Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260

CJMS. **10**(2)(2021), 254-xx

A Meshless Method for Numerical Solutions of Non-Homogeneous Differential Equation with Variable Delays

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Abstract. This paper is devoted to solve a class of differential equation with simultaneously combining variable coefficients and variable delays namely variable-delay differential equations (VD-DEs). For this purpose, a numerical method is proposed in which the unknown function and its derivative are approximated with the basis of interpolating Multiquadric radial basis functions (MQ-RBFs) at arbitrary collocation points. According to the existing mechanism, the synchronization problem is recast to a system of algebraic equations. In the other hand, the proposed method provides a very adjustable framework for approximation according to the discretization and due to a board range of arbitrary nodes. Finally, some illustrative examples are given to verify the validity and applicability of the new technique.

Keywords: Radial basis function, Multiquadric function, Collocation method, Variable delay, Variable coefficients, Algebraic equation.

2000 Mathematics subject classification: 26A33; 34A08; 34K37.

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Received: 30 March 2020
Revised: 08 May 2020
Accepted: 12 May 2020

1. Introduction

In examining a physical process from a mathematical point of view, the assumption used for a large range of dynamical systems is that the intended process behavior depends only on its current state. However, there are situations where this assumption is not valid and the use of a classical model in their analysis and design will result in poor performance of these systems. In these cases, the influence of former state system will also be considered on their behavior [24, 13, 14, 30] which is referred to delay differential equations (DDEs). A generalization of classic DDEs when the dynamics system is described by variable-delays is considered as VDDEs. It is worth noting that the research on VDDEs is completely new and numerical studies of these problems is still at an early stage of growth. The reason to formulate and solve VDDEs has recently been answered affirmatively turn into significant increasing of these problems in life sciences [24]. In this paper, we intend to obtain a new numerical approach for approximating the solution of the following VDDE:

$$
\begin{aligned}
&y'(x) = b_0(x) + b_1(x)y(x) + \sum_{k=2}^{m} b_k(x)y(x-\delta_k(x)), \quad x_0 \le x \le x_f \qquad (1.1)\\
&y(x) = \gamma(x), \quad \eta \le x \le x_0
\end{aligned}
$$

where the coefficient $b_k(x)$ and the delays $\delta_k(x)$ are given continuous functions, $k = 0, 1, \cdots, m$, $\delta_k(x) \ge 0$ for $x \ge x_0$ and $\eta = \inf_{x_0 \le x \le x_f}\{x - \delta_k(x)\}$. The initial data consists of a bounded and continuous function $\gamma(x)$ on $[\eta, x_0]$.

Although, many computational methods have been proposed for solving DDEs, [25, 3, 19], only a small number of these methods have so far been generalized for solving VDDEs. Asymptotic behavior of the solutions to a differential equation with variable delays is considered in [1, 4, 20, 28, 31]. But, most of the mentioned type of VDDEs have not been analytical solutions; hence, numerical methods are needed to get approximate solutions. In [21], the authors applied a matrix collocation method to solve the differential equation (1.1) based on Morgan-Voyce polynomials. In [23], the authors have used the method based on hybrid Taylor and Lucas polynomials to solve this equation.

Getting through the techniques for finding solutions of differential equations, RBFs is a relatively new technique. Many researchers have come up with this technique because it has better accuracy, stability, efficiency and simplicity of implementation over other methods, exemplified, RBF meshless method used by Liu et. al. in [18], Dehghan et. al. in [5] and Ahmadi et. al. in [2]. In addition, the local RBF method has been investigated in [29] to solve the variable-order time fractional diffusion equation. Also, RBFs are actively used for solving KdV equation in [6, 8]. A local RBFs collocation method based on the MQ-RBFs for solving nonlinear coupled Burgers equations is presented in [22]. Kumar and Yadav in [15] provide RBF neural network techniques for solving differential equations of various kinds. Some other work related to this field may be found in [12, 7, 16]. The overall aim of this paper is using MQ-RBF collocation method to obtain the approximation solutions of problem (1.1). With the suggested technique, the VDDEs (1.1) are reformulated into a system of algebraic equations with discrete parameters. To assess the quality of numerical solutions we applied the Gauss elimination method for obtaining the unknown coefficients.

We cover this article with next subdivision. Preliminary concepts of RBF and collocation method are given in section 2. The function approximation and the operational matrix of the derivatives, are discussed in section 3. Also, we present the collocation scheme based on MQ-RBf to solve problem (1.1) in this section. In section 4, several numerical examples and comparisons between our results and those obtained by other methods are intended for resolution. Conclusions are presented in section 5.

2. RBF Collocation Method

A collocation method based on RBFs interpolation has been introduced to solve the VDDE (1.1).

2.1. **RBF Definition.** If the value of a function depends only on its distance from the origin, then we have an RBF with real-valued, that gives meaning $\phi(x) = \phi(\|x\|)$, or a distance from another point like c, as a center, in which $\phi(x, c) = \phi(\|x - c\|)$ where $\|.\|$ is the Euclidean norm, as usual. Now, we define an approximation based on RBFs as follows.

Definition 2.1. Let $\phi(r)$, $r \geq 0$, be an RBF function with distinct centers $x_0, x_1, \cdots, x_N$ and the data $f_i = f(x_i)$, $i = 0, 1, \cdots, N$. An approximation based on RBFs takes the form:

$$s(x) = \sum_{i=0}^{N} \lambda_i \phi(\|x - x_i\|_2), \tag{2.1}$$

where λ_i are the unknown RBF coefficients are selected in such a way that $s(x_i) = f_i$.

Some Types of RBFs: Commonly used types of RBFs include the following forms in which $r = \|x-x_i\|$ and the shape parameter ε controls their flatness [11]:

- Piecewise Smooth:
- $\phi(r) = r^3$ Cubic RBF
- $\phi(r) = r^5$ Quintic RBF
- $\phi(r) = r^2 log(r)$ Tthin Plate spline (TPS) RBF
- $\phi(r) = (1-r)^m + p(r)$ Wendland functions where p is a polynomial

- Infinitely Smooth:
- $\phi(r) = \sqrt{1+(\varepsilon r)^2}$ Multiquadric (MQ) RBF
- $\phi(r) = \dfrac{1}{1+(\varepsilon r)^2}$ Inverse Quadratic (IQ) RBF
- $\phi(r) = e^{-(\varepsilon r)^2}$ Gaussian RBF

2.2. **RBF Collocation Method.** Now, we briefly introduce the RBFs collocation method. Consider the following boundary value problem:

$$Lu = f \quad in\, \Omega \tag{2.2}$$

$$u = g \quad on\, \partial\Omega \tag{2.3}$$

in which $\Omega \subseteq R^d$, d shows the dimension of the problem and L is a linear differential operator. We distinguish in our notation center $X = \{x_1, ..., x_N\}$ and the collocation points $\Xi = \{\alpha_1, ..., \alpha_N\}$. Then we have the approximate solution of (2.2)-(2.3) in the form:

$$\tilde{u}(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|), \tag{2.4}$$

where λ_i, $i = 1, 2, \cdots, N$, are unknown coefficients that determined by collocation, ϕ is a RBF, $\|.\|$ is the Euclidean norm and x_i is the centers of the RBFs.

Now, let Ξ divided into two subsets. One subset contains N_I centers, Ξ_1, where Eq.(2.2) is enforced and the other subset contains N_B centers, Ξ_2, where boundary conditions are enforced. The collection matrix, obtained by applying the collection points in the differential equation and boundary conditions, will be as follows:

$$A = \begin{bmatrix} A_I \\ A_B \end{bmatrix},$$

in which, $A_I = L\phi(\|\alpha - x_j\|)_{\alpha=\alpha_i}$,$\alpha_i \in \Xi_1$,$x_j \in X$, and $A_B = L\phi(\|\alpha - x_j\|)_{\alpha=\alpha_i}$, $\alpha_i \in \Xi_2$, $x_j \in X$. The unknown coefficients λ_i will be obtained by solving the linear system $A\lambda = F$, where F is a vector included $f(\alpha_i)$, $\alpha_i \in \Xi_1$, and $g(\alpha_i)$, $\alpha_i \in \Xi_2$.

3. Main Matrix Relation and Method of Solution

The main purpose of this paper is to obtain an approximate solution for VDDE (1.1) based on the following MQ-RBF:

$$y(x) \simeq y_N(x) = \sum_{j=1}^{N} \lambda_j \varphi(\|x - x_j\|), \tag{3.1}$$

where λ_j, $j = 1, 2, \cdots, N$, are unknown coefficients that determined by collocation method, φ is an RBF, $\|.\|$ is the Euclidean norm, N is a sufficient big constant. and x_j, $j = 1, 2, \cdots, N$ are the centers of the RBFs.

Now, we can rewrite the solution function (3.1) in the following matrix form:

$$y(x) \simeq y_N(x) = \Lambda\Phi(x), \tag{3.2}$$

where $\Lambda = [\lambda_1, \lambda_2, \cdots, \lambda_N]$ are the unknown coefficients and $\Phi(x) = [\varphi_1(x), \varphi_2(x), \cdots, \varphi_N(x)]^T$. In addition, to approximate the delay term in Eq. (1.1) we will have:

$$y(x - \delta_k(x)) \simeq \sum_{j=1}^{N} \lambda_j \varphi(\|x - \delta_k(x) - x_j\|) = \Lambda\Phi(x - \delta_k(x)). \tag{3.3}$$

Also, for the matrix form of the first derivative $y^{'}(x)$ we have:

$$y^{'}(x) \simeq \sum_{j=1}^{N} \lambda_j \varphi^{'}(\|x - x_j\|) = \Lambda\Phi^{'}(x). \tag{3.4}$$

For simplicity, without loss of generality we assume that

$$\begin{aligned} x - \delta_k(x) &\leq x_0, \quad \text{for} \quad k = 2, 3, \cdots, p \\ x - \delta_k(x) &> x_0, \quad \text{for} \quad k = p+1, \cdots, m. \end{aligned} \tag{3.5}$$

Then, we can rewrite the VDDE (1.1) as follows:

$$\begin{aligned} &y^{'}(x) - b_1(x)y(x) - \sum_{k=p+1}^{m} b_k(x)y(x - \delta_k(x)) = b_0(x) + \sum_{k=2}^{p} b_k(x)\gamma(x) \\ &y(x_0) = \gamma(x_0). \end{aligned} \tag{3.6}$$

By substituting the relations (3.1)-(3.4) into problem (3.6), we have

$$\begin{aligned} &\Lambda\Phi^{'}(x) - b_1(x)\Lambda\Phi(x) - \sum_{k=p+1}^{m} b_k(x)\Lambda\Phi(x - \delta_k(x)) = b_0(x) + \sum_{k=2}^{p} b_k(x)\gamma(x), \\ &\Lambda\Phi(x_0) = \gamma(x_0). \end{aligned} \tag{3.7}$$

Now, for solving the VDDE (1.1), we need to find the unknown coefficients presented in (3.1). For this purpose, we used the collocation points $x_i = x_0 + ih,\ i = 1, 2, \cdots, N$ with $h = (x_f - x_0)/N$. Based on the above approximations and also by employing the collocation points mentioned above, the dynamic equation (1.1) is transformed into the following compact form:

$$\Psi\Lambda = \mathbf{G} \tag{3.8}$$

where Ψ is a $N \times N$ matrix such that

$$\Psi_{ij} = \begin{cases} \varphi(\|x_i - x_j\|), \qquad i = 1, \\ \varphi^{'}(\|x_i - x_j\|) - b_1(x_i)\varphi(\|x_i - x_j\|) \\ \qquad - \sum_{k=p+1}^{m} b_k(x_i)\varphi(\|x_i - \delta_k(x_i) - x_j\|),\ i = 2, \cdots, N \end{cases} \tag{3.9}$$

for $j = 1, 2, \cdots, N$ and

$$\mathbf{G} = \begin{bmatrix} \gamma(x_1) \\ b_0(x_2) + \sum_{k=2}^{p} b_k(x_2)\gamma(x_2) \\ b_0(x_3) + \sum_{k=2}^{p} b_k(x_3)\gamma(x_3) \\ \vdots \\ b_0(x_N) + \sum_{k=2}^{p} b_k(x_N)\gamma(x_N) \end{bmatrix}_{N\times 1}$$

To solve system (3.8), we can adopt the Gauss elimination method to find unknowns $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_N$. This will also give the approximation solution (3.1).

3.1. **Error estimation.** Let us suppose that $E(t) = y(x) - \tilde{y}(x)$ is the error function where $y(x)$ is the exact solution of the VDDE (1.1) and $\tilde{y}(x)$ is that the approximate solution of this equation that is given by (3.1) . Therefore,

$$\tilde{y}^{'}(x) - b_0(x) - b_1(x)\tilde{y}(x) - \sum_{k=2}^{m} b_k(x)\tilde{y}(x - \delta_k(x)) = R_1(x), \ \ x_0 \le x \le x_f \tag{3.10}$$

$$\tilde{y}(x) - \gamma(x) = R_2(x), \quad \eta \le x \le x_0$$

Now, by subtracting (1.1) from (3.10), we have:

$$\begin{aligned} (y^{'} - \tilde{y}^{'})(x) - b_0(x) - b_1(x)(y - \tilde{y})(x) - \sum_{k=2}^{m} b_k(x)(y - \tilde{y})(x - \delta_k(x)) \\ = -R_1(x), \ \ x_0 \le x \le x_f \\ y(x) - \tilde{y}(x) = -R_2(x), \quad \eta \le x \le x_0 \end{aligned} \tag{3.11}$$

Now, the error function $E(t)$ is established by the following equation:

$$E^{'}(x) - b_0(x) - b_1(x)E(x) - \sum_{k=2}^{m} b_k(x)E(x-\delta_k(x)) = R_1(x), \quad x_0 \leq x \leq x_f \tag{3.12}$$

$$E(x) = R_2(x), \quad \eta \leq x \leq x_0$$

in which, $R_1(t)$ and $R_2(t)$ are known functions in the collocation points. So, to find approximate error, we can follow the same method mentioned in this section.

4. Numerical Examples

In this section, some examples are given to demonstrate the accuracy and efficiency of the proposed method. The shape parameter is chosen as $\varepsilon = N/32$. Here, the error between the exact solution $y(x)$ and the approximate solution $\tilde{y}(t)$, found using our method, namely the absolute error and computed as follows:

$$Error\{y(x), \tilde{y}(x)\} = \|y(x) - \tilde{y}(x)\|_{\infty}, \quad x \in [x_0, x_f]. \tag{4.1}$$

Example 4.1. As an applicable problem, we consider the problem of the effect of noise on light which is reflected from laser to mirror with a constant delay as follows:

$$y^{'}(x) = -\frac{1}{\epsilon}y(x) + \frac{1}{\epsilon}y(x)y(x-1), \tag{4.2}$$

with the initial condition $y(x) = 0.8$, $-1 \leq x \leq 0$. Figure 1 illustrates the optoelectronic device used by Saboureau et al. [27]. The feedback operates on the pump of the laser by using part of the output light which is injected into a photodetector connected to the pump. The delay of the feedback is controlled by changing the length of the optical path. Applying the RBF collocation method for this boundary problem leads to the Figure 2.

Example 4.2. Consider the following VDDE:

$$2y^{'}(x) - xy(x) + xe^{2x^2}y(x-x^2) = 4e^{2x}, \quad 0 \leq x \leq 1 \tag{4.3}$$

with the initial condition $y(0) = 1$. The exact solution of this problem is $y(x) = e^{2x}$. Figure 3 shows the behavior of the numerical solutions by the proposed method for different values of N together with the exact solution of problem (4.3). The absolute error of this approximation is shown in Figure 4. To evaluate the efficiency and performance of MQ-RBFs, a comparison is made between the absolute errors obtained by our method at $x = 0.5$ and different values of N with the latest results that achieved by the Morgan-Voyce polynomials [21] in Table 1. Comparing

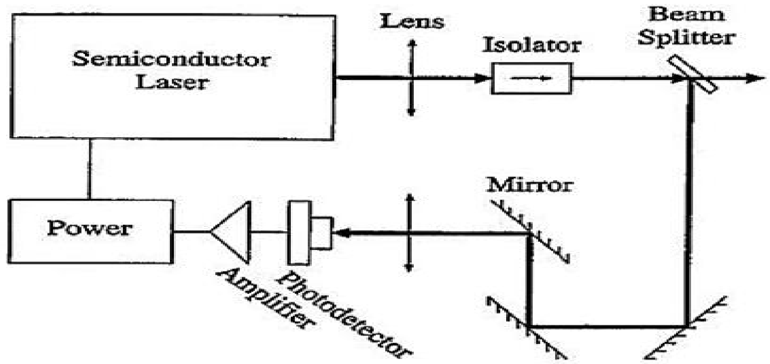

FIGURE 1. Semiconductor laser subject to an optoelectronic feedback.

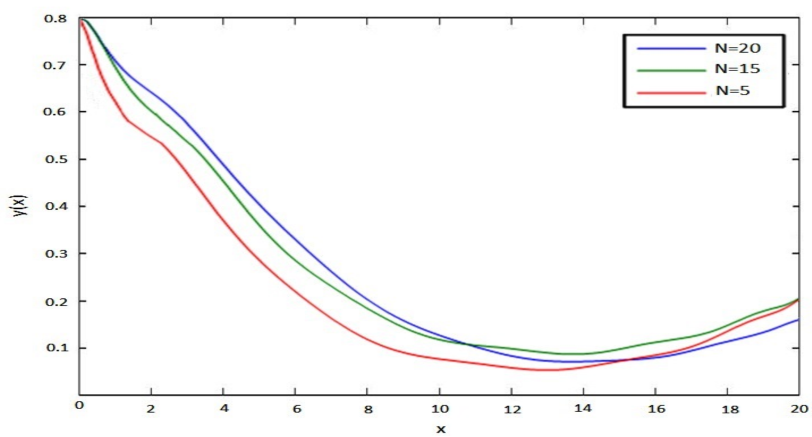

FIGURE 2. The numerical solution of Example 4.1 at different values of N with $\epsilon = 0.1$.

these results reveals that the accuracy of the RBFs collocation method is higher than other methods.

Example 4.3. Consider the following VDDE with different delays in the form

$$y^{'}(x) = (x^2+x-1)e^{-x} - xy(x-ln(x+1)) - e^{-x^2}y(x-x^2) + y(x), \quad 0 \le x \le 1 \tag{4.4}$$

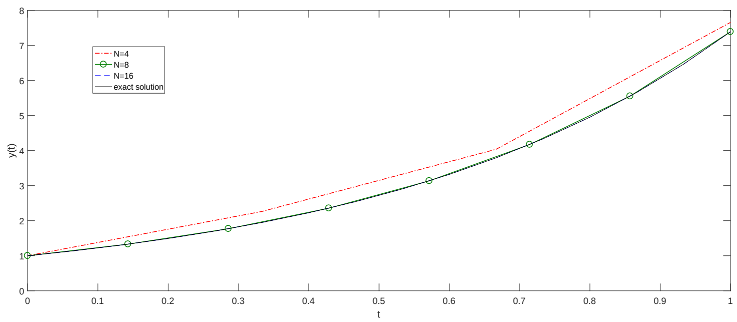

FIGURE 3. Comparison of he exact and approximation solutions with different choose of N for Example 4.2.

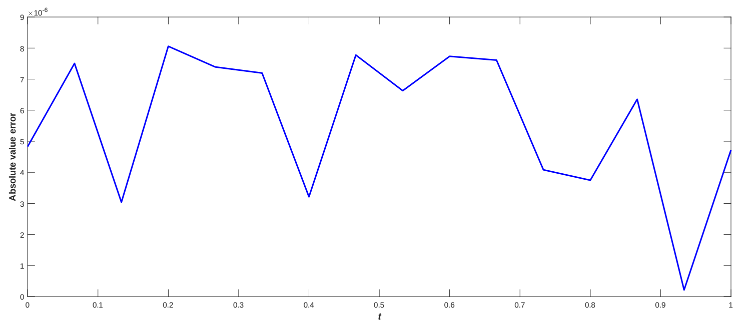

FIGURE 4. Absolute error of $y(x)$ for Example 4.2.

TABLE 1. Comparison the absolute error at different choices of N for Example 4.2.

N	3	4	12
This study	1.7456×10^{-4}	1.6743×10^{-5}	1.5826×10^{-6}
Method in [21]	2.70441×10^{-1}	2.31845×10^{-2}	5.42593×10^{-3}

with the initial condition $y(0) = 1$. The exact solution of this problem is $y(x) = e^{-x}$. In Figure 5, we plotted the exact solution and the approximated solution of $y(x)$ for $N = 16$. As expected, we can observe that the approximated solution converge to the exact values. The absolute errors of the presented method for this example, shown in Figure 6. To evaluate the efficiency and performance of the RBF collocation method presented in Section 2 is better than those introduced in other literatures, a comparison is made between the absolute errors obtained by our

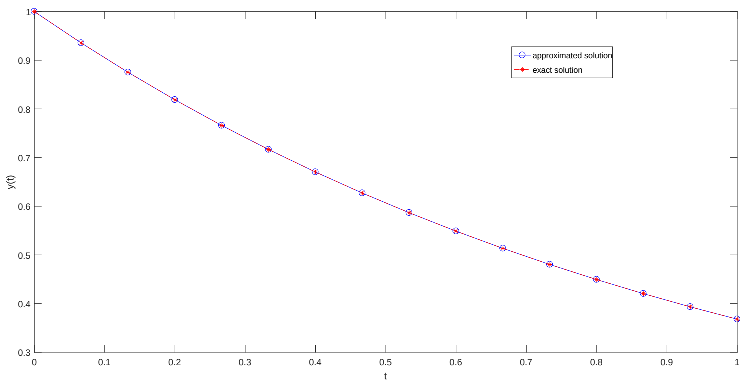

FIGURE 5. Comparison of the exact and approximation solutions for Example 4.3.

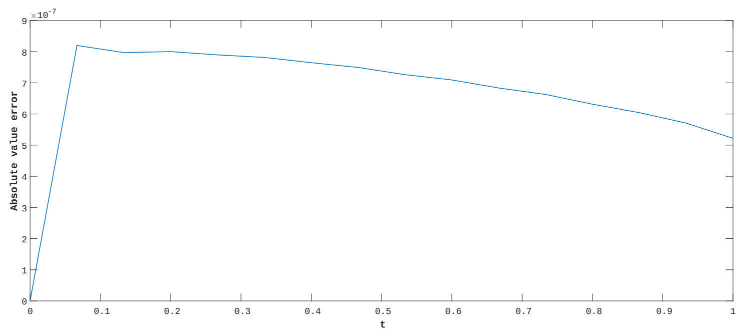

FIGURE 6. Absolute errors of $y(x)$ for Example 4.3.

TABLE 2. Comparison the absolute errors at different choices of x for Example 4.3.

x	This study	Method in [21]
0	0.5784×10^{-9}	0
0.2	0.8001×10^{-7}	2.86131×10^{-3}
0.4	0.7645×10^{-7}	3.17959×10^{-3}
0.6	0.7092×10^{-7}	2.86876×10^{-3}
0.8	0.6314×10^{-7}	2.76306×10^{-3}
1	0.5218×10^{-7}	2.81276×10^{-3}

method for $N = 8$ at different values of t with results that achieved by other researchers in Table 2.

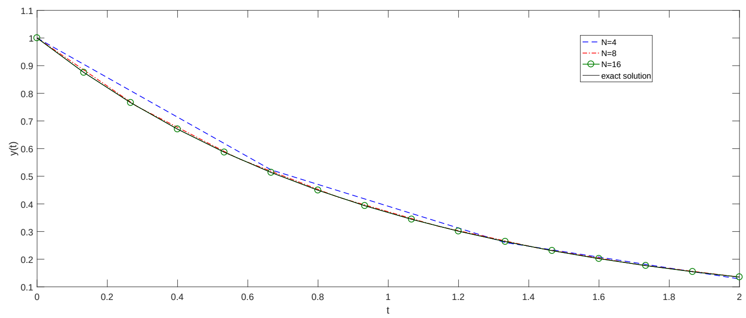

FIGURE 7. Comparison of exact and approximation solutions with different choose of N for Example 4.4.

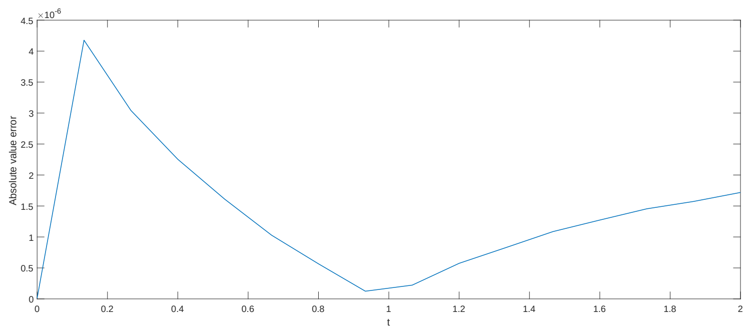

FIGURE 8. Absolute error of $y(x)$ for Example 4.4.

Example 4.4. Consider the VDDE with nonlinear variable-delay as follows

$$y^{'}(x) = (x^2+1)e^{-x} - y(x - ln(x^2+1)) - y(x), \quad 0 \leq x \leq 2 \qquad (4.5)$$

with the initial condition $y(0) = 1$. The exact soluion of this problem is $y(x) = e^{-x}$. Applying the RBF collocation method for this variable-delay problem with different values of N leads to Figure 7. The absolute errors of the presented method for this example with $N = 6$, shown in Figure 8. The accuracy of the MQ-RBF collocation method can be easily concluded from the results reported in Table 3.

Example 4.5. Consider the time-varying delay system described by

$$\begin{bmatrix} y_1^{'}(x) \\ y_2^{'}(x) \end{bmatrix} = \begin{bmatrix} x & 1 \\ x & 2x \end{bmatrix} \begin{bmatrix} y_1^{'}(x-\delta_1(x)) \\ y_2^{'}(x-\delta_1(x)) \end{bmatrix} + \begin{bmatrix} 2 & x \\ x^2 & 0 \end{bmatrix} \begin{bmatrix} y_1^{'}(x-\delta_2(x)) \\ y_2^{'}(x-\delta_2(x)) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x), \qquad (4.6)$$

Table 3. Comparison the absolute error at different choices of x for Example 4.4.

x	This study	Method in [21]	Method in [23]
0	4.1327×10^{-9}	5.55112×10^{-15}	6.70000×10^{-8}
0.2	1.5456×10^{-6}	3.10453×10^{-6}	1.08938×10^{-2}
0.4	9.7075×10^{-7}	2.03288×10^{-6}	2.09254×10^{-2}
0.6	5.6447×10^{-7}	1.09160×10^{-6}	1.97319×10^{-2}
0.8	2.2283×10^{-7}	4.48652×10^{-7}	1.25567×10^{-2}
1	4.9257×10^{-8}	1.14014×10^{-7}	6.26909×10^{-3}
1.2	2.5183×10^{-7}	5.72331×10^{-7}	2.41340×10^{-3}
1.4	4.3048×10^{-7}	9.17880×10^{-7}	2.71723×10^{-3}
1.6	5.7571×10^{-7}	1.22873×10^{-6}	1.09889×10^{-2}
1.8	6.4797×10^{-7}	1.43489×10^{-6}	8.32068×10^{-3}
2	7.7180×10^{-7}	1.44145×10^{-6}	5.64328×10^{-2}

in which $\delta_1(x) = \frac{1}{3}$, $\delta_2(x) = \frac{2}{3}$, $y_1(x) = y_2(x) = u(x) = 0$ for $x \in [-\frac{2}{3}, 0]$ and $u(x) = 2x + 1$ for $x > 0$. The exact solutions of problem (4.6) are:

$$y_1(x) = \begin{cases} 0 & 0 \le x < \frac{1}{3}, \\ \frac{7}{162} - \frac{2}{9}x + \frac{1}{6}x^2 + \frac{1}{3}x^3 & \frac{1}{3} \le x < \frac{2}{3}, \\ \frac{11}{162} - \frac{58}{243}x + \frac{31}{162}x^2 + \frac{1}{9}x^3 + \frac{7}{72}x^4 + \frac{1}{6}x^5 & \frac{2}{3} \le x \le 1, \end{cases}$$

and

$$y_2(x) = \begin{cases} x + x^2 & 0 \le x < \frac{1}{3}, \\ \frac{5}{486} + x + \frac{7}{9}x^2 + \frac{2}{9}x^3 + \frac{1}{2}x^4 & \frac{1}{3} \le x < \frac{2}{3}, \\ \frac{1}{486} + x + \frac{200}{243}x^2 + \frac{20}{81}x^3 + \frac{29}{72}x^4 - \frac{1}{9}x^5 + \frac{1}{6}x^6 & \frac{2}{3} \le x \le 1. \end{cases}$$

The absolute errors of the presented method for $y_1(x)$ and $y_2(x)$ with different values of N are shown in Figure 9. As can be seen, by increasing the number of N, the absolute error of the approximate solutions are decreased.

5. Conclusion

In this work, we introduced a new technique based on the collocation method to solve a class of VDDEs with variable coefficients and variable delays. Our design uses the unknown variable as a linear combination of the MQ-RBF. In the next step, the context of these basis functions for delay and derivative approximations, allow us to reduce the VDDE to a system of algebraic equations for choosing the coefficients and parameters optimally. If the exact solution of the problem is not known,

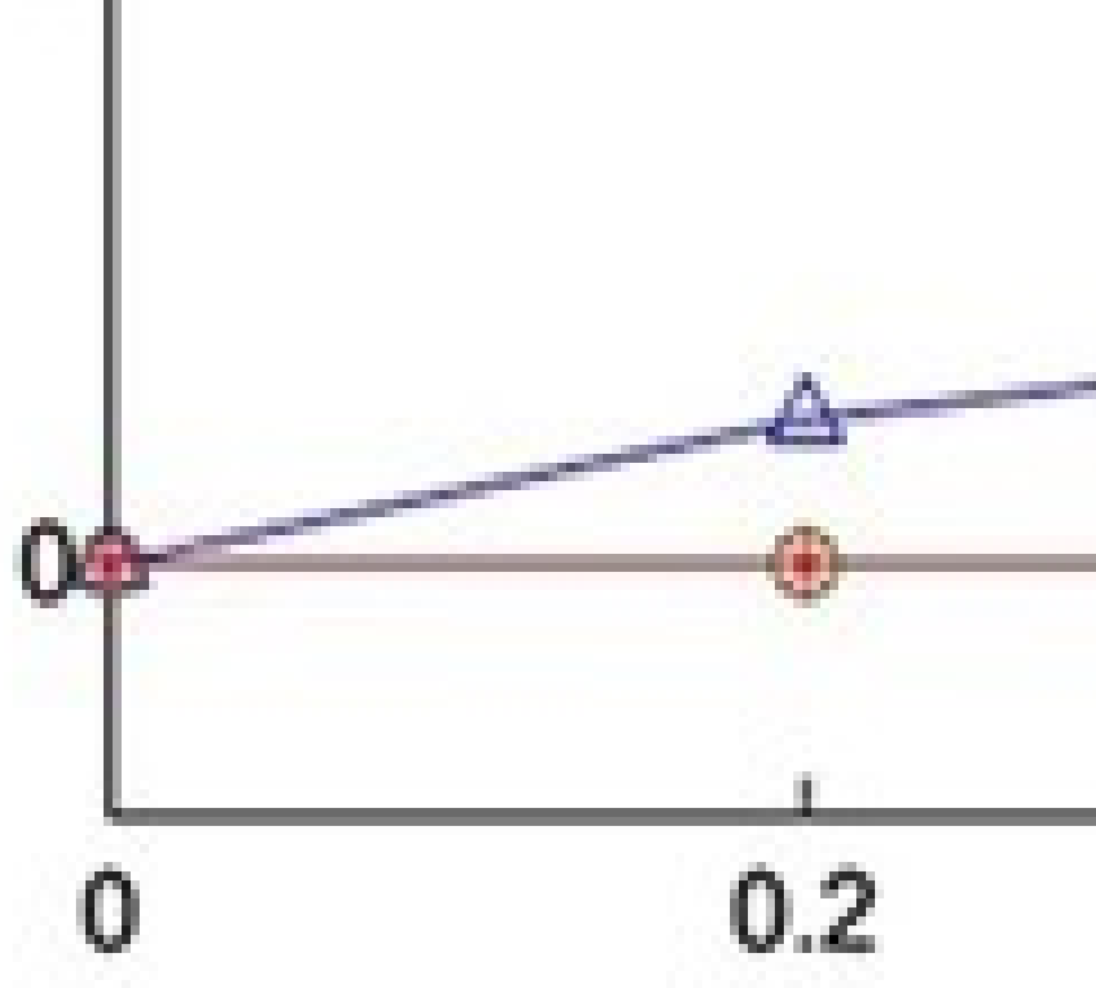

FIGURE 9. Absolute error of $y_1(x)$ and $y_2(x)$ with $N = 5, 15, 20$ for Example 4.5.

by using this technique it is possible to estimate the error function and also to reduce the error due to the residual function. It is seen that, the accuracy improves, when N is increased. The numerical results obtained from examples confirm the efficiency, accuracy, and high performance of this scheme. In the future, this method will be extended to high-order fractional differential-difference equations and their systems, but some modifications are required. Most of the mentioned systems in [9, 10, 26, 17] need to consider that delays and numerical methods are highly recommended for these systems.

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