu+\frac{1}{2}})) & \to & \Pi(\mathcal{D}_{\lambda,\nu;\mu+\frac{1}{2}}), & A \mapsto \Pi(\Pi \circ A), \\ \mathrm{Hom}_{\mathrm{diff}}(\Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \Pi(\mathcal{F}_{\mu+\frac{1}{2}})) & \to & \Pi(\mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu+\frac{1}{2}}) & A \mapsto \Pi(\Pi \circ A \circ (\Pi \otimes \Pi), \\ \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \mathcal{F}_\mu) & \to & \Pi(\mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu}), & A \mapsto \Pi(A \circ (Id \otimes \Pi)), \\ \mathrm{Hom}_{\mathrm{diff}}(\Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \mathcal{F}_\nu, \mathcal{F}_\mu) & \to & \Pi(\mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu}), & A \mapsto \Pi(A \circ (\Pi \otimes Id)). \end{cases}$$

□

Proposition 4.3. *Any 1-cocycle $\Upsilon \in \mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1);\mathfrak{D}_{\lambda,\nu;\mu})$, is decomposed into (Υ',Υ'') in $\mathrm{Hom}(\mathfrak{aff}(1);\mathfrak{D}_{\lambda,\nu;\mu})\oplus\mathrm{Hom}(\mathcal{H};\mathfrak{D}_{\lambda,\nu;\mu})$. Υ' and Υ'' are*
solutions of the following equations:

$$\Upsilon'([X_{g_1},X_{g_2}])-\mathfrak{L}^{\lambda,\nu;\mu}_{X_{g_1}}\Upsilon'(X_{g_2})+\mathfrak{L}^{\lambda,\nu;\mu}_{X_{g_2}}\Upsilon'(X_{g_1})=0, \tag{4.4}$$

$$\Upsilon''([X_g,X_\theta])-\mathfrak{L}^{\lambda,\nu;\mu}_{X_g}\Upsilon''(X_\theta)+\mathfrak{L}^{\lambda,\nu;\mu}_{X_\theta}\Upsilon'(X_g)=0, \tag{4.5}$$

$$\Upsilon'([X_\theta,X_\theta])+2\mathfrak{L}^{\lambda,\nu;\mu}_{X_\theta}\Upsilon''(X_\theta)=0. \tag{4.6}$$

where, $g,g_1,g_2\in\mathbb{R}_1[x]$.

Proof. The equations (4.4), (4.5) and (4.6) are equivalent to the fact that Υ is a 1-cocycle. For any $X_F, X_G\in\mathfrak{aff}(1|1)$,

$$\delta\Upsilon(X_F,X_G):=(-1)^{|F||\Upsilon|}\mathfrak{L}^{\lambda,\nu;\mu}_{X_F}\Upsilon(X_G)-(-1)^{|G|(|F|+|\Upsilon|)}\mathfrak{L}^{\lambda,\nu;\mu}_{X_G}\Upsilon(X_F)-\Upsilon([X_F,X_G])=0.$$

□

Now, in order to compute $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})$, we need first to describe the $\mathfrak{aff}(1)$-relative cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{aff}(1),\mathfrak{D}_{\lambda,\nu;\mu})$. So, we shall need the following description of some $\mathfrak{aff}(1)$-invariant mappings.

Lemma 4.4. *Let*

$$A\ :\ \mathcal{H}\times\mathcal{F}_\lambda\times\mathcal{F}_\nu\to\mathcal{F}_\mu,(hdx^{-\frac{1}{2}},fdx^\lambda,gdx^\nu)\mapsto A(h,f,g)dx^\mu$$

be a trilinear differential operator. If a is $\mathfrak{aff}(1)$-invariant then

$$\mu=\lambda+\nu-\frac{1}{2}+k,\ \textit{where}\ k\in\mathbb{N}.$$

and the following relation holds

$$A_k(h,f,g)=\sum_{i+j=k}\gamma_{i,j}hf^{(i)}g^{(j)}dx^{\lambda+\nu-\frac{1}{2}+k}.$$

Proof. Any trilinear differential operator $A\ :\ \mathcal{H}\times\mathcal{F}_\lambda\times\mathcal{F}_\nu\to\mathcal{F}_\mu$ can be expressed as

$$A_k(h,f,g)=\sum_{i+j=k}\gamma_{i,j}hf^{(i)}g^{(j)},$$

where, the $\gamma_{i,j}$ are smooth functions.The invariance property of A with respect any vector fields X_F reads:

$$F(A(h,f,g))'+\mu F'A(h,f,g)=A(Fh'-\frac{1}{2}F'h,f,g)+A(h,Ff'+\lambda F'f,g)+A(h,f,Fg'+\nu F'g). \tag{4.7}$$

The invariance with respect the vector field $X_1 = \partial_x$ yields that A must be expressed with constant coefficients. Consider terms in $F'hf^{(i)}g^{(j)}$ in (4.7), we get

$$\mu = \lambda + \nu - \frac{1}{2} + k.$$

□

Lemma 4.5. *The 1-cocycle Υ is a coboundary for $\mathfrak{aff}(1|1)$ if and only if its restriction to $\mathfrak{aff}(1)$ is a coboundary for $\mathfrak{aff}(1)$.*

Proof. It is easy to see that if Υ is a coboundary for $\mathfrak{aff}(1|1)$ then its restriction to $\mathfrak{aff}(1)$ is a coboundary for $\mathfrak{aff}(1)$. Now, assume that, Υ' is a coboundary for $\mathfrak{aff}(1)$, that is, there existe $\widetilde{A} \in \mathcal{D}_{\lambda,\nu;\mu}$ such that for all g polynomial in the variable x with degree at most 1
$\Upsilon(X_g) = \mathfrak{L}^{\lambda,\nu;\mu}_{X_g}\widetilde{A}$. By replacing Υ by $\Upsilon - \delta\widetilde{A}$, we can suppose that
$\Upsilon\,|_{\mathfrak{aff}(1)}= 0$. But, in this case, the map Υ is $\mathfrak{aff}(1)$-invariant must satisfy, for g polynomial with degree 0 or 1, the following equation

$$\mathfrak{L}^{\lambda,\nu;\mu}_{X_g}\Upsilon(X_\theta) - \Upsilon([X_g, X_\theta]) = 0, \tag{4.8}$$

$$\mathfrak{L}^{\lambda,\nu;\mu}_{X_\theta}\Upsilon(X_\theta) + \mathfrak{L}^{\lambda,\nu;\mu}_{X_\theta}\Upsilon(X_\theta) = 0, \tag{4.9}$$

where $g \in \mathbb{R}_2[X]$.
1) If Υ is an even 1-cocycle then Υ is decomposed into four trilinear maps:

$$\begin{cases} \Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \mathcal{F}_\nu \to \mathcal{F}_\mu, \\ \Pi(\mathcal{H}) \otimes \mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}) \to \mathcal{F}_\mu, \\ \Pi(\mathcal{H}) \otimes \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \to \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\ \Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}) \to \Pi(\mathcal{F}_{\mu+\frac{1}{2}}). \end{cases}$$

The equation (4.7) is nothing but the $\mathfrak{aff}(1)$-invariance property of these maps. Therfore, the expressions of these maps are given by Lemma 4.4, in fact, the change of parity functor Π commutes with the $\mathfrak{aff}(1)$-action. So, we must have $\mu = \lambda+\nu+k$, where $k+1 \in \mathbb{N}$, otherwise, the operator Υ is identically the zero map.More precisely:
If $\mu = \lambda + \nu + k$, where $k \in \mathbb{N}^*$, we have

$$\Upsilon_k(X_{h\theta})(\theta f, g) = \sum_{i=0}^{k} a_i h f^{(i)} g^{(k-i)} \tag{4.10}$$

$$\Upsilon_k(X_{h\theta})(f, \theta g) = \sum_{i=0}^{k} b_i h f^{(i)} g^{(k-i)} \tag{4.11}$$

$$\Upsilon_k(X_{h\theta})(f,g)=\theta\sum_{i=0}^{k+1}c_ihf^{(i)}g^{(k-i+1)} \tag{4.12}$$

$$\Upsilon_k(X_{h\theta})(\theta f,\theta g)=\theta\sum_{i=0}^{k}d_ihf^{(i)}g^{(k-i)} \tag{4.13}$$

The maps Υ_k must satisfy the equation (4.8). More precisely, the maps Υ_k must satisfy the following four equation

$$\begin{aligned}
&\theta(\Upsilon_k(X_\theta)(\theta f,g))'+2\Upsilon_k(X_\theta)(\frac{1}{2}f,g)-2\Upsilon_k(X_\theta)(\theta f,\frac{1}{2}\theta g')=0,\\
&\theta(\Upsilon_k(X_\theta)(f,\theta g))'+2\Upsilon_k(X_\theta)(\frac{1}{2}\theta f',\theta g)+2\Upsilon_k(X_\theta)(f,\frac{1}{2}g)=0,\\
&\partial_\theta(\Upsilon_k(X_\theta)(f,g))+2\Upsilon_k(X_\theta)(\frac{1}{2}\theta f',g)+2\Upsilon_k(X_\theta)(f,\frac{1}{2}\theta g')=0,\\
&\partial_\theta(\Upsilon_k(X_\theta)(\theta f,\theta g))+2\Upsilon_k(X_\theta)(\frac{1}{2}f,\theta g)-2\Upsilon_k(X_\theta)(\theta f,\frac{1}{2}g)=0.
\end{aligned}$$

By a direct , but very hard , computation we show that Υ_k is a coboundary.

2) Similarly, if Υ is an odd 1-cocycle then Υ is decomposed into four components:

$$\begin{cases}
\Pi(\mathcal{H})\otimes\mathcal{F}_\lambda\otimes\mathcal{F}_\nu\to\mathcal{F}_\mu,\\
\Pi(\mathcal{H})\otimes\Pi(\mathcal{F}_{\lambda+\frac{1}{2}})\otimes\Pi(\mathcal{F}_{\nu+\frac{1}{2}})\to\mathcal{F}_\mu,\\
\Pi(\mathcal{H})\otimes\Pi(\mathcal{F}_{\lambda+\frac{1}{2}})\otimes\mathcal{F}_\nu\to\Pi(\mathcal{F}_{\mu+\frac{1}{2}}),\\
\Pi(\mathcal{H})\otimes\mathcal{F}_\lambda\otimes\Pi(\mathcal{F}_{\nu+\frac{1}{2}})\to\Pi(\mathcal{F}_{\mu+\frac{1}{2}}).
\end{cases}$$

The equation (4.7) is nothing but the $\mathfrak{aff}(1)$-invariance property of these maps. Therfore, the expressions of these maps are given by Lemma 4.4. So, we must have $\mu=\lambda+\nu+k-\frac{1}{2}$, where $k\in\mathbb{N}$, otherwise, the operator Υ is identically the zero map. If $\mu=\lambda+\nu+k-\frac{1}{2}$, where $k\in\mathbb{N}$, we show, as in the previous case that Υ is coboundary. □

4.2. **Proof of Theorem 4.1.**

Proof. The first cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})$ is decomposed into odd and an even subspaces:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})=\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})_0\oplus\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})_1.$$

We compute each part separetly.

1. Let Υ be a non trivial even 1-cocycle for $\mathfrak{aff}(1|1)$ in $\mathfrak{D}_{\lambda,\nu;\mu}$. The

restriction of Υ on $\mathfrak{aff}(1)$ is with values in $(\mathfrak{D}_{\lambda,\nu;\mu})_{\overline{0}}$ which is isomorphic, as $\mathfrak{aff}(1)$-module, to

$$\mathcal{D}_{\lambda,\nu;\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu} \oplus \mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu+\frac{1}{2}} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu+\frac{1}{2}},$$

the restriction of Υ on $\Pi(\mathcal{H})$ is with values in $(\mathfrak{D}_{\lambda,\nu;\mu})_{\overline{1}}$ which is isomorphic, as $\mathfrak{aff}(1)$-module, to

$$\Pi(\mathcal{D}_{\lambda,\nu;\mu+\frac{1}{2}} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu} \oplus \mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu+\frac{1}{2}}).$$

Hereafter, $F = f_0 + f_1\theta$ and $G = g_0 + g_1\theta$ where $f_0, f_1, g_0, g_1 \in C^\infty(\mathbb{R})$.
The restriction of Υ on $\mathfrak{aff}(1)$ is given by

$$\Upsilon\mid_{\mathfrak{aff}(1)} = A_0 + A_1 + ... + A_k,$$

where,

$$A_0(X_h, F, G) = \beta_{01}\mathfrak{c}_{01}(X_h, f_0, g_0) + \beta_{02}\mathfrak{c}_{02}(X_h, f_1, g_1) + \theta\beta_{03}\mathfrak{c}_{03}(X_h, f_1, g_0) + \theta\beta_{04}\mathfrak{c}_{04}(X_h, f_0, g_1),$$

$$A_1(X_h, F, G) = \beta_{11}\mathfrak{c}_{11}(X_h, f_0, g_0) + \beta_{12}\mathfrak{c}_{12}(X_h, f_1, g_1) + \theta\beta_{13}\mathfrak{c}_{13}(X_h, f_1, g_0) + \theta\beta_{14}\mathfrak{c}_{14}(X_h, f_0, g_1),$$

$$A_k(X_h, F, G) = \beta_{k1}\mathfrak{c}_{k1}(X_h, f_0, g_0) + \beta_{k2}\mathfrak{c}_{k2}(X_h, f_1, g_1) + \theta\beta_{k3}\mathfrak{c}_{k3}(X_h, f_1, g_0) + \theta\beta_{k4}\mathfrak{c}_{k4}(X_h, f_0, g_1),$$

where, $\mathfrak{c}_{0i}$, $\mathfrak{c}_{1i}$, ... ,$\mathfrak{c}_{ki}$ are as thase defined in $H^1(\mathfrak{aff}(1), \mathcal{D}_{\lambda,\nu;\mu})$ and β_{0i}, β_{1i}, ..., β_{ki} in $\mathbb{R}$.
The restriction of Υ on $\Pi(\mathcal{H})$ is given by

$$\begin{aligned}\Upsilon\mid_{\Pi(\mathcal{H})} = \;& \sum_{i+j=k} r_{1,i,j} f_1^{(i)} g_0^{(j)} + \sum_{i+j=k} r_{2,i,j} f_0^{(i)} g_1^{(j)} \\ & +\theta\big[\sum_{i+j=k+1} r_{3,i,j} f_0^{(i)} g_0^{(j)} + \sum_{i+j=k} r_{4,i,j} f_1^{(i)} g_1^{(j)}\big]\end{aligned}$$

By the 1-cocycle relation:
$\delta\Upsilon(X_h, X_\theta)(F, G) = 0$, we prove that

$$\begin{cases} r_{1,i,j}(\mu - \lambda - \nu - k) - \frac{1}{2}(\beta_{j4} - \beta_{j2} - \beta_{j1}) = 0 \\ r_{2,i,j}(\mu - \lambda - \nu - k) - \frac{1}{2}(\beta_{j3} - \beta_{j1} + \beta_{j2}) = 0 \\ r_{3,i,j}(\mu - \lambda - \nu - k) - \frac{1}{2}(2\beta_{j1} - \beta_{j3} - \beta_{j4}) = 0 \\ r_{4,i,j}(\mu - \lambda - \nu - k) - \frac{1}{2}(2\beta_{j2} + \beta_{j3} - \beta_{j4}) = 0 \end{cases}$$

where $i, j \in \{0, ..., k\}$.

From Lemma 4.5 that the dimension of $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\nu;\mu})$ is equal to the number of parameters β_{jl} where $l \in \{1, 2, 3, 4\}$.
Thus we have $dim\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda,\nu;\mu}) = 2k + 1$.

2. Let Υ be a non trivial odd 1-cocycle for $\mathfrak{aff}(1|1)$ in $\mathfrak{D}_{\lambda,\nu;\mu}$.
The restriction of Υ on $\mathfrak{aff}(1)$ is with values in $(\mathfrak{D}_{\lambda,\nu;\mu})_{\overline{1}}$ which is isomorphic, as $\mathfrak{aff}(1)$-module, to

$$\Pi(\mathcal{D}_{\lambda,\nu;\mu+\frac{1}{2}} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu} \oplus \mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu+\frac{1}{2}}).$$

the restriction of Υ on $\Pi(\mathcal{H})$ is with values in $(\mathfrak{D}_{\lambda,\nu;\mu})_{\bar{0}}$ which is isomorphic, as $\mathfrak{aff}(1)$-module, to

$$\mathcal{D}_{\lambda,\nu;\mu}\oplus\mathcal{D}_{\lambda+\frac{1}{2},\nu+\frac{1}{2};\mu}\oplus\mathcal{D}_{\lambda,\nu+\frac{1}{2};\mu+\frac{1}{2}}\oplus\mathcal{D}_{\lambda+\frac{1}{2},\nu;\mu+\frac{1}{2}},$$

Hereafter, $F=f_0+f_1\theta$ and $G=g_0+g_1\theta$,where, $f_0,f_1,g_0,g_1\in C^\infty(\mathbb{R})$.
The restriction of Υ on $\mathfrak{aff}(1)$ is given by

$$\Upsilon\mid_{\mathfrak{aff}(1)}=B_0+B_1+...+B_k,$$

where,

$$B_0(X_h,F,G)=\gamma_{01}\mathfrak{c}_{03}(X_h,f_1,g_0)+\gamma_{02}\mathfrak{c}_{04}(X_h,f_0,g_1)+\theta\gamma_{03}\mathfrak{c}_{01}(X_h,f_0,g_0)+\theta\gamma_{04}\mathfrak{c}_{02}(X_h,f_1,g_1),$$

$$B_1(X_h,F,G)=\gamma_{11}\mathfrak{c}_{13}(X_h,f_1,g_0)+\gamma_{12}\mathfrak{c}_{14}(X_h,f_0,g_1)+\theta\gamma_{13}\mathfrak{c}_{11}(X_h,f_0,g_0)+\theta\gamma_{14}\mathfrak{c}_{12}(X_h,f_1,g_1),$$

$$B_k(X_h,F,G)=\gamma_{k1}\mathfrak{c}_{k3}(X_h,f_1,g_0)+\gamma_{k4}\mathfrak{c}_{k2}(X_h,f_0,g_1)+\theta\gamma_{k3}\mathfrak{c}_{k1}(X_h,f_0,g_0)+\theta\gamma_{k4}\mathfrak{c}_{k2}(X_h,f_1,g_1),$$

where, $\mathfrak{c}_{0i}$, $\mathfrak{c}_{1i}$, ... ,$\mathfrak{c}_{ki}$ are as thase defined in $H^1(\mathfrak{aff}(1),\mathcal{D}_{\lambda,\nu;\mu})$ and γ_{0i}, γ_{1i}, ..., γ_{ki} in $\mathbb{R}$.
The restriction of Υ on $\Pi(\mathcal{H})$ is given by

$$\begin{aligned}\Upsilon\mid_{\Pi(\mathcal{H})}= &\sum_{i+j=k} q_{1,i,j}f_0^{(i)}g_0^{(j)}+\sum_{i+j=k-1} q_{2,i,j}f_1^{(i)}g_1^{(j)}\\ &+\theta\big[\sum_{i+j=k+1} q_{3,i,j}f_1^{(i)}g_0^{(j)}+\sum_{i+j=k} q_{4,i,j}f_0^{(i)}g_1^{(j)}\big]\end{aligned}$$

By the 1-cocycle relation:
$\delta\Upsilon(X_h,X_\theta)(F,G)=0$, we prove that

$$\begin{cases} q_{1,i,j}(\mu-\lambda-\nu-k+\frac{1}{2})-\frac{1}{2}(\gamma_{j3}+\gamma_{j2}+\gamma_{j1})=0\\ q_{2,i,j}(\mu-\lambda-\nu-k+\frac{1}{2})-\frac{1}{2}(\gamma_{j4}-\gamma_{j1}+\gamma_{j2})=0\\ q_{3,i,j}(\mu-\lambda-\nu-k+\frac{1}{2})-\frac{1}{2}(2\gamma_{j1}+\gamma_{j3}-\gamma_{j4})=0\\ q_{4,i,j}(\mu-\lambda-\nu-k+\frac{1}{2})-\frac{1}{2}(2\gamma_{j2}+\gamma_{j3}+\gamma_{j4})=0\end{cases}$$

where $i,j\in\{0,...,k\}$.

From Lemma 4.5 that the dimension of $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})$ is equal to the number of parameters γ_{jl} where $l\in\{1,2,3,4\}$.
Thus we have $dim\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(1|1),\mathfrak{D}_{\lambda,\nu;\mu})=2k+2$.

□

Acknowledgments

We would like to thank Mabrouk Ben AMMAR and Claude Roger for helpful discussions.

References

[1] I. Basdouri, M. Ben Ammar, *Cohomology of* $\mathfrak{osp}(1|2)$ *Acting on Linear Differential Operators on the Supercercle* $S^{1|1}$. Letters in Mathematical Physics(2007)81:239–251.

[2] I. Basdouri, M. Ben Ammar, N. Ben Fraj, M. Boujelbene and K. Kammoun, *Cohomology of the Lie Superalgebra of Contact Vector Fields on* $\mathbb{R}^{1|1}$ *and Deformations of the Superspace of Symbols* . Journal of Nonlinear Math Physics, Vol. 16, No. 4 (2009) 1–37.

[3] Imed Basdouri, Ismail Laraiedh, Othmen Ncib, *The Linear* $\mathfrak{aff}(n|1)-$*Invariant Differential Operators on Weighted Densities on the superspace* $\mathbb{R}^{1|n}$ *and* $\mathfrak{aff}(n|1)-$*Relative Cohomology.* International Journal of Geometric Methods in Modern Physics, Volume 10, Issue 04, April 2013.

[4] Sofiane Bouarroudj, *Cohomology of the vector fields Lie algebras on* $\mathbb{R}P^1$ *acting on bilinear differential operators*, International Journal of Geometric Methods in Modern Physics, Vol. 2, No. 1 (2005) 23–40.

[5] S. Bouarroudj, V. Ovsienko, *Three cocycles on* $Diff(S^1)$ *generalizing the Schwarzian derivative*, Internat. Math. Res. Notices 1998, No.1, 25–39.

[6] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, **Plenum Publ. New York**, 1986.

[7] B. L. Feigin, D. B. Fuchs, *Homology of Lie algebras on vector fields on the line*, Funkts. Anal. Prilozhen., 16, No. 2, 47–63 (1982).

[8] C. Chevalley and S.Eilenberg, cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc. 63(1948), 85-124.

[9] E. Cartan, Sur les nombres de Betti des espaces de groupes clos, C. R. Acad. Sci. (Paris) 187(1928), 196-198.

[10] H. Cartan and S. Eilenberg, Homological Algebra, Princeton U. Press, 1956.

[11] C. Weible, History of homological Algebra, in History of topology, 797-836, North-Holland, Amsterdam, 1999.

[12] A. Fialowski, Deformations of Lie algebras, Math. USSR-Sb. **55** (1986), 467-473.

[13] A. Fialowski, An example of formal deformations of Lie algebras, Deformation Theory of Algebras and Structures and Appl., Kluwer (1988), 375-401.