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A method based on the meshless approach for the numerical solution of the singularly perturbed differential-difference equation arising in the modeling of neuronal variability

Fahimeh Akhavan Ghassabzade [1] and Jafar Saberi-Nadjafi [2] and Ali Reza Soheili [2]

[1] Department of Mathematics, Faculty of Sciences, University of Gonabad, Gonabad, Iran

[1,2] Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract. In this paper, an efficient procedure based on the multiquadric radial basis functions (RBFs) collocation method is applied for the numerical solution of the singularly perturbed differential-difference (SPDDE) equation. The method is coupled with the Residual subsampling algorithm for support adaptivity. The problem considered in this paper shows turning point behavior which is added to the complexity in the construction of numerical approximation to the solution of the problem. The proposed algorithm is very simple to perform. Some numerical examples are given to validate the computational efficacy of the suggested numerical scheme.

Keywords: differential-difference equation, neuronal variability, multiquadric collocation method, radial basis function.

2000 Mathematics subject classification: 65L10, 65L11; Secondary 65L50, 65L70.

[1]Corresponding author: akhavan_ gh@yahoo.com
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1. Introduction

Singularly perturbed differential-difference equations (SPDDEs) have many applications in several fields like in the study of human pupil light reflex [1] and the study of bistable mechanisms [2]. These equations can also be used in the modeling of physiological processes or diseases [3, 4, 5], neuronal variability and so on. The mathematical model of obtaining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites can give rise to a first-exit time problem. In Stein's model, the distribution representing inputs is taken as a Poisson process with exponential decay between synaptic inputs[6]. If there are inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, then the problem for the expected first-exit time V, given the initial membrane potential , can be formulated as a general boundary- value problem for linear second-order differential-difference equation (DDE) [7]

$$\frac{\sigma^2}{2}V''(x)+(\mu-x)V'(x)+\omega_e V(x+\tau_e)+\omega V(x-\tau_i)-(\omega_e+\omega_i)V(x)=-1, \tag{1.1}$$

with boundary condition $V(x)\equiv 0$ for $x\notin(x_1,x_2)$. Where the values $x=x_1$ and $x=x_2$ correspond to the inhibitory reversal potential and the threshold value of membrane potential for action potential generation, respectively. The first-order derivative term corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as a Poisson process with mean rates ω_e and ω_i , respectively, and generate jumps in the membrane potential of amounts τ_e and τ_i , respectively, which are small quantities and could be dependent on voltage. In the past, less consideration had been given to the numerical solutions of SPDDEs. But, in the recent years, there has been increasing attention in the treatment of such problems. The study of such kind of boundary value problems, first, was presented by Lange and Miura. [7, 8, 9, 10]. Then, many authors [11, 12, 13] solved SPDDEs with the aid of numerical techniques. Kadalbajoo and Sharma [14] presented a numerical scheme for the SPDDEs. But their study just includes the case when the convection coefficient has the same sign throughout the domain and the shifts are of $o(\varepsilon)$. Using Taylor approximations, they could eliminate the terms which involve the small shifts in the problem. In [15], author has solved singularly perturbed turning point problem by using a parameter-uniform method. An ε-uniform numerical method was used for solving third order singularly perturbed delay differential equations in [16]. In [17], the authors have used the higher order nonuniform grids for singularly perturbed convection-diffusion-reaction problems. A

meshless method is used in [18], for solving boundary layer's SPDDE.
In this paper, we state a model problem for a general BVP for singularly perturbed DDE including both types of shifts (negative as well as positive shifts) on $\Omega \in (a_0, b_0)$

$$\varepsilon V''(x) + a(x)V'(x) - b(x)V(x) + c(x)V(x-\delta) + d(x)V(x+\eta) = f(x), \tag{1.2}$$

Subject to interval conditions

$$V(x) = \psi(x) \quad for \quad a_0 - \delta \leqslant x \leqslant a_0, \tag{1.3}$$

$$V(x) = \gamma(x) \quad for \quad b_0 \leqslant x \leqslant b_0 + \eta, \tag{1.4}$$

where δ and η are delay and advance arguments respectively, $0 < \varepsilon \ll 1$ and $a(x)$, $b(x)$, $c(x)$,$d(x)$,$f(x)$, $\psi(x)$ and $\gamma(x)$ are sufficiently smooth functions. When the shifts δ and η are both zero, the solution of the corresponding ODE displays a layer behavior or turning point (i.e. the points of the domain where $a(x) = 0$) behavior depending upon the coefficient of convection term, if $a(x)$ does not change the sign or change the sign on Ω. The solution generally blows up exponentially in the interior turning point case. So obtaining the solution is more complicated than the cases without interior turning points.
To possesses a unique solution exhibiting an interior layer, we consider the following assumptions:

(i) $a(0) = 0, a^{'}(0) > 0$.
(ii) $b(x) \geq b_0 > 0$ for all $-1 \leq x \leq 1$
(iii) $|a^{'}(x)| \geq \dfrac{|a^{'}(0)|}{2}$ for all $-1 \leq x \leq 1$, which ensures that there is no other turning pointin the domain.

Related to the biological phenomena mentioned above, here we are interested to present a numerical study. Our numerical study is based on the multiquadric RBF collocation method developed by Kansa [19, 20], as we know so far no one paid attention to developing that for solving SPDDEs.
During the last decade, researchers have tried to develop a group of meshless or meshfree methods which are based on radial basis functions (RBFs). In particular, the multiquadric collocation method has deserved much notification as a robust numerical method for the interpolation problems and solving differential equations [21, 22, 23, 24, 25]. This method has a notable benefit over traditional methods such as finite element, finite difference, and finite volume. Because it does not need any meshes in the domain, and it approximates the solution utilizing the radial basis functions (RBFs) on a set of nodes scattered in the problem domain.

2. An outline of MQ RBF collocation method

2.1. **MQ RBF.** Given a set of N distinct points $\{x_i \in \omega, i = 1, \ldots, N\}$, where ω is a bounded domain in R^d. These points are named centers. An RBF is a function $\Phi(r; c), r = \|x - x_i\|_2$, whose value depends only on the distance from some center points. There is a large class of RBF. The basis function used by Hardy was the quadric surfaces

$$\phi(r; c) = \sqrt{c^2 + r^2}, \tag{2.1}$$

where c is a shape parameter. The RBF (2.1) is called the multiquadric or MQ RBF. It has become common to redefine the MQ (2.1) by first letting $c = \dfrac{1}{\epsilon}$ and neglecting the scaling factor $\dfrac{1}{\epsilon}$, which result in

$$\phi(r; \epsilon) = \sqrt{1 + \epsilon^2 r^2}. \tag{2.2}$$

Now, we employ the Integrate multiquadric RBF. Integrate RBF methods integrate the original RBF with respect to r, one or more times, to get a new basis function in hope of restoring or even improving the convergence of the RBF methods [26]. The notation $\phi^n(r)$ represent an RBF that has been integrated $(n > 0)$ n times with respect to r:

$$\Phi^{''}(r) = \phi(r) = \sqrt{1 + \epsilon^2 r^2},$$

$$\Phi^{'}(r) = \phi^1(r) = \frac{(\epsilon r\sqrt{1 + \epsilon^2 r^2} + sinh^{-1}(\epsilon r))}{2\epsilon},$$

$$\Phi(r) = \phi^2(r) = \frac{(-2 + \epsilon^2 r^2)\sqrt{1 + \epsilon^2 r^2} + 3\epsilon r sinh^{-1}(\epsilon r)}{6\epsilon^2}.$$

The integrated MQ basis functions are referred to as IMQ1 and IMQ2 to indicate how many times they have been integrated. Due to the exponential convergence and superior performance of the IMQ2. Here, the IMQ2 will be used.

2.2. **Collocation method.** Now, we shortly introduce the RBFs collocation method. Let $\Omega \subseteq R^d$, consider the following boundary value problem (BVP)

$$Lu = f \quad in\, \Omega, \tag{2.3}$$

$$u = g \quad on\, \partial\Omega, \tag{2.4}$$

where L is a linear differential operator and d is the dimension of the problem. For nonlinear operators, some kind of linearization will be needed to seek the solution iteratively. We distinguish in our notation

centers $X = \{x_1, ..., x_N\}$ and the collocation points $\Xi = \{\alpha_1, ..., \alpha_N\}$. We seek the approximate solution $u\ (x)$ of (2.3) and (2.4) in the form

$$\tilde{u}(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|), \tag{2.5}$$

where λ_i's coefficients to be determined by collocation, ϕ is a radial basis function, $\|.\|$ is the Euclidean norm, and x_i is the center of the radial basis function. Now, let Ξ is divided into two subsets. One subset contains N_I centers, Ξ_1, where Eq.(2.3) is enforced and the other subset contains N_B centers, Ξ_2, where boundary conditions are enforced. The collocation matrix that is obtained by matching the differential equation and the boundary condition at the collocation points has the following form:

$$A = \begin{bmatrix} A_I \\ A_B \end{bmatrix},$$

where $A_I = L\phi(\|\alpha - x_j\|)_{\alpha=\alpha_i}$,$\alpha_i \in \Xi_1$,$x_j \in X$, and $A_B = L\phi(\|\alpha - x_j\|)_{\alpha=\alpha_i}$, $\alpha_i \in \Xi_2$, $x_j \in X$. The unknown coefficients λ_i are determined by solving the linear system $A\lambda = F$, where F is a vector consisting $f(\alpha_i)$, $\alpha_i \in \Xi_1$, and $g(\alpha_i)$, $\alpha_i \in \Xi_2$.

3. Application of the MQ RBF collocation method

In this section, we are interested in solving singularly perturbed differential difference equation (1.2) by the MQ RBF collocation method. For this purpose, we rewrite the equations (1.2)-(1.4) as follows

$$\varepsilon V''(x) + a(x)V'(x) - b(x)V(x) + d(x)V(x+\eta) = f(x) - c(x)\psi(x-\delta), \tag{3.1}$$
$$a_0 < x \leqslant a_0 + \delta,$$

$$\varepsilon V''(x) + a(x)V'(x) - b(x)V(x) + c(x)V(x-\delta) + d(x)V(x+\eta) = f(x), \tag{3.2}$$
$$a_0 + \delta < x < b_0 - \eta,$$

$$\varepsilon V''(x) + a(x)V'(x) - b(x)V(x) + c(x)V(x-\delta) = f(x) - d(x)\gamma(x+\eta), \tag{3.3}$$
$$b_0 - \eta \leqslant x < b_0,$$

$$V(a_0) = \psi(a_0), \tag{3.4}$$

$$V(b_0) = \gamma(b_0). \tag{3.5}$$

Then, we choose N equally spaced nodes , $i = 1, 2, \cdots, N$ in $\bar{\Omega}$ and approximate the solution of equation (1.2) by

$$V^N(x) = \sum_{i=1}^{N} \lambda_i \Phi(\|x - x_i\|) + \lambda_{N+1} + \lambda_{N+2} x. \tag{3.6}$$

Using the collocation method to ensure that the approximation satisfies in equations (3.1)-(3.5), one obtains the following linear system with unknown coefficients $\lambda_1, \lambda_2, ..., \lambda_{N+1}, \lambda_{N+2}$:

$$\sum_{j=1}^{N} \{\varepsilon \Phi''(r_{ij}) + a(x_i)\Phi'(r_{ij}) - b(x_i)\Phi(r_{ij}) + d(x_i)\Phi(\|x_i + \eta - x_j\|)\}\lambda_j +$$
$$(d(x_i) - b(x_i))\lambda_{N+1} + (a(x_i) - b(x_i)x_i + d(x_i)(x_i + \eta))\lambda_{N+2} = f(x_i) -$$
$$c(x_i)\psi(x_i - \delta), \quad a_0 < x_i \leqslant a_0 + \delta, \tag{3.7}$$

$$\sum_{j=1}^{N} \{\varepsilon \Phi''(r_{ij}) + a(x_i)\Phi'(r_{ij}) - b(x_i)\Phi(r_{ij}) + c(x_i)\Phi(\|x_i - \delta - x_j\|) +$$
$$d(x_i)\Phi\|x_i + \eta - x_j\|\}\lambda_j + (c(x_i) + d(x_i) - b(x_i))\lambda_{N+1} + (c(x_i)(x_i - \delta) +$$
$$d(x_i)(x_i + \eta) - b(x_i)x_i + a(x_i))\lambda_{N+2} = f(x_i), \quad a_0 < x_i < b_0 - \eta, \tag{3.8}$$

$$\sum_{j=1}^{N} \{\varepsilon \Phi''(r_{ij}) + a(x_i)\Phi'(r_{ij}) - b(x_i)\Phi(r_{ij}) + c(x_i)\Phi(\|x_i - \delta - x_j\|)\}\lambda_j +$$
$$(c(x_i) - b(x_i))\lambda_{N+1} + (c(x_i)(x_i - \delta) - b(x_i)x_i + a(x_i))\lambda_{N+2} = f(x_i) -$$
$$d(x_i)\gamma(x_i + \eta), \quad b_0 - \eta \leqslant x_i < b_0, \tag{3.9}$$

$$\sum_{j=1}^{N} \lambda_j \Phi(r_{1j}) + \lambda_{N+1} + \lambda_{N+2}\, a_0 = \psi(a_0), \tag{3.10}$$

$$\sum_{j=1}^{N} \lambda_j \Phi(r_{Nj}) + \lambda_{N+1} + \lambda_{N+2}\, b_0 = \gamma(b_0), \tag{3.11}$$

where $r_{ij} = \|x_i - x_j\|$. The above system has N equations and $N+2$ unknowns. To obtain the $N+2$ unknown coefficients $\lambda_1, \lambda_2, ..., \lambda_{N+1}, \lambda_{N+2}$, we force two supplementary collocation conditions at x_1 and x_2 in (3.7) and (3.9), respectively, to get a $(N+2) \times (N+2)$ system of linear equations for the λ_j . Next, this system must be solved to obtain the unknown coefficients. So, we have used the Gaussian elimination method to solve such a system. Following, we will be using an algorithm introduced by Driscoll and Heryudono [27]. The idea is to monitor the residual R to the differential equation at midpoints and iteratively refine the point set

until R drops below some User-defined threshold. The reader is referred to the original paper for details.
Remark that this provides the solution in the form of a function that can be evaluated everywhere in $[a_0, b_0]$. Not additional interpolation is required.

4. Error Estimation

Let us call $e(x) = V(x) - \tilde{V}(x)$ as the error function. Where $V(x)$ is the exact solution of (1.2)-(1.4) and $\tilde{V}(x)$ is approximation of $V(x)$. Hence,

$$\varepsilon\tilde{V}''(x) + a(x)\tilde{V}'(x) - b(x)\tilde{V}(x) + c(x)\tilde{V}(x-\delta) + d(x)\tilde{V}(x+\eta) - f(x) = R(x), \quad a_0 < x < b_0, \tag{4.1}$$

$$\tilde{V}(x) - \psi(x) = R_1(x), \quad a_0 - \delta \leqslant x \leqslant a_0, \tag{4.2}$$

$$\tilde{V}(x) - \gamma(x) = R_2(x), \quad b_0 \leqslant x \leqslant b_0 + \eta, \tag{4.3}$$

$$\varepsilon V''(x) + a(x)V'(x) - b(x)V(x) + c(x)V(x-\delta) + d(x)V(x+\eta) - f(x) = 0, \quad a_0 < x < b_0, \tag{4.4}$$

$$V(x) - \psi(x) = 0, \quad a_0 - \delta \leqslant x \leqslant a_0, \tag{4.5}$$

$$V(x) - \gamma(x) = 0, \quad b_0 \leqslant x \leqslant b_0 + \eta. \tag{4.6}$$

By subtracting eqs. (4.4)-(4.6) from eqs. (4.1)-(4.3), we have

$$\varepsilon(V''(x) - \tilde{V}''(x)) + a(x)(V'(x) - \tilde{V}'(x)) - b(x)(V(x) - \tilde{V}(x)) + c(x)(V(x-\delta) - \tilde{V}(x-\delta)) + d(x)(V(x+\eta) - \tilde{V}(x+\eta)) - f(x) = -R(x), \quad a_0 < x < b_0. \tag{4.7}$$

$$V(x) - \tilde{V}(x) = -R_1(x), \quad a_0 - \delta \leqslant x \leqslant a_0, \tag{4.8}$$

$$V(x) - \tilde{V}(x) = -R_2(x), \quad b_0 \leqslant x \leqslant b_0 + \eta. \tag{4.9}$$

Now, the error function $e(x)$ is satisfying below equation:

$$\varepsilon e''(x) + a(x)e'(x) - b(x)e(x) + c(x)e(x-\delta) + d(x)e(x+\eta) = -R(x), \ a_0 < x < b_0$$

$$e(x) = -R_1(x); \qquad a_0 - \delta \leqslant x \leqslant a_0$$

$$e(x) = -R_2(x); \qquad b_0 \leqslant x \leqslant b_0 + \eta$$

Since the function $R(x)$ is known, so to find $e(x)$, we follow a similar method introduced in section 3.

5. Numerical experiments

In this section, two examples are provided to demonstrate the accuracy and efficiency of the suggested method. The accuracy of the RBFs solution depends heavily on the choice of a parameter C in radial basis function. We use shape Parameter $c_j = 0.815d_j$ [28], for the IMQ2 basis, where d_j is the distance from j^{th} point to its nearest neighbor. We use User-defined thresholds $\theta_c = 1e-10$ and $\theta_r = 1e-6$ in the RSM algorithm for the first example and $\theta_c = 1e-8$ and $\theta_r = 1e-4$ for the next example. The computation is carried out using 16 digits precision. All codes were written in Matlab 2012a on a 2.30 MHz Alpha Machine with 4GB RAM. Since the exact solutions are not known for the considered examples, we devise double mesh principle [29] for reporting the errors as follows:

$$E_{\varepsilon}^{N} = \max_{0 \leqslant i \leqslant N} |V_N^i - V_{2N}^{2i}|.$$

where V_N^i and V_{2N}^{2i} are the numerical solutions by taking N and $2N$ points, respectively.

Example 5.1. [30]
Consider the following problem with turning point at $x = 0.5$

$$\varepsilon V''(x) + 2(x-0.5)\left[(1+0.3121)(x-0.5)\right]V'(x) - \left[\frac{4}{3} + 0.2764(x-0.5)\right]V(x) +$$
$$0.2V(x-\delta) + \frac{1}{8}V(x+\eta) = x, \qquad x \in (0,1), \tag{5.1}$$

and under interval conditions

$$V(x) = 0, \quad -\delta \leqslant x \leqslant 0, \tag{5.2}$$

$$V(x) = 0, \ \ 1 \leqslant x \leqslant 1+\eta \tag{5.3}$$

Example 5.2. [30]
In this example, we consider the following singularly perturbed differential-difference equation with turning point at $x = 0.5$

$$\varepsilon V''(x) + (x-0.5)\left[3 + 4(x-0.5)\right]V'(x) - 2V(x) + 4(x-0.5)^2 V(x-\delta) +$$
$$V(x+\eta) = 1, \qquad x \in (0,1), \tag{5.4}$$

and under interval conditions

$$V(x) = 0, \quad -\delta \leqslant x \leqslant 0, \tag{5.5}$$

$$V(x) = 0, \ \ 1 \leqslant x \leqslant 1+\eta \tag{5.6}$$

TABLE

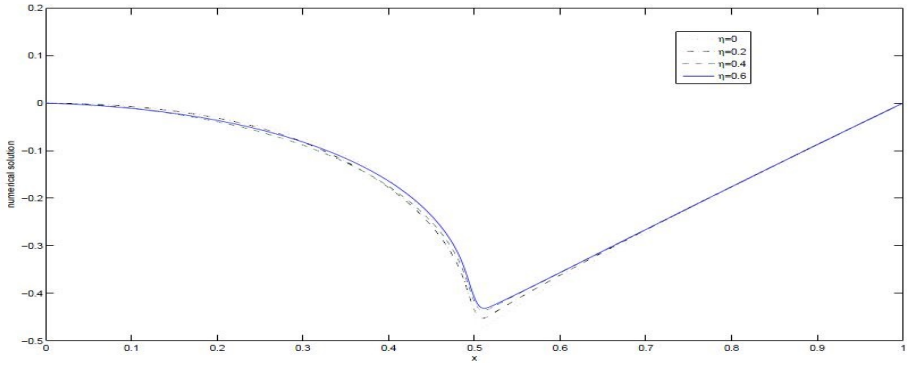

FIGURE 1. The numerical solution of example 5.1 for $\delta = 0$, $\varepsilon = 0.0001$.

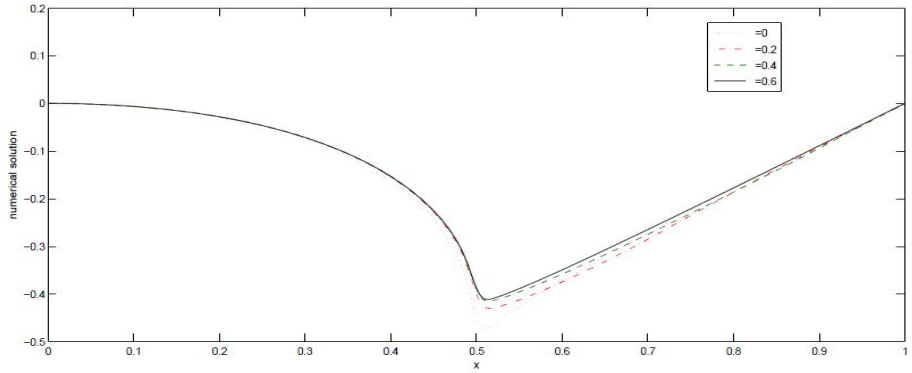

FIGURE 2. The numerical solution of example 5.1 for $\eta = 0$, $\varepsilon = 0.0001$.

TABLE 1. Maximum double mesh error and condition number $\delta = 0.4$ and$\eta = 0.2$for example 5.1

ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N	219	278	246	258	356	823
condition number	$9.6672e+7$	$7.3312e+07$	$1.4326e+08$	$1.4374e+09$	$1.6193e+10$	$8.6453e+11$
Error	$4.1029e-09$	$1.7766e-08$	$3.0649e-08$	$4.1071e-08$	$4.4940e-08$	$2.8904e-08$

TABLE 2. Maximum double mesh error and condition number $\delta = 0.4$ and$\eta = 0.6$ for example 5.2

ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N	67	62	62	116	245	489
condition number	$1.1124e+06$	$1.2569e+06$	$7.1005e+06$	$1.6042e+08$	$2.5505e+09$	$1.4634e+11$
Error	$9.2548e-07$	$2.4228e-06$	$2.4935e-06$	$1.0661e-06$	$1.1823e-06$	$3.7634e-06$

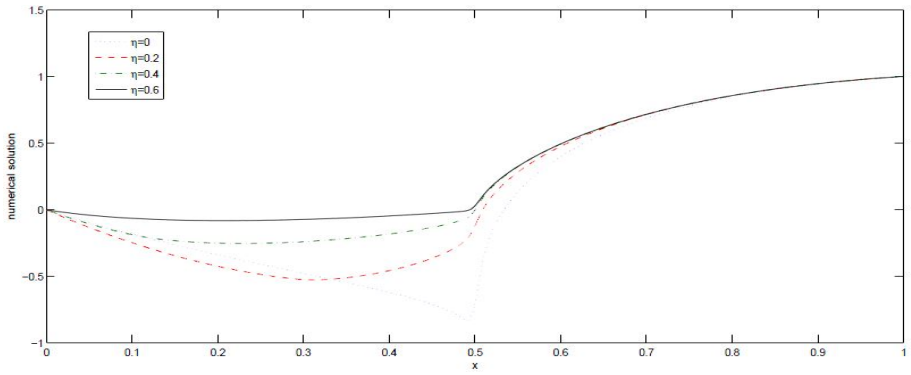

FIGURE 3. The numerical solution of example 5.2 for $\delta = 0$, $\varepsilon = 0.0001$.

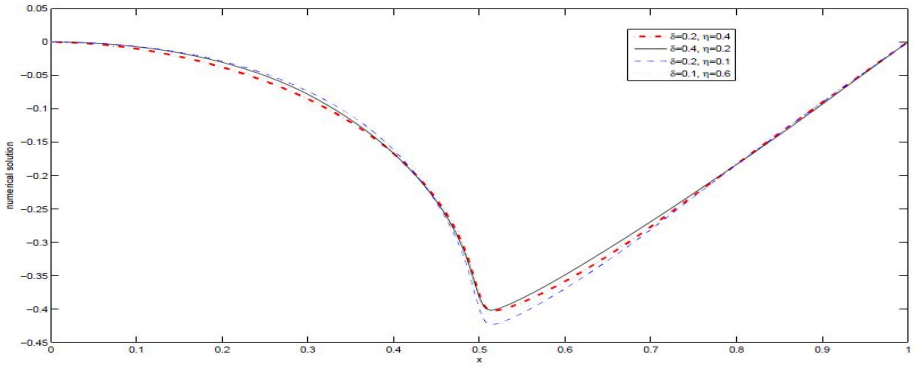

FIGURE 4. The numerical solution of example 5.1 for $\varepsilon = 0.0001$.

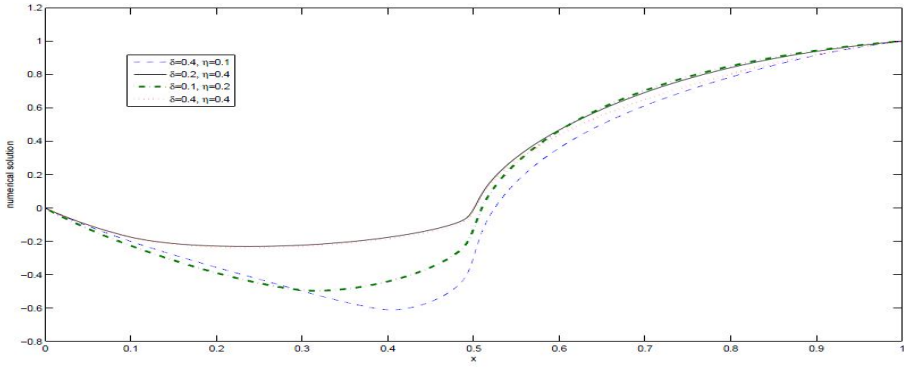

FIGURE 5. The numerical solution of example 5.2 for $\varepsilon = 0.0001$.

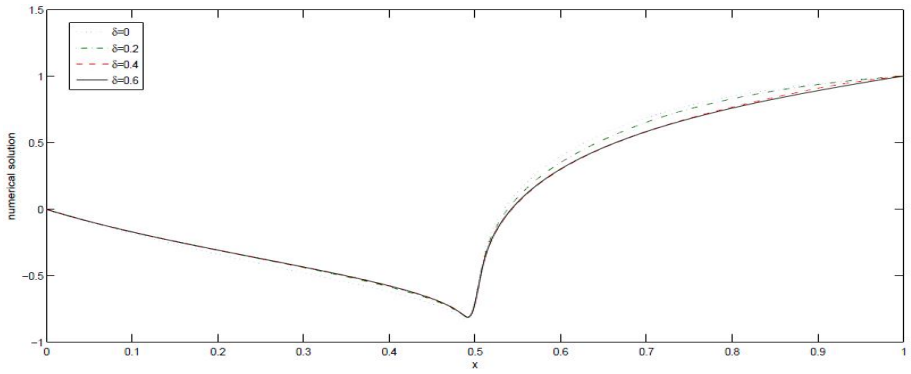

FIGURE 6. The numerical solution of example 5.2 for $\eta = 0$, $\varepsilon = 0.0001$.

TABLE 3. Maximum double mesh error and condition number $\delta = 0.4$ and$\eta = 0.4$ for example 5.1

ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N	220	304	250	246	340	812
condition number	$9.6084e+07$	$7.4826e+07$	$2.7152e+08$	$2.6427e+09$	$3.2992e+10$	$6.7549e+11$
Error	$5.2294e-09$	$909686e-09$	$2.1576e-08$	$3.5343e-08$	$3.5997e-08$	$3.6412e-07$

TABLE 4. Maximum double mesh error and condition number $\delta = 0.1$ and$\eta = 0.2$ for example 5.2

ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N	84	81	89	135	304	705
condition number	$1.3760e+06$	$1.6907e+06$	$1.2369e+07$	$1.8561e+08$	$1.1871e+10$	$2.8245e+11$
Error	$5.5738e-07$	$8.0986e-07$	$2.2758e-06$	$1.8824e-06$	$1.6348e-06$	$2.1759e-06$

TABLE 5. Maximum double mesh error and condition number $\delta = 0.2$ and$\eta = 0.1$ for example 5.1

ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N	22	274	273	231	346	779
condition number	$1.0071e+08$	$7.6722e+07$	$1.7690e+08$	$1.5609e+09$	$2.0149e+10$	$4.5486e+11$
Error	$4.7502e-09$	$2.2302e-08$	$3.6247e-08$	$4.4338e-08$	$5.0160e-08$	$3.1013e-08$

TABLE 6. Maximum double mesh error and condition number $\delta = 0.2$ and$\eta = 0.1$ for example 5.2

ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N	81	83	89	142	287	670
condition number	$1.3303e+06$	$1.5048e+06$	$7.4858e+06$	$1.1838e+08$	$7.2471e+09$	$2.8346e+11$
Error	$7.6704e-07$	$1.6981e-06$	$2.0527e-06$	$1.1613e-06$	$1.6922e-06$	$3.8664e-06$

6. Discussion

In this paper, we present the MQ RBF collocation method combined with the Residual Subsampling algorithm by Driscoll Heryudono for node adaptivity for solving singularly perturbed differential–difference equations including positive as well as negative shifts in the reaction term. Two numerical examples have been carried out. We start by N=15 equidistance node $\theta_c = 1e-10$ and $\theta_r = 1e-6$ for first example,$\theta_c = 1e-8$ and $\theta_r = 1e-4$ for next example and Shape parameter $c_j = 0.815d_j$, where d_j is the distance from j^{th} point to its nearest neighbor. The shape parameters are adjusted after each iteration to prevent the condition number for different values. Numerical results show that the present method has high accuracy for different values of ε, δ and η. Moreover, the implementation of the method is straightforward to code. The effect of shift on the solution of the problem has been investigated by drawing the diagram of the numerical solution for different values of shifts. Figures 1 and 3 show the numerical solutions for examples 5.1 and 5.2, respectively, when the problem has only a positive shift. Figures 2 and 4 also represent the numerical solutions for examples 5.1 and 5.2, respectively, when the problem has only a negative shift. The effect of mixed type of shift on the solution has been demonstrated in figures 5 and 6. As the diagrams show, the interior layer shifts in accordance with the changes in advance and delay. Double mesh error and condition number of the linear system obtained from the collocation method are tabulated in tables 1-6. We recognize from the Tables that condition number for all cases is less than 10^{16}, hence the roundoff error of the 16-digits computation is not the cause for the deterioration of solution.

7. Competing interests

The authors declare that they have no competing interests.

8. Authors' contributions

FAG, GSN, AS performed the research and all three wrote the paper. All authors read and approved the final manuscript.

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