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The converse of Baer's theorem for two-nilpotent variety

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Abstract. In this paper the generalization of the converse of Baer's theorem for two-nilpotent variety of class row (n, m) is carried out. Baer proved that finiteness of $G/Z_n(G)$ implies that $\gamma_{n+1}(G)$ is finite. Hekster proved the converse of the Baer's theorem with the assumption that G can be finitely generated. The Baer's theorem can be considered as a result of a classical theorem by Schur denoting that finiteness of $G/Z(G)$ leads to the finiteness of G'. The converse of the Baer's theorem has been proved conditionally by Taghavi et al. (2019), as well. In the Main Theorem, we prove that, if $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$ and $\gamma_{m,n+i}(G)$ is finite for some $n, i, m \geq 0$. Then $G/Z_{n,m}(G)$ is finite in which $\gamma_{m,n}(G)$ and $Z_{n,m}(G)$ denote verbal and marginal subgroups with respect to two-nilpotent variety of class row (n, m). Thus the generalization of the converse of Baer's theorem for two-nilpotent variety of groups valids by considering $i = 0$. In this article some other results are attained by the converse of the Baer's theorem. It is also concluded that when $n = m = 1$. Similar results are obtained for variety of the soluble groups. In addition, the converse of the Schur's theorem which proved by Halasi and Podoski is concluded in this paper, for two-nilpotent variety. We have also obtained some similar results of Chakaneh et al. (2019) for (n, m)-isoclinic family of groups and $(1, m)$-stem groups.

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1. Introduction

Let G be an arbitrary group. In 1904, Schur proved that if the center of a group G has finite index, then the derived subgroup of G is finite [10]. In 1952, Baer [1], extended this theorem and showed that if $G/Z_n(G)$ has finite order for a group G, then $\gamma_{n+1}(G)$ is also finite, where $Z_n(G)$ and $\gamma_{n+1}(G)$ are the $(n+1)$-th term of the upper central series and the $(n+1)$-th term of the lower central series of G, respectively. Infinite extra special p-groups show that the converse of Schur's theorem is not true in general. In 1956, Hall [5] proved a partial converse of the Baer's theorem. He gave a bound for $G/Z_{2n}(G)$ in terms of the order of $\gamma_{n+1}(G)$. A result of Hekster [8], 1986, shows that if G is a finitely generated group such that $\gamma_{n+1}(G)$ is finite, then $G/Z_n(G)$ is finite. In 2014, Hatamian et al. [6] generalized this result by obtaining the same conclusion under the weaker hypothesis denoting that $Z_{2n}(G)/Z_n(G)$ is finitely generated. In 2019, Y. Taghavi et al. [11] proved the converse of Baer's theorem as follows: If G is a group such that for some $n, i \geq 1$, $\gamma_{n+1}(G) \cap Z_n(G) = 1$ and $\gamma_{n+i}(G)$ is finite, then $G/Z_n(G)$ is finite. Chakaneh et al. [3] showed that for a group G with $\Phi(G) \cap Z(G) = 1$, the finiteness of $\gamma_{n+1}(G)$ implies the finiteness of $G/Z(G)$. Also, by this assumption, they proved that the existence of the isomorphism between the center factors of two groups sufficed for those groups to be isoclinic.

In our present paper, first we state some properties of two-nilpotent variety. The Main Theorem of the paper is: Let G be a group such that $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$ and $\gamma_{m,n+i}(G)$ be finite for some $n, i, m \geq 0$. Then $G/Z_{n,m}(G)$ is finite. Thus, the generalization of the converse of Baer's theorem for two-nilpotent variety of groups of class row (n, m) holds for these groups. Finally, we will express some of its applications for (n, m)-isoclinic family of groups and $(1, m)$-stem groups.

2. NOTATION AND RELATIONS

First, we study the relation between the k-th center and marginal subgroup in two-nilpotent variety.

Let F_∞ be a free group on a countably infinite set $\{x_1, x_2, \ldots\}$ and V be an arbitrary subset of F. Suppose that $v = x_{l_1}^{i_1} \ldots x_{l_r}^{i_r} \in V$ where $i_j = \pm 1$, $1 \leq j \leq r$ and $y_1, \ldots, y_t$ be distinct elements in $x_{l_1}, \ldots, x_{l_r}$.

Consider t arbitrary distinct elements $g_1, \ldots, g_t \in G$. By uniformly replacement of g_i in x_{i_j}, some elements of G are obtained that is said to be *the value of the word v at* $(g_1, \ldots, g_t)$. The subgroup of G generated by all values of G of words in V is called the *verbal* subgroup of G determined by $V(G)$,

$$V(G) = \langle v(g_1, g_2, \ldots, g_n) | g_i \in G, 1 \leq i \leq n, n \in \mathbb{N}, v \in V\rangle.$$

The *marginal* subgroup of G with respect to V denoted by $V^*(G)$ is defined by

$$V^*(G) = \{a \in G | v(g_1, \ldots, g_{i-1}, g_i a, g_{i+1}, \ldots, g_k) = v(g_1, \ldots, g_k); \\ v \in V, g_i \in G, 1 \leq i \leq k, k \in \mathbb{N}\}.$$

Verbal and marginal subgroups are fully-invariant and characteristic subgroups, respectively. Also the class of groups with respect to V, i.e. $\mathcal{V} = \{G | V(G) = 1\}$, is called the *variety* of groups defined by the set V, (see [8] for more details).

The following lemma will be useful for the rest of this paper. For more information see [8].

Lemma 2.1. *(*[8]*, Proposition 2.3) Let G be a group with normal subgroup N and $\mathcal{V}$ be a variety. If $N \cap V(G) = 1$, then $N \subseteq V^*(G)$ and $V^*(G/N) = V^*(G)/N$.*

In particular, if $V = \{[x, y]\}$, in which $[x, y] = x^{-1}y^{-1}xy$ then $V(G) = G'$, $V^*(G) = Z(G)$ and $\mathcal{V} = \mathcal{A}_b$ is the variety of abelian groups.

In special case, for any $c \in \mathbb{N}$, when $V = \{[x_1, \ldots, x_{c+1}]\}$ then $V(G) = \gamma_{c+1}(G)$, $V^*(G) = Z_c(G)$ and $\mathcal{V} = \mathcal{N}_c$ is the variety of nilpotent groups of the class at most c.

The structure of verbal and marginal subgroups of a group with respect to the two-nilpotent variety of the class row (m, n) are described as follows.

Suppose that $V = \left\{\gamma_{m,n} := \left[[x_{1_1}, \ldots, x_{m+1_1}], \ldots, [x_{1_{n+1}}, \ldots, x_{m+1_{n+1}}]\right]\right\}$ and $\mathcal{V}$ is the variety of the two-nilpotent groups of the class row (m, n), $\mathcal{V} = \mathcal{N}_{m,n}$. Given a group G, we have $V(G) = \gamma_{n+1}\left(\gamma_{m+1}(G)\right)$ and there are two following trivial series

$$G = \gamma_1(\gamma_1(G)) \supseteq G' = \gamma_2(\gamma_1(G)) \supseteq \cdots \supseteq \gamma_{n+1}(\gamma_1(G)) = \gamma_{n+1}(G) \\ \supseteq \gamma_{n+1}(\gamma_2(G)) \supseteq \cdots \supseteq \gamma_{n+1}(\gamma_{m+1}(G)),$$

and

$$G = \gamma_1(\gamma_1(G)) \supseteq G' = \gamma_1(\gamma_2(G)) \supseteq \cdots \supseteq \gamma_1(\gamma_{m+1}(G)) = \gamma_{m+1}(G) \\ \supseteq \gamma_2(\gamma_{m+1}(G)) \supseteq \cdots \supseteq \gamma_{n+1}(\gamma_{m+1}(G))$$

We denote the marginal subgroup of a group G with respect to $\mathcal{N}_{n,m}$, by $Z_{n,m}(G)$. The following lemma shows a description for the elements of $Z_{n,m}(G)$.

Lemma 2.2. *Let G be a group. $a \in Z_{n,m}(G)$ if and only if*

$$\big[[a, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] = 1,$$

for all $g_{i_j} \in G$, if $j = 1$, then $2 \le i \le m+1$ and if $2 \le j \le n+1$, then $1 \le i \le m+1$.

Proof. Suppose that $a \in Z_{n,m}(G)$. Then by considering $g_i = 1$ in definition of marginal subgroup we have

$\big[[a1, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] =$
$\big[[1, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] = 1.$

Now, if $\big[[a, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] = 1$, then for each $g_{i_j} \in G$, $1 \le i \le m+1$, $1 \le j \le n+1$ we can write

$$\begin{aligned}
&\big[[ag_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \big[[[a, g_{2_1}]^{g_{1_1}}[g_{1_1}, g_{2_1}], g_{3_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \Big[\Big[[[a, g_{2_1}]^{g_{1_1}}, g_{3_1}]^{[g_{1_1}, g_{2_1}]}[g_{1_1}, g_{2_1}, g_{3_1}], g_{4_1}, \ldots, g_{m+1_1}\Big], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\Big] \\
&= \ldots \\
&= \big[[a, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big]^k \big[[g_{1_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \big[[g_{1_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big],
\end{aligned}$$

for some $k \in G$. Thus if a is in the arbitrary place then

$$\begin{aligned}
&\big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_i}, \ldots, g_{j-1_i}, ag_{j_i}, g_{j+1_i}, \ldots, g_{m+1_i}], \\
&\qquad\qquad \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, \big[[ag_{j_i}, [g_{1_i}, \ldots, g_{j-1_i}]]^{-1}, g_{j+1_i}, \ldots, g_{m+1_i}\big], \\
&\qquad\qquad \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \Big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, \Big[[ag_{j_i}, [g_{1_i}, \ldots, g_{j-1_i}]^{-1}]^{[g_{1_i}, \ldots, g_{j-1_i}]}, g_{j+1_i}, \ldots, g_{m+1_i}\Big] \\
&\qquad\qquad, \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\Big] \\
&= \big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [ag_{j_i}, l_{2_i}, \ldots, l_{m+1_i}]^c, \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \ldots \\
&= \big[[ag_{j_i}, h_{2_1}, \ldots, h_{m+1_1}], \ldots, [h_{1_{n+1}}, \ldots, h_{m+1_{n+1}}]\big]^l \\
&= \big[[g_{j_i}, h_{2_1}, \ldots, h_{m+1_1}], \ldots, [h_{1_{n+1}}, \ldots, h_{m+1_{n+1}}]\big]^l \\
&= \big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{j_i}, l_{2_i}, \ldots, l_{m+1_i}]^c, \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \ldots \\
&= \big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [ag_{j_i}, l_{2_i}, \ldots, l_{m+1_i}]^c, \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big] \\
&= \Big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, \Big[[g_{j_i}, [g_{1_i}, \ldots, g_{j-1_i}]^{-1}]^{[g_{1_i}, \ldots, g_{j-1_i}]}, g_{j+1_i}, \ldots, g_{m+1_i}\Big], \\
&\qquad\qquad \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big]
\end{aligned}$$

$$= \big[[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, \big[[g_{j_i}, [g_{1_i}, \ldots, g_{j-1_i}]]^{-1}, g_{j+1_i}, \ldots, g_{m+1_i}\big] \\ , \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big]$$

$$= [[g_{1_1}, g_{2_1}, \ldots, g_{m+1_1}], \ldots, [g_{1_i}, \ldots, g_{j-1_i}, g_{j_i}, g_{j+1_i}, \ldots, g_{m+1_i}], \\ \ldots, [g_{1_{n+1}}, \ldots, g_{m+1_{n+1}}]\big],$$

for some elements $l, c \in G$, i.e. $a \in Z_{n,m}(G)$ and the assertion holds. □

So, we can write

$$V^*(G) = Z_{n,m}(G) = \{a \in G | [a,_m G] \subseteq Z_n(\gamma_{m+1}(G))\} .$$

For example, for all $n, m \geq 0$,

$V^*(G) = Z_{n,0}(G) = \{a \in G | [a, g_1, \ldots, g_n] = 1, g_i \in G, 1 \leq i \leq n\} = Z_n(G)$

and

$$Z_{0,m}(G) = \{a \in G | [a, g_1, \ldots, g_m] = 1, g_i \in G, 1 \leq i \leq m\} = Z_m(G).$$

Similarly

$$Z_{1,1}(G) = \{a \in G | \, [a, g_1, [g_2, g_3]] = 1, \forall g_i \in G, 1 \leq i \leq 3\},$$

and

$$Z_{n,1}(G) = \{a \in G | \, \big[[a, g], [g_{1_1}, g_{2_1}], \ldots, [g_{1_{n+1}}, g_{2_{n+1}}]\big] = 1, g \in G, \\ g_{i_j} \in G, i = 1, 2, j = 1, \ldots, n+1\}.$$

Also

$$Z_{1,m}(G) = \{a \in G | \big[[a, g_2, \ldots, g_{m+1}], [g'_1, \ldots, g'_{m+1}]\big] = 1, g_i, g'_j \in G, \\ 2 \leq i \leq m+1, 1 \leq j \leq m+1\}.$$

In fact, there are the following chains

$$1 = Z_0(G) \subseteq Z_1(G) = Z(G) \subseteq \cdots \subseteq Z_n(G) = Z_{n,0}(G) \\ \subseteq Z_{n,1}(G) \subseteq \cdots \subseteq Z_{n,k}(G) \subseteq \cdots ,$$

$$1 = Z_0(G) \subseteq Z_1(G) = Z(G) \subseteq \cdots \subseteq Z_m(G) = Z_{0,m}(G) \\ \subseteq Z_{1,m}(G) \subseteq \cdots \subseteq Z_{l,m}(G) \subseteq \cdots ,$$

such that $m, n, k, l \geq 0$. In the following theorem, we survey the relation between k-th center and marginal subgroup in two-nilpotent variety. The following theorem plays an important role in our main theorem.

Theorem 2.3. *For each $k, n, m \geq 0$,*

$$Z_k\left(\frac{G}{Z_{n,m}(G)}\right) \subseteq \frac{Z_{n,m+k}(G) \cap Z_{n+k,m}(G)}{Z_{n,m}(G)}.$$

Proof. Let $\bar{x} = xZ_{n,m}(G) \in Z_k\left(\frac{G}{Z_{n,m}(G)}\right)$, so $[x, g_1, \ldots, g_k] \in Z_{n,m}(G)$ for given $g_1, \ldots, g_k \in G$. Hence $[[x, g_1, \ldots, g_k],_m G] \subseteq Z_n(\gamma_{m+1}(G))$ and so

$$[[x, g_1, \ldots, g_k, g_{k+1}, \ldots, g_{k+m}], b_1, \ldots, b_n] = 1, \qquad (*)$$

for each $g_{k+1},\dots,g_{k+m}\in G$ and $b_i\in\gamma_{m+1}(G), 1\le i\le m+1$. Since $\gamma_{m+k+1}(G)\subseteq\gamma_{m+1}(G)$, one can conclude that $[x,_{m+k} G]\subseteq Z_n\left(\gamma_{m+k+1}(G)\right)$ and $x\in Z_{n,m+k}(G)$. On the other hand, the relation $(*)$ can be considered as

$$[[[x,g_1,\dots,g_m],g_{m+1},\dots,g_{k+m}],b_1,\dots,b_n]=1.$$

Thus we can conclude $[x,_m G]\subseteq Z_{n+k}(\gamma_{m+1}(G))$ or $x\in Z_{n+k,m}(G)$. Hence

$$\bar{x}\in\frac{Z_{n,m+k}(G)\cap Z_{n+k,m}(G)}{Z_{n,m}(G)}\ ,$$

and the assertion holds. □

3. Main results

In this section, we show that the converse of the Baer's theorem is true for variety of two-nilpotent groups of class row (n,m), which is a special case of the main theorem. Moreover, we investigate the converse of Baer's theorem for variety of two-nilpotent groups with trivial Frattini subgroups and state some properties of (n,m)-isoclinism families of these groups and $(1,m)$-stem groups.

The following statements generalize Theorem 2.3 and Lemma 2.5 in [7] for two-nilpotent variety.

Lemma 3.1. *Let G be a group and $N\trianglelefteq G$. Then for each $m\ge 0$*
(a) If $N\cap Z_{1,m}(G)=1$ then $N\cap Z_{n,m}(G)=1$, for all $n\ge 1$.
(b) If $N\cap\gamma_{m,n}(G)=1$, for some $n\ge 0$ then $N\le Z_{n,m}(G)$.

Proof. (a) Use induction on n. Let $x\in N\cap Z_{n,m}(G)$, so $[x,_m G]\subseteq Z_n(\gamma_{m+1}(G))$. Hence for each $g_{i_j}\in G, 1\le i\le n+1$ and $1\le j\le m+1$, we can write

$$[[x,g_{1_2},\dots,g_{1_{m+1}}],[g_{2_1},\dots,g_{2_{m+1}}],\dots,[g_{n+1_1},\dots,g_{n+1_{m+1}}]]=1.$$

Thus

$$[[x,g_{1_2},\dots,g_{1_{m+1}}],[g_{2_1},\dots,g_{2_{m+1}}]]\in Z_{n-1}(\gamma_{m+1}(G)).$$

One can see that

$$[[x,g_{1_2},\dots,g_{1_{m+1}}],[g_{2_1},\dots,g_{2_{m+1}}],_m G]\subseteq Z_{n-1}(\gamma_{m+1}(G)).$$

Since N is normal and by using Lemma 2.2 we have

$$[[x,g_{1_2},\dots,g_{1_{m+1}}],[g_{2_1},\dots,g_{2_{m+1}}]]\in Z_{n-1,m}(G)\cap N=1.$$

So $[x,g_{1_2},\dots,g_{1_{m+1}}]\in Z(\gamma_{m+1}(G))$ and this implies $[x,_m G]\subseteq Z(\gamma_{m+1}(G))$. Thus $x\in Z_{1,m}(G)\cap N=1$, as required.
(b) The assertion is an special case of Lemma 2.1 for $\mathcal{V}=\mathcal{N}_{n,m}$. □

Definition 3.2. Let G be an arbitrary group. If there are ordinals n, m such that $Z_{n,m}(G) = Z_{n+1,m}(G) = Z_{n+2,m}(G) = \cdots$, we call this terminal subgroup as *two-hypercenter* of G.

In the other words, consider upper central series

$$1 = Z_0(G) \subseteq Z_1(G) = Z(G) \subseteq \cdots \subseteq Z_m(G) = Z_{0,m}(G) \subseteq Z_{1,m}(G) \subseteq \cdots \subseteq Z_{l,m}(G) \subseteq \cdots,$$

such that $m, n, l \geq 0$. This series need not reach G, but if G is finite, the series terminates at a subgroup called the two-hypercenter.

We have seen in [11] that the converse of Baer's theorem holds with condition $\gamma_{n+1}(G) \cap Z_n(G) = 1$. In fact this result may be generalized to two-nilpotent variety of groups as follows.

Theorem 3.3. *Let $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$, for some $n, m \in \mathbf{N}$. Then $Z_{n,m}(G)$ is the two-hypercenter of G.*

Proof. Trivially $\gamma_{m,n}(G) \cap Z_{1,m}(G) \subseteq \gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$. This implies $\gamma_{m,n}(G) \cap Z_{n+i,m}(G) = 1$ for each $i \geq 1$, by Lemma3.1(a). Thus $Z_{n+i,m}(G) \subseteq Z_{n,m}(G)$ by Lemma 3.1 (b). Hence $Z_{n,m}(G) = Z_{n+i,m}(G)$, as required. □

The following famous theorem of Hall [5] is essential in this research. It presents a relation between upper and lower central series.

Theorem 3.4. *Let G be a group and $\gamma_{n+1}(G)$ be finite. Then $G/Z_{2n}(G)$ is finite.*

We are now equipped to prove the main theorem of this article.

Theorem 3.5. *Let G be a group such that $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$ and $\gamma_{m,n+i}(G)$ be finite for some $n, i, m \geq 0$. Then $G/Z_{n,m}(G)$ is finite.*

Proof. Since $\gamma_{m,n+i}(G) = \gamma_{n+i+1}(\gamma_{m+1}(G))$ is finite, by Theorem 3.4

$$\gamma_{m+1}(G)/Z_{2(n+i)}(\gamma_{m+1}(G)),$$

is finite. On the other hands by normality of center, if $x \in Z_{2(n+i)}(\gamma_{m+1}(G))$ then $[x,_m G] \leq Z_{2(n+i)}(\gamma_{m+1}(G))$ or $x \in Z_{2(n+i),m}(G)$. It means

$$Z_{2(n+i)}(\gamma_{m+1}(G)) \leq Z_{2(n+i),m}(G).$$

Thus the epimorphism $\dfrac{\gamma_{m+1}(G)}{Z_{2(n+i)}(\gamma_{m+1}(G))} \longrightarrow \dfrac{\gamma_{m+1}(G)Z_{2(n+i),m}(G)}{Z_{2(n+i),m}(G)}$ implies that

$$\frac{\gamma_{m+1}(G)Z_{2(n+i),m}(G)}{Z_{2(n+i),m}(G)} = \gamma_{m+1}\left(\frac{G}{Z_{2(n+i),m}(G)}\right)$$

is finite. Again by Theorem 3.4, one can conclude that

$$\frac{G/Z_{2(n+i),m}(G)}{Z_{2m}G/Z_{2(n+i),m}(G)},$$

is finite. By Theorem 2.3, there exists the following epimorphism

$$\frac{G/Z_{2(n+i),m}(G)}{Z_{2m}\left(\dfrac{G}{Z_{2(n+i),m}(G)}\right)} \longrightarrow \frac{G/Z_{2(n+i),m}(G)}{\dfrac{Z_{2(n+i),3m}(G)\cap Z_{2(n+i)+2m,m}(G)}{Z_{2(n+i),m}(G)}}.$$

Theorem 3.3, implies that $Z_{2(n+i)+2m,m}(G) = Z_{n,m}(G)$. Also $Z_{n,m}(G) \subseteq Z_{n,3m}(G) \subseteq Z_{2(n+i),3m}(G)$. Hence

$$\frac{G}{Z_{2(n+i),3m}(G)\cap Z_{2(n+i)+2m,m}(G)} = \frac{G}{Z_{n,m}(G)},$$

is finite and the result holds. □

In special case if $n = 0$, the above result is a generalization of the following theorem.

Theorem 3.6. *(*[11]*, Theorem 4.1) Let G be a group such that $\gamma_{n+1}(G)\cap Z_n(G) = 1$ and $\gamma_{n+i}(G)$ be finite for some $n, i \geq 0$. Then $G/Z_n(G)$ is finite.*

The converse of Baer's theorem for two-nilpotent variety of groups valids by considering $i = 0$ in Theorem 3.5.

Theorem 3.7. *Let G be a group such that $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$ and $\gamma_{m,n}(G)$ be finite for some $n, m \geq 0$. Then $G/Z_{n,m}(G)$ is finite.*

Also, we know always $\gamma_{m,n}(G) \cap Z_{n,m}(G) \subseteq \Phi(G)$, in which $\Phi(G)$ is the Frattini subgroup of G. Thus we can conclude that the following corollary where is a generalization of the main results in [9, 4].

Corollary 3.8. *Let G be a group such that $\Phi(G) = 1$ and $\gamma_{m,n+i}(G)$ be finite for some $m, n, i \geq 0$. Then $G/Z_{n,m}(G)$ is finite.*

When $n = m = 1$ we have similar results for the soluble variety.

Corollary 3.9. *Let $\mathcal{V} = \mathcal{S}_2$ be the variety of soluble group of class at most 2. If G is a group such that $G'' \cap V^*(G) = 1$ and G'' is finite then $G/V^*(G)$ is finite.*

Corollary 3.10. *Let $\mathcal{V} = \mathcal{S}_2$ be the variety of soluble group of class at most 2. If G is a group such that $\Phi(G) = 1$ and G'' is finite then $G/V^*(G)$ is finite.*

Now, we try to give some structural properties of the center factor group, $G/Z_{1,m}(G)$, for a group G. But first we need the next lemma.

Lemma 3.11. *Let G be a group such that $\gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$. Then $Z_{1,m}(G)$ is the two-hypercenter of G.*

Proof. By Lemma 3.1, $\gamma_{m,1}(G) \cap Z_{n,m}(G) = 1$ for each $n > 0$ and therefore $Z_{1,m}(G) = Z_{n,m}(G)$. Hence $Z_{1,m}(G)$ is the two-hypercenter of G. □

Combining Lemma 3.11 and Corollary 3.8 show that the converse of Baer's theorem is true for two-nilpotent variety of class row (n, m).

Corollary 3.12. *Let G be a group such that $\Phi(G) = 1$ and $\gamma_{m,n}(G)$ be finite for some positive integer n. Then $G/Z_{1,m}(G)$ is finite.*

We need the concept of $\mathcal{V}$-isologism. Isologism in fact expresses isoclinism with respect to a certain variety. In this way for each variety of groups an equivalence relation on the class of all groups arises.

Definition 3.13. ([8]) Let $\mathcal{V}$ be a variety and G and H be two groups. A *$\mathcal{V}$-isologism between G and H* is a pair of isomorphisms (α, β) with $\alpha : G/V^*(G) \xrightarrow{\simeq} H/V^*(H)$ and $\beta : V(G) \xrightarrow{\simeq} V(H)$, such that for all $s > 0$, all $v(x_i, \dots, x_s) \in V(F_\infty)$ and all $g_1, \dots, g_s \in G$, it holds that $\beta(v(g_1, \dots, g_s)) = v(h_1, \dots, h_s)$, whenever $h_i \in \alpha(g_i V^*(G))(1 \leq i \leq s)$. It is written by $G \underset{\mathcal{V}}{\sim} H$ and said that G and H are $\mathcal{V}$-isologism. In this case, we write $G \underset{(n,m)}{\sim} H$ instead of $G \underset{\mathcal{N}_{n,m}}{\sim} H$.

The following proposition states a relationship between the (n, m)-isologism of two groups and one of their center factors.

Proposition 3.14. *Let G and H be two groups such that $\gamma_{m,1}(H) \cap Z_{1,m}(H) = \gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$. Then $G \underset{(1,m)}{\sim} H$ if and only if $G/Z_{1,m}(G) \simeq H/Z_{1,m}(H)$.*

In the sequel, it is shown that each (n, m)-isoclinic family of groups that $\gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$, for each $m \geq 0$, is contained in an $(1, m)$-isoclinic family when $n \geq 1$. The following theorem is a generalization of Theorem 10 in [3].

Theorem 3.15. *Let G and H be two groups such that $\gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$ and $G/Z_{n,m}(G) \simeq H/Z_{n,m}(H)$. Then $G \underset{(1,m)}{\sim} H$.*

Proof. . By Lemma 3.11 we can write $G/Z_{1,m}(G) \simeq H/Z_{1,m}(H)$. Now, the result follows using Proposition 3.14. □

The following result shows that, under some condition, $(1, m)$-isoclinic implies an (n, m)-isoclinic.

Corollary 3.16. *Let G and H be two groups such that $\gamma_{m,1}(H) \cap Z_{1,m}(H) = \gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$. Then $G \underset{(1,m)}{\sim} H$ if and only if $G \underset{(n,m)}{\sim} H$.*

Now, we recall that a group S is an n-*stem* group if it satisfies $Z(S) \subseteq \gamma_{n+1}(S)$. An *stem* group is a 1-stem group. The existence of at least one stem group in each isoclinic family was proved by Hall [5]. We define an (n,m)-stem group S to be $Z_{n,m}(S) \subseteq \gamma_{m,n}(S)$, for each $m \geq 0$. Thus by considering $m = 0$, $n = 1$ we can conclude $Z(S) \subseteq S'$, that is S is an stem group. Also, each $(n,0)$-stem group is actually an n-stem group.

Lemma 3.17. *Let G be a group such that $\gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$ and S be a $(1,m)$-stem group $(1,m)$-isoclinic to G, for each $m \geq 0$. Then $Z_{n,m}(S)$ is trivial for each n.*

Proof. Since $G \underset{(1,m)}{\sim} S$, we have $\gamma_{m,1}(G) \cap Z_{1,m}(G) \simeq \gamma_{m,1}(S) \cap Z_{1,m}(S)$. Thus by Lemma 3.1(a), $\gamma_{m,1}(S) \cap Z_{n,m}(S) = 1$. By Lemma 3.1(b), we can conclude that $Z_{n,m}(S) \subseteq Z_{1,m}(S)$ and so $Z_{n,m}(S) = Z_{1,m}(S)$. Since S is a $(1,m)$-stem group, therefore $Z_{n,m}(S) = Z_{1,m}(G) = \gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$, as required. □

By considering $m = 0$, Lemma 3.17 is a generalization of Chakaneh et. al. [3] as follows.

Corollary 3.18. *([3], Theorem 8) Let G be a group such that $G' \cap Z(G) = 1$ and S be an stem group isoclinic to G. Then $Z_n(S)$ is trivial for each n.*

If (α, β) is a $\mathcal{V}$-isologism between G and H, then it is not difficult to see that β induces an isomorphism from $V^*(G) \cap V(G)$ onto $V^*(H) \cap V(H)$, (Theorems 4.3 and 4.5 in [8]). The following lemma is a conclusion of them for two-nilpotent variety.

Lemma 3.19. *Let G and H be two groups and (α, β) be an $\mathcal{N}_{n,m}$-isologism between G and H. Then for all $i \geq 0$*
(a) $\alpha(\gamma_{m,i+1}(G)Z_{n,m}(G)/Z_{n,m}(G)) = \gamma_{m,i+1}(H)Z_{n,m}(H)/Z_{n,m}(H)$.
(b) $\beta(\gamma_{m,n}(G) \cap Z_{i,m}(G)) = \gamma_{m,n}(H) \cap Z_{i,m}(H)$.

We are going to prove an important theorem on the generalization of converse of Baer's theorem on two-nilpotent variety.

Theorem 3.20. *Let G and H be two groups such that $G \underset{(n,m)}{\sim} H$, $\gamma_{m,n+i}(G)$ be finite for some positive integer i and $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$. Then $H/Z_{n,m}(H)$ is finite. In particular, H satisfies the converse of Baer's theorem, for two-nilpotent variety.*

Proof. It follows from Lemma 3.19 and Theorem 3.5. Let two groups are $\mathcal{N}_{n,m}$-isologism. If each of them satisfies the Theorem 3.5 then for another also holds. □

Now, by Corollary 3.16, we can deduce the following result with a weaker condition for the Theorem 3.20.

Corollary 3.21. *Let G and H be two groups such that $G \underset{(1,m)}{\sim} H$, $\gamma_{m,n+i}(G)$ be finite for some positive integer i and $\gamma_{m,1}(G) \cap Z_{1,m}(G) = 1$. Then $H/Z_{n,m}(H)$ is finite. In particular, H satisfies the converse of Baer's theorem, for two-nilpotent variety.*

Proof. By Lemma 3.11 and Corollary 3.16, we have $G \underset{(n,m)}{\sim} H$ and $\gamma_{m,n}(G) \cap Z_{n,m}(G) = 1$. Now, the result follows using Theorem 3.20. □

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