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F-Sums of Graphs and their Reformulated-Zagreb Indices

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Abstract. The reformulated Zagreb index $EM_1(G)$ of a simple graph G is defined as the sum of the terms $(d_u + d_v - 2)^2$ over all edges uv of G. In this paper, we study the reformulated Zagreb indices for the F-sums of some special well-known graphs such as subdivision and total graph which is introduced by Eliasi and Taeri [4].

Keywords: Reformulated Zagreb index, subdivision, total graph.

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1. Introduction

All the graphs considered in this paper are connected and simple. For vertex $u \in V(G)$, the degree of the vertex u in G, denoted by $d_G(u)$, is the number of edges incident to u in G. A *topological index* of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular

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structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [6]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index. Two of these topological indices are known under various names, the most commonly used ones are the first and second Zagreb indices.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestić [8]. They are defined as $M_1(G) = \sum\limits_{u\in V(G)} d_G(u)^2$ and $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. Note that the first Zagreb index may also written as $M_1(G) = \sum\limits_{uv\in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [1].

Furtula and Gutman in [5] recently investigated this index and named this index as *forgotten topological index* or F-index and showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation coefficients greater than 0.95. The F-index of a graph G is defined as $F(G) = \sum\limits_{u\in V(G)} d_G^3(u)$.

Recently, Shirdel et al.[11] introduced a variant of the first Zagreb index called hyper-Zagreb index. The *hyper-Zagreb index* of G is denoted by $HM(G)$ and defined as $HM(G) = \sum\limits_{uv\in E(G)} (d(u)+d(v))^2$. Milicević et al. [10] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees $EM_1(G) = \sum\limits_{e\in E(G)} d(e)^2$, where $d(e)$ denotes the degree of the edge e in G, which is defined by $d(e) = d(u) + d(v) - 2$ with $e = uv$. The use of these descriptors in QSPR study was also discussed in their report [10]. Reformulated Zagreb index, particularly its upper/lower bounds has attracted recently the attention of many mathematicians and computer scientists, see [3, 9, 10, 12, 13]. In this paper, we study the reformulated Zagreb indices for the F-sums of some special well-known graphs such as subdivision and total graph

2. Preliminaries

For a connected graph G, there are four related graphs as follows:
(i) The subdivision graph $S(G)$ is the graph obtained from G by replacing each edge of G by a path of length two.

$(ii) R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.
(iii) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.
(iv) The total graph $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The *Cartesian product,* $G \square H$, of the graphs G and H has the vertex set $V(G \square H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \square H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \square H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \square H$. Eliasi and Taeri [4] introduced the following four operations of the graphs G_1 and G_2 based on the Cartesian product of these graphs.

Let F be one of the symbols S, R, Q or T. The F*-sum* $G +_F H$ is a graph with the set of vertices $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) of $G +_F H$ are adjacent if and only if $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in F(G)$. The Zagreb indices of the F-sum of graphs are obtained by Deng et al. [2]. The F-index of four operations on some special graphs are computed by Ghobadi and Ghorbaninejad [7]. Eliasi and Taeri[4] have obtained the Wiener index of four new sums of graphs.

3. Reformulated Zagreb indices of F-sums of graphs

First we compute the reformulated Zagreb index of the graph $G_1 +_S G_2$.

Theorem 3.1. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(G_1 +_S G_2) = n_1 HM(G_2) + n_2 F(G_1) + 8m_2 M_1(G_1) + (10m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 16m_1$.

Proof. Let $\{x_1, x_2, \ldots, x_{n_1}\}$ and $\{y_1, y_2, \ldots, y_{n_2}\}$ be the vertex sets of G_1 and G_2, respectively. From the definition of reformulated Zagreb index and the structure of the graph $G_1 +_S G_2$, we have

$$\begin{aligned}
& EM_1(G_1 +_S G_2) \\
= & \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1+_S G_2)} \Big(d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_2,y_2)) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_1,y_2)) - 2\Big)^2 \\
& + \sum_{x_1x_2\in E(S(G_1))} \sum_{y_1\in V(G_2)} \Big(d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_2,y_1)) - 2\Big)^2 \\
= & A_1 + A_2, \qquad (3.1)
\end{aligned}$$

where A_1 and A_2 are the sums of the terms, in order.

We shall calculate A_1 and A_2 of (3.1) separately.

First we calculate the sum $A_1 = \sum\limits_{x_1\in V(G_1)} \sum\limits_{y_1y_2\in E(G_2)} \Big(d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_1,y_2)) - 2\Big)^2$. For each vertex (x_i, y_j) in $G_1 +_S G_2$, the degree of (x_i, y_j) is $d_{G_1}(x_i) + d_{G_2}(y_j)$. Thus

$$\begin{aligned}
& A_1 = \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1}(x_1) + d_{G_2}(y_1) + d_{G_1}(x_1) + d_{G_2}(y_2) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(2d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(4d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 \\
& +4d_{G_1}(x_1)(d_{G_2}(y_1) + d_{G_2}(y_2)) - 8d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2)) + 4\Big).
\end{aligned}$$

From the definitions of first and hyper-Zagreb indices, we obtain:

$A_1 = 4m_2M_1(G_1) + n_1HM(G_2) + (8m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 16m_1.$

Next we find the value of the sum A_2.

$$\begin{aligned}
A_2 = & \sum_{x_1x_2\in E(S(G_1))}\sum_{y_1\in V(G_2)}\Big(d_{G_1+_S G_2}((x_1,y_1))+d_{G_1+_S G_2}((x_2,y_1))-2\Big)^2 \\
= & \sum_{y_1\in V(G_2)}\sum_{x_1x_2\in E(S(G_1))}\Big(d_{S(G_1)}(x_1)+d_{G_2}(y_1)+d_{S(G_1)}(x_2)-2\Big)^2 \\
= & \sum_{y_1\in V(G_2)}\sum_{x_1x_2\in E(S(G_1))}\Big((d_{S(G_1)}(x_1)+d_{S(G_1)}(x_2))^2+(d_{G_2}(y_1)-2)^2 \\
& +2(d_{S(G_1)}(x_1)+d_{S(G_1)}(x_2))(d_{G_2}(y_1)-2)\Big) \\
= & n_2HM(S(G_1))+2m_1(M_1(G_2)+4n_2-8m_2)+2(2m_2-2n_2)M_1(S(G_1)).
\end{aligned} \tag{3.2}$$

Using $M_1(S(G_1)) = M_1(G_1) + 4m_1$ and $HM(S(G_1)) = 4M_1(G_1) + F(G_1) + 8m_1$ in (3.2), we obtain

$$A_2 = 4m_2M_1(G_1)+n_2F(G_1)+2m_1M_1(G_2).$$

Adding A_1 and A_2, we obtain the required result.

Next we obtain the reformulated Zagreb index of the graph $G_1 +_R G_2$.

Theorem 3.2. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(G_1 +_R G_2) = 4n_2HM(G_1)+n_1HM(G_2)+4n_2F(G_1)+(40m_2-8n_2)M_1(G_1)+(22m_1-4n_1)M_1(G_2)+4m_1n_2-48m_1m_2$.

Proof. By the definition of reformulated Zagreb index and the structure of $G_1 +_R G_2$,

$$\begin{aligned}
& EM_1(G_1+_R G_2) \\
= & \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1+_R G_2)}\Big(d_{G_1+_R G_2}((x_1,y_1))+d_{G_1+_R G_2}((x_2,y_2))-2\Big)^2 \\
= & \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big(d_{G_1+_R G_2}((x_1,y_1))+d_{G_1+_R G_2}((x_1,y_2))-2\Big)^2 \\
& +\sum_{x_1x_2\in E(R(G_1))}\sum_{y_1\in V(G_2)}\Big(d_{G_1+_R G_2}((x_1,y_1))+d_{G_1+_R G_2}((x_2,y_1))-2\Big)^2 \\
= & A_1+A_2,
\end{aligned} \tag{3.3}$$

where A_1 and A_2 are the sums of the terms, in order.

We shall obtain the value of A_1 and A_2 of (3.3) separately.

$$\begin{aligned} A_1 &= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big(d_{G_1+_RG_2}((x_1,y_1))+d_{G_1+_RG_2}((x_1,y_2))-2\Big)^2 \\ &= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big(d_{R(G_1)}(x_1)+d_{G_2}(y_1)+d_{R(G_1)}(x_1)+d_{G_2}(y_2)-2\Big)^2 \\ &= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big(4d_{G_1}(x_1)+(d_{G_2}(y_1)+d_{G_2}(y_2))-2\Big)^2 \\ &= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big((4d_{G_1}(x_1)-2)^2+(d_{G_2}(y_1)+d_{G_2}(y_2))^2 \\ &\quad +2(4d_{G_1}(x_1)-2)(d_{G_2}(y_1)+d_{G_2}(y_2))\Big) \end{aligned}$$

From the definitions of first Zagreb index and hyper-Zagreb index, we have

$$A_1 = 16m_2M_1(G_1)+n_1HM(G_2)+(16m_1-4n_1)M_1(G_2)+4n_1m_2-32m_1m_2.$$

Next we obtain the sum A_2.

$$\begin{aligned} A_2 &= \sum_{x_1x_2\in E(R(G_1))}\sum_{y_1\in V(G_2)}\Big(d_{G_1+_RG_2}((x_1,y_1))+d_{G_1+_RG_2}((x_2,y_1))-2\Big)^2 \\ &= \sum_{y_1\in V(G_2)}\sum_{x_1x_2\in V(G_1)}\Big(d((x_1,y_1))+d((x_2,y_1))-2\Big)^2 \\ &\quad + \sum_{y_1\in V(G_2)}\sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}}\Big(d(x_1,y_1)+d(x_2,y_1)-2\Big)^2 \\ &= A_2'+A_2'', \end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
A_2' &= \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(d(x_1,y_1)+d(x_2,y_1))-2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(d_{R(G_1)}(x_1)+d_{G_2}(y_1)+d_{R(G_1)}(x_2)-2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(2d_{G_1}(x_1)+2d_{G_1}(x_2)+2d_{G_2}(y_1)-2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(4(d_{G_1}(x_1)+d_{G_1}(x_2))^2+(2d_{G_2}(y_1)-2)^2 \\
&\quad +4(d_{G_1}(x_1)+d_{G_1}(x_2))(2d_{G_2}(y_1)-2)\Big) \\
&= 4n_2HM(G_1)+4m_1M_1(G_2)+(16m_2-8n_2)M_1(G_1)+4m_1n_2-16m_1m_2.
\end{aligned}
$$

and

$$
\begin{aligned}
A_2'' &= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}} \Big(d(x_1,y_1)+d(x_2,y_1)-2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}} \Big(d_{R(G_1)}(x_1)+d_{G_2}(y_1)+d_{R(G_1)}(x_2)-2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}} \Big(2d_{G_1}(x_1)+d_{G_2}(y_1)\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{x_1\in V(G_1)} d_{G_1}(x_1)\Big(4(d_{G_1}(x_1))^2+(d_{G_2}(y_1))^2+4d_{G_1}(x_1)d_{G_2}(y_1)\Big) \\
&= 4n_2F(G_1)+2m_1M_1(G_2)+8m_2M_1(G_1).
\end{aligned}
$$

From A_2' and A_2'', we have

$A_2 = 4n_2HM(G_1)+4n_2F(G_1)+6m_1M_1(G_2)+(24m_2-8n_2)M_1(G_1)+4m_1n_2-16m_1m_2$.

Using (3.3) and the sums A_1,A_2, we obtain the desired result.

Next we find the reformulated Zagreb index of $G_1 +_Q G_2$.

Theorem 3.3. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(G_1 +_Q G_2) = 2n_2HM(G_1)+n_2HM(L(G_1))+n_1HM(G_2)+7n_2F(G_1)+(16m_2-26n_2)M_1(G_1)+(12m_1-4n_1)M_1(G_2)+12n_2M_2(G_1)4n_1m_2+20n_2m_1-32m_1m_2$.

Proof. By the definition of reformulated Zagreb index,

$$
\begin{aligned}
& EM_1(G_1 +_Q G_2) \\
= & \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1+_Q G_2)} \Big(d_{G_1+_Q G_2}((x_1,y_1)) + d_{G_1+_Q G_2}((x_2,y_2)) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1+_Q G_2}((x_1,y_1)) + d_{G_1+_Q G_2}((x_1,y_2)) - 2\Big)^2 \\
& + \sum_{x_1x_2\in E(Q(G_1))} \sum_{y_1\in V(G_2)} \Big(d_{G_1+_Q G_2}((x_1,y_1)) + d_{G_1+_Q G_2}((x_2,y_1)) - 2\Big)^2 \\
= & A_1 + A_2, \qquad (3.5)
\end{aligned}
$$

where A_1 and A_2 are the sums of the terms, in order.

We shall calculate A_1 and A_2 of 3.5 separately.

$$
\begin{aligned}
& A_1 = \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1+_Q G_2}((x_1,y_1)) + d_{G_1+_Q G_2}((x_1,y_2)) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{Q(G_1)}(x_1) + d_{G_2}(y_1) + d_{Q(G_1)}(x_1) + d_{G_2}(y_2) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(2d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big((4d_{G_1}(x_1)^2 - 8d_{G_1}(x_1) + 4) + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 \\
& +2(2d_{G_1}(x_1) - 2)(d_{G_2}(y_1) + d_{G_2}(y_2))\Big) \\
= & m_2(4M_1(G_1) - 16m_1 + 4n_1) + n_1HM(G_2) + (8m_1 - 4n_1)M_1(G_2) \\
= & n_1HM(G_2) + 4m_2M_1(G_1) + (8m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 16m_1m_2.
\end{aligned}
$$

$$
\begin{aligned}
& A_2 = \sum_{x_1x_2\in E(Q(G_1))} \sum_{y_1\in V(G_2)} \Big(d_{G_1+_Q G_2}((x_1,y_1)) + d_{G_1+_Q G_2}((x_2,y_1)) - 2\Big)^2 \\
= & \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1)) \\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d(x_1,y_1) + d(x_2,y_1)) - 2\Big)^2 \\
& + \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1)) \\ x_1,\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d(x_1,y_1) + d(x_2,y_1) - 2\Big)^2 \\
= & A_2' + A_2'',
\end{aligned}
$$

where

$$\begin{aligned}
A_2' &= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d((x_1,y_1)) + d((x_2,y_1)) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d_{Q(G_1)}(x_1) + d_{G_2}(y_1) + d_{Q(G_1)}(x_2) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d_{G_1}(x_1) + d_{G_2}(y_1) + d_{Q(G_1)}(x_2) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d_{G_1}(x_1)^2 + (d_{G_2}(y_1)-2)^2 + d_{Q(G_1)}(x_2)^2 \\
&\quad +2d_{G_1}(x_1)(d_{G_2}(y_1)-2) + 2d_{G_1}(x_1)d_{Q(G_1)}(x_2) + 2(d_{G_2}(y_1)-2)d_{Q(G_1)}(x_2)\Big) \\
&= \sum_{y_1\in V(G_2)} \sum_{x_1\in V(G_1)} d_{G_1}(x_1)\Big(d_{G_1}(x_1)^2 + +(d_{G_2}(y_1)-2)^2 + 2d_{G_1}(x_1)(d_{G_2}(y_1)-2)\Big) \\
&\quad + \sum_{y_1\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d_{Q(G_1)}(x_2)^2 + 2d_{G_1}(x_1)d_{Q(G_1)}(x_2) \\
&\quad +2(d_{G_2}(y_1)-2)d_{Q(G_1)}(x_2)\Big) \\
&= n_2F(G_1) + 2m_1(M_1(G_2) + 4n_2 - 8m_2) + 2(2m_2 - 2n_2)M_1(G_1) \\
&\quad + \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2 d_{Q(G_1)}(x_2)^2 + (2n_2 d_{G_1}(x_1) + 2(2m_2 - 2n_2))d_{Q(G_1)}(x_2)\Big).
\end{aligned}$$

One can see that for a vertex $x_2 \in V(Q(G_1)) - V(G_1)$, $d_{Q(G_1)}(x_2) = d_{G_1}(x) + d_{G_1}(w)$, where $x_2 = xw \in E(G_1)$. Thus

$$\begin{aligned}
A_2' &= n_2F(G_1) + 2m_1(M_1(G_2) + 4n_2 - 8m_2) + 2(2m_2 - 2n_2)M_1(G_1) \\
&\quad + \sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2(d_{G_1}(x) + d_{G_1}(w))^2 \\
&\quad +(2n_2 d_{G_1}(x_1) + 2(2m_2 - 2n_2))(d_{G_1}(x) + d_{G_1}(w))\Big) \\
&= n_2F(G_1) + 2m_1(M_1(G_2) + 4n_2 - 8m_2) + 2(2m_2 - 2n_2)M_1(G_1) \\
&\quad +2n_2HM(G_1) + 2n_2(F(G_1) + 2M_2(G_1)) + 4(2m_2 - 2n_2)M_1(G_1) \\
&= 2n_2HM(G_1) + 3n_2F(G_1) + (12m_2 - 12n_2)M_1(G_1) \\
&\quad +2m_1M_1(G_2) + 4n_2M_2(G_1) + 8m_1n_2 - 16m_1m_2.
\end{aligned}$$

$$
\begin{aligned}
A_2'' &= \sum_{y_1 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(Q(G_1)) \\ x_1,\, x_2 \in V(Q(G_1)) - V(G_1)}} \Big(d(x_1,y_1) + d(x_2,y_1) - 2\Big)^2 \\
&= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(Q(G_1)) \\ x_1,\, x_2 \in V(Q(G_1)) - V(G_1)}} \Big(d_{Q(G_1)}(x_1) + d_{Q(G_1)}(x_2) - 2\Big)^2 \\
&= n_2 \sum_{w_iw_j, w_jw_k \in E(G_1)} \Big(d_{G_1}(w_i) + d_{G_1}(w_j) + d_{G_1}(w_j) + d_{G_1}(w_k) - 2\Big)^2 \\
&= n_2 \sum_{X_iX_j \in E(L(G_1))} \Big(d_{L(G_1)}(X_i) + d_{L(G_1)}(X_j) + 2\Big)^2,
\end{aligned}
$$

where X_i and X_j are vertices of the line graph $L(G_1)$

$$
\begin{aligned}
&= n_2 \sum_{X_iX_j \in E(L(G_1))} \Big((d_{L(G_1)}(X_i) + d_{L(G_1)}(X_j))^2 + 4 + 4(d_{L(G_1)}(X_i) + d_{L(G_1)}(X_j))\Big) \\
&= n_2\Big(HM(L(G_1)) + 4n_2M_1(L(G_1)) + 4\,|E(L(G_1))|\,\Big).
\end{aligned}
$$

Since $|E(L(G_1))| = \frac{M_1(G_1)}{2} - m_1$ and $M_1(L(G_1)) = F(G_1) - 4M_1(G_1) + 2M_2(G_1) + 4m_1$. Thus $A_2'' = n_2HM(L(G_1)) + 4n_2F(G_1) + 8n_2M_2(G_1) - 14n_2M_1(G_1) + 12n_2m_1$.

From the sums A_2' and A_2'', we have $A_2 = 2n_2HM(G_1) + n_2HM(L(G_1)) + 7n_2F(G_1) + (12m_2 - 26n_2)M_1(G_1) + 2m_1M_1(G_2) + 12n_2M_2(G_1) + 20n_2m_1 - 16m_1m_2$.

Adding A_1 and A_2, we get the desired result.

Finally, we obtain the reformulated Zagreb index of $G_1 +_T G_2$.

Theorem 3.4. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then $EM_1(G_1 +_T G_2) = 6n_2HM(G_1) + n_2HM(L(G_1)) + n_1HM(G_2) + 12n_2F(G_1) + 16n_2M_2(G_1) + (48m_2 - 38n_2)M_1(G_1) + (22m_1 - 4n_1)M_1(G_2) + 24m_1n_2 + 4n_1m_2 - 64m_1m_2$.*

Proof. By the definition of reformulated Zagreb index,

$$
\begin{aligned}
& EM_1(G_1 +_T G_2) \\
&= \sum_{(x_1,y_1)(x_2,y_2) \in E(G_1 +_T G_2)} \Big(d_{G_1 +_T G_2}((x_1,y_1)) + d_{G_1 +_T G_2}((x_2,y_2)) - 2\Big)^2 \\
&= \sum_{x_1 \in V(G_1)} \sum_{y_1y_2 \in E(G_2)} \Big(d_{G_1 +_T G_2}((x_1,y_1)) + d_{G_1 +_T G_2}((x_1,y_2)) - 2\Big)^2 \\
&\quad + \sum_{x_1x_2 \in E(T(G_1))} \sum_{y_1 \in V(G_2)} \Big(d_{G_1 +_T G_2}((x_1,y_1)) + d_{G_1 +_T G_2}((x_2,y_1)) - 2\Big)^2.
\end{aligned}
$$

Note that $E(T(G_1)) = E(G_1) \cup E(S(G_1)) \cup E(L(G_1))$. Thus

$$
\begin{aligned}
& EM_1(G_1 +_T G_2) \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1+_T G_2}((x_1,y_1)) + d_{G_1+_T G_2}((x_2,y_2)) - 2\Big)^2 \\
& + \sum_{y_1\in V(G_2)} \Big(\sum_{\substack{x_1x_2\in E(T(G_1)) \\ x_1,x_2\in V(G_1)}} + \sum_{\substack{x_1x_2\in E(T(G_1)) \\ x_1\in V(G_1),\, x_2\in V(T(G_1))-V(G_1)}} \\
& + \sum_{\substack{x_1x_2\in E(T(G_1)) \\ x_1,\, x_2\in V(T(G_1))-V(G_1)}} \Big)\Big(d_{G_1+_T G_2}((x_1,y_1)) + d_{G_1+_T G_2}((x_2,y_1)) - 2\Big)^2 \\
= & A_1 + A_2 + A_3 + A_4, \qquad (3.6)
\end{aligned}
$$

where A_1 to A_4 are the sums of the terms, in order.

We shall calculate A_1 to A_4 of 3.6 separately. A similar arguments of A_1 and A_2' in Theorem 3.2, we have

$$
\begin{aligned}
& A_1 = \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1+_T G_2}((x_1,y_1)) + d_{G_1+_T G_2}((x_1,y_2)) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(2d_{T(G_1)}(x_1) + d_{G_2}(y_1) + d_{G_2}(y_2) - 2\Big)^2 \\
= & \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(4d_{G_1}(x_1) + d_{G_2}(y_1) + d_{G_2}(y_2) - 2\Big)^2 \\
= & 16m_2M_1(G_1) + n_1HM(G_2) + (16m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 32m_1m_2.
\end{aligned}
$$

and

$$
\begin{aligned}
& A_2 = \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in E(G_1)} \Big(d_{G_1+_T G_2}((x_1,y_1)) + d_{G_1+_T G_2}((x_2,y_1) - 2)\Big)^2 \\
= & \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in E(G_1)} \Big(d_{T(G_1)}(x_1) + 2d_{G_2}(y_1) + d_{T(G_1)}(x_2) - 2)\Big)^2 \\
= & \sum_{y_1\in V(G_2)} \sum_{x_1x_2\in E(G_1)} \Big(2d_{G_1}(x_1) + 2d_{G_2}(y_1) + 2d_{G_1}(x_2) - 2)\Big)^2 \\
= & 4n_2HM(G_1) + 4m_1M_1(G_2) + (16m_2 - 8n_2)M_1(G_1) + 4m_1n_2 - 16m_1m_2.
\end{aligned}
$$

$$
\begin{aligned}
A_3 = & \sum_{y_1 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(T(G_1)) \\ x_1 \in V(G_1),\, x_2 \in V(T(G_1)) - V(G_1)}} \Big(d_{G_1 +_T G_2}((x_1, y_1)) + d_{G_1 +_T G_2}((x_2, y_1)) - 2 \Big)^2 \\
= & \sum_{y_1 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(T(G_1)) \\ x_1 \in V(G_1),\, x_2 \in V(T(G_1)) - V(G_1)}} \Big(d_{T(G_1)}(x_1) + d_{G_2}(y_1) + d_{T(G_1)}(x_2) - 2 \Big)^2 \\
= & \sum_{y_1 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(T(G_1)) \\ x_1 \in V(G_1),\, x_2 \in V(T(G_1)) - V(G_1)}} \Big(4d_{G_1}(x_1)^2 + (d_{G_2}(y_1) - 2)^2 + d_{T(G_1)}(x_2)^2 \\
& + 4d_{G_1}(x_1)(d_{G_2}(y_1) - 2) + 2(d_{G_2}(y_1) - 2)d_{T(G_1)}(x_2) + 4d_{G_1}(x_1)d_{T(G_1)}(x_2) \Big) \\
= & \sum_{y_1 \in V(G_2)} \sum_{x_1 \in V(G_1)} d_{G_1}(x_1) \Big(4d_{G_1}(x_1)^2 + (d_{G_2}(y_1) - 2)^2 + 4d_{G_1}(x_1)(d_{G_2}(y_1) - 2) \Big) \\
& + \sum_{y_1 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(T(G_1)) \\ x_1 \in V(G_1),\, x_2 \in V(T(G_1)) - V(G_1)}} \Big(d_{T(G_1)}(x_2)^2 + 2(d_{G_2}(y_1) - 2)d_{T(G_1)}(x_2) \\
& + 4d_{G_1}(x_1)d_{T(G_1)}(x_2) \Big) \\
= & \; 4n_2F(G_1) + 2m_1(M_1(G_2) + 4n_2 - 8m_2) + 4M_1(G_1)(2m_2 - 2n_2) \\
& + \sum_{\substack{x_1x_2 \in E(T(G_1)) \\ x_1 \in V(G_1),\, x_2 \in V(T(G_1)) - V(G_1)}} \Big(n_2(d_{G_1}(u) + d_{G_1}(v))^2 \\
& + 2(d_{G_2}(y_1) - 2)(d_{G_1}(u) + d_{G_1}(v)) + 4d_{G_1}(x_1)(d_{G_1}(u) + d_{G_1}(v)) \Big) \\
= & \; 2n_2HM(G_1) + 8n_2F(G_1) + 8n_2M_2(G_1) + (16m_2 - 16n_2)M_1(G_1) + 2m_1M_1(G_2) \\
& + 8m_1n_2 - 16m_1m_2.
\end{aligned}
$$

A similar argument of A_2'' in Theorem 3.3 , we obtain

$$
\begin{aligned}
A_4 = & \sum_{y_1 \in V(G_2)} \sum_{\substack{x_1x_2 \in E(Q(G_1)) \\ x_1, x_2 \in V(Q(G_1)) - V(G_1)}} \Big(d_{G_1 +_T G_2}((x_1, y_1)) + d_{G_1 +_T G_2}((x_2, y_2)) \Big)^2 \\
= & \; n_2HM(L(G_1)) + 4n_2F(G_1) + 8n_2M_2(G_1) - 14n_2M_1(G_1) + 12n_2m_1.
\end{aligned}
$$

Adding the sums A_1 to A_4, we get the desired result.

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