Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(1)(2022), 126-137 (Research paper)

Stability of Sacks-Uhlenbeck Biharmonic Maps

Mehdi Kazemi Torbaghan [1] and Morteza Mirmohammad Rezaii [2]
[1] Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran
[2] Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran.

Abstract. In this paper, the first and second variation formulas of the Sacks-Uhlenbeck bienergy functional is obtained. As an application, instability and non-existence theorems for Sacks-Uhlenbeck biharmonic maps are given.

Keywords: Harmonic maps, Biharmonic maps, Stability, Calculus of variations, Sacks-Uhlenbeck biharmonic maps.

2000 Mathematics subject classification: 53C35, 58E20; Secondary 53C43.

1. Introduction

Harmonic maps between Riemannian manifolds were first introduced by Eells and Sampson, in 1964. They showed that any map $\phi_0 : (M, g) \longrightarrow (N, h)$ from any compact Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature can be deformed into a harmonic maps. This is so-called the fundamental existence theorem for harmonic maps, [2]. In view of physics, harmonic maps have been studied in various fields of physics, such as super conductor, ferromagnetic material, liquid crystal, etc., [8, 9].

[1] Corresponding author: m.kazemi@ub.ac.ir
Received: 16 May 2020
Revised: 17 July 2020
Accepted: 18 July 2020

Biharmonic maps, as an extension of harmonic maps, were first studied by Jiang in 1986, [5]. In recent decade, many scholars have done research on this topic. These maps play a key role in describing the model of fluid dynamics and elasticity, [1, 3, 4, 12].
In 1981, Sacks and Uhlenbeck introduced the notion of Sacks-Uhlenbeck harmonic maps. Let $\psi : (M, g) \longrightarrow (P, \rho)$ be a smooth map between Riemannian manifolds. The *Sacks-Uhlenbeck energy functional* of ψ for $\alpha > 1$, is denoted by $E_\alpha(\psi)$ and defined as follows:

$$E_\alpha(\psi) := \int_M (1+ \mid d\psi \mid^2)^\alpha dv_g, \tag{1.1}$$

where dv_g is the volume element of (M, g) and $\mid d\psi \mid$ denotes the Hilbert-Schmidt norm of the differential map $d\psi \in \Gamma(T^*M \times \psi^{-1}TP)$ with respect to g and ρ. E_α satisfies Morse theory and Ljusternik-Schnirelman theory if $\alpha > 1$. The critical points of E_α are called the *α-Sacks-Uhlenbeck harmonic maps.* Sacks and Uhlenbeck used the critical maps s_α of E_α, to derive an existence theory for harmonic maps of orientable surfaces into Riemannian manifolds. They showed that convergence of the critical points of E_α is sufficient to produce at least one harmonic map of sphere into Riemannian manifold. Note that every harmonic map from a sphere is in fact a conformal minimal immersion, [11]. There have been extensive studies in this area (see for instance, [6, 7, 10]).
In this paper, following the ideas in [5], we investigate Sacks-Uhlenbeck biharmonic maps between Riemannian manifolds as an extension of Sacks-Uhlenbeck harmonic maps. More precisely, we use the methods provided in [5, 11] to study the new type of biharmonicity. Our main results are included in Section 3. In particular, instability and non-existence theorems for Sacks-Uhlenbeck biharmonic maps are given.
The organization of this paper is as follows. In section 2, the concepts of harmonic, biharmonic maps are reviewed and some essential formulas which are necessary for this paper are given. In the third section, the first and second variation formulas for Sacks-Uhlenbeck biharmonic maps is obtained. Then, the stability of Sacks-Uhlenbeck biharmonic maps from a compact Riemannian manifold into an arbitrary Riemannian manifold with constant positive sectional curvature is studied.

2. Preliminaries

In this section, we recall some basic concepts which will be used later. For more details, see ([5, 11]). Let $\psi : (M, g) \longrightarrow (P, \rho)$ be a smooth map between Riemannian manifolds. Throughout this paper, we consider (M, g) as a compact Riemannian manifold. Denote the Levi-Civita connections on M and P by ∇ and ∇^P, respectively. Moreover, the induced connection on the pullback bundle $\psi^{-1}TP$ is denoted by ∇^ψ and defined by $\nabla^\psi_X W = \nabla^P_{d\psi(X)} W$, for any $X \in \chi(M)$ and $W \in \Gamma(\psi^{-1}TP)$. The energy functional of ψ is defined as follows

$$E(\psi) = \frac{1}{2}\int_M \mid d\psi \mid^2 dv_g. \tag{2.1}$$

The corresponding Euler-Lagrange equation of the energy functional E is given by

$$\tau(\psi) := \nabla^\psi_{e_i} d\psi(e_i) - d\psi(\nabla_{e_i} e_i) = 0, \tag{2.2}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on M (here henceforward we sum over repeated indices). $\tau(\psi)$ is called the *tension field* of ψ. In terms of the Euler-Lagrange equation, a map ψ is called harmonic if $\tau(\psi) = 0$, [2]. Furthermore, ψ is said to satisfy the *conservation law* if

$$\rho(\tau(\psi), d\psi(X)) = 0, \tag{2.3}$$

for any $X \in \chi(M)$, [5].
Biharmonic maps $\psi : (M, g) \longrightarrow (P, \rho)$ are critical points of the bienergy functional

$$E_2(\psi) = \frac{1}{2}\int_M \mid \tau(\psi) \mid^2 dv_g. \tag{2.4}$$

The Euler- Lagrange equation associated to E_2 is, [5]:

$$\tau_2(\psi) := -J^\psi(\tau(\psi)) = 0, \tag{2.5}$$

where J^ψ denotes the Jacobi operator of ψ on the sections of the pullback bundle $\psi^{-1}TP$, defined by

$$J^\psi(V) := -\Delta^\psi V - trace_g R^P(d\psi, V)d\psi, \qquad V \in \Gamma(\psi^{-1}TP), \tag{2.6}$$

here $R^P(X,Y) = [\nabla^P X, \nabla^P Y] - \nabla^P_{[X,Y]}$ is the curvature operator on P, and Δ^ψ is the rough Laplacian on the sections of $\psi^{-1}TP$ which is defined with respect to a local orthonormal frame field $\{e_i\}$ on M as follows:

$$-\Delta^\psi V := \nabla^\psi_{e_i}\nabla^\psi_{e_i} V - \nabla^\psi_{\nabla_{e_i} e_i} V. \tag{2.7}$$

Theorem 2.1. *(the second variation formula of the bienergy functional, see* [5]*) Let $\psi : (M, g) \longrightarrow (P, \rho)$ be a biharmonic map, and $\{\psi_t\}$ be an arbitrary smooth variation of ψ such that $\psi_0 = \psi$. suppose that $V = \frac{d\psi_t}{dt} \mid_{t=0}$. Then the second variation formula of $\frac{1}{2}E_2(\psi_t)$ is given as*

$$\frac{1}{2}\frac{d^2}{dt^2} \mid_{t=0} E_2(\psi_t) = \int_M l_\psi(V) dv_g, \tag{2.8}$$

where

$$\begin{aligned}
l_\psi(V) :=& \mid -\Delta^\psi V - R^P(d\psi(e_i),V)d\psi(e_i)\mid^2 \\
&- \rho\Big(V, (\nabla^\psi_{e_i}R^P)(d\psi(e_i),\tau(\psi))V + (\nabla^\psi_{\tau(\psi)}R^P)(d\psi(e_i),V)d\psi(e_i) \\
&+ R^P(\tau(\psi),V)\tau(\psi) + 2R^P(d\psi(e_i),V)\nabla^\psi_{e_i}\tau(\psi) \\
&+ 2R^P(d\psi(e_i),\tau(\psi))\nabla^\psi_{e_i}V\Big). \qquad (2.9)
\end{aligned}$$

ψ is said to be *stable biharmonic* if $\frac{d^2}{dt^2}\mid_{t=0} E_2(\psi_t) \geq 0$ for every smooth variation $\{\psi_t\}$ of ψ, [5]. In particular, ψ is called *strongly stable biharmonic* if $l_\psi(V)$ is non-negative for every variational vector field V along ψ.

3. Main Results

In this section, first of all, we find the first and second variation formulas of Sacks-Uhlenbeck bienergy functional for $\alpha > 1$. Then, instability and non-existence theorems for Sacks-Uhlenbeck biharmonic maps are given.

Definition 3.1. A smooth map $\psi : (M,g) \longrightarrow (P,\rho)$ is said to be α-*Sacks-Uhlenbeck biharmonic map* if ψ is a critical point of the α-Sacks-Uhlenbeck bienergy functional, $E_{\alpha,2}$, defined as follows

$$E_{\alpha,2}(\psi) := \int_M (1+\mid \tau(\psi)\mid^2)^\alpha dv_g, \qquad (3.1)$$

where $\mid \tau(\psi)\mid$ denotes the Hilbert-Schmidt norm of the tension field of ψ.

In the following theorem, we calculate the first variation formula corresponding to the α-Sacks-Uhlenbeck bienergy functional.

Theorem 3.2. *(The first variation formula) Let $\psi : (M,g) \longrightarrow (P,\rho)$ be a smooth map, and $\psi_t : M \longrightarrow P$ $(-\epsilon < t < \epsilon)$ be a smooth variation of ψ such that $\psi_0 = \psi$. Then*

$$\frac{d}{dt}\mid_{t=0} E_{\alpha,2}(\psi_t) = \int_M \rho(\tau_{\alpha,2}(\psi),V)dv_g, \qquad (3.2)$$

where $V = \frac{d\psi_t}{dt}\mid_{t=0}$ and $\tau_{\alpha,2}(\psi)$ is defined by

$$\begin{aligned}
\tau_{\alpha,2}(\psi) :=& -\Delta^\psi[2\alpha(1+\mid\tau(\psi)\mid^2)^{\alpha-1}\tau(\psi)] \\
&- trace_g R^P(d\psi, 2\alpha(1+\mid\tau(\psi)\mid^2)^{\alpha-1}\tau(\psi))d\psi \qquad (3.3)
\end{aligned}$$

Proof. Let $\Psi : M \times (-\epsilon, \epsilon) \longrightarrow P$ be a smooth map defined by $\Psi(t, x) = \psi_t(x)$. Here $M \times (-\epsilon, \epsilon)$ is equipped with the product metric. Denote the induced connections on $T(M \times (-\epsilon, \epsilon)), \Psi^{-1}TP$ and $T^*(M \times (-\epsilon, \epsilon)) \otimes \Psi^{-1}TP$ by $\bar{\nabla}, \nabla^{\Psi}$ and $\tilde{\nabla}$, respectively. Moreover, let X and $\frac{\partial}{\partial t}$ be smooth vector fields on M and $(-\epsilon, \epsilon)$, respectively. The canonical extension of $\frac{\partial}{\partial t}$ and X to $M \times (-\epsilon, \epsilon)$ are denoted by $\frac{\partial}{\partial t}$ and X again. According to the above notations, we have

$$\bar{\nabla}_{\frac{\partial}{\partial t}} e_k - \bar{\nabla}_{e_k} \frac{\partial}{\partial t} = [\frac{\partial}{\partial t}, e_k] = 0. \tag{3.4}$$

Now, by considering the following equations

$$\nabla^{\Psi}_{\frac{\partial}{\partial t}} d\Psi(e_i) - \nabla^{\Psi}_{e_i} d\Psi(\frac{\partial}{\partial t}) = d\Psi[\frac{\partial}{\partial t}, e_i] = 0, \tag{3.5}$$

and

$$\nabla^{\Psi}_{\frac{\partial}{\partial t}} d\Psi(\bar{\nabla}_{e_i} e_i) - \nabla^{\Psi}_{\bar{\nabla}_{e_i} e_i} d\Psi(\frac{\partial}{\partial t}) = d\Psi[\frac{\partial}{\partial t}, \bar{\nabla}_{e_i} e_i] = 0, \tag{3.6}$$

we get

$$\begin{aligned} &\frac{\partial}{\partial t}(1+ \mid \tau(\psi_t) \mid^2)^{\alpha} \\ &= \rho(\nabla^{\Psi}_{e_i} \nabla^{\Psi}_{e_i} d\Psi(\frac{\partial}{\partial t}) - \nabla^{\Psi}_{\bar{\nabla}_{e_i} e_i} d\Psi(\frac{\partial}{\partial t}), 2\alpha(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)) \\ &+ \rho(R^P(d\Psi(\frac{\partial}{\partial t}), d\Psi(e_i)) d\Psi(e_i), 2\alpha(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)) \end{aligned} \tag{3.7}$$

Let Q_t and Z_t be two vector fields on M, defined by

$$g(Q_t, W) = \rho(\nabla^{\Psi}_W d\Psi(\frac{\partial}{\partial t}), 2\alpha(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t))$$

and

$$g(Z_t, W) = \rho(d\Psi(\frac{\partial}{\partial t}), 2\alpha \nabla^{\Psi}_W [(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)])$$

for any $W \in \chi(M)$. By calculating the divergence of Q_t and Z_t, it can be obtained that

$$\begin{aligned} div(Q_t) &= \rho(\nabla^{\Psi}_{e_i} \nabla^{\Psi}_{e_i} d\Psi(\frac{\partial}{\partial t}) - \nabla^{\Psi}_{\bar{\nabla}_{e_i} e_i} d\Psi(\frac{\partial}{\partial t}), 2\alpha(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)) \\ &+ \rho(\nabla^{\Psi}_{e_i} d\Psi(\frac{\partial}{\partial t}), 2\alpha \nabla^{\Psi}_{e_i} [(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)]), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} div(Z_t) = \rho(d\Psi(\frac{\partial}{\partial t}), &\nabla^{\Psi}_{e_i} \nabla^{\Psi}_{e_i} [2\alpha(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)] \\ &- \nabla^{\Psi}_{\bar{\nabla}_{e_i} e_i} [2\alpha(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)]) \\ &+ \rho(\nabla^{\Psi}_{e_i} d\Psi(\frac{\partial}{\partial t}), 2\alpha \nabla^{\Psi}_{e_i} [(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} \tau(\psi_t)]). \end{aligned} \tag{3.9}$$

By (3.7), (3.8)and (3.9), we have

$$
\begin{aligned}
&\frac{d}{dt}E_{\alpha,2}(\psi_t)\mid_{t=0}\\
&=\int_M \frac{\partial}{\partial t}(1+\mid\tau(\psi_t)\mid^2)^\alpha\mid_{t=0}d\upsilon_g\\
&=\int_M\Bigg(\rho(R^P(d\Psi(\frac{\partial}{\partial t}),d\Psi(e_i))d\Psi(e_i),\\
&2\alpha(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\tau(\psi_t))+div(Q_t)-div(Z_t)\\
&+\rho(d\Psi(\frac{\partial}{\partial t}),\nabla^\Psi_{e_i}\nabla^\Psi_{e_i}[2\alpha(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\tau(\psi_t)]\\
&-\nabla^\Psi_{\nabla_{e_i}e_i}[2\alpha(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\tau(\psi_t)])\Bigg)\mid_{t=0}d\upsilon_g\\
&=\int_M\Bigg(\rho(d\Psi(\frac{\partial}{\partial t}),-\Delta^\Psi[2\alpha(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\tau(\psi_t)])\\
&+div(Q_t)-div(Z_t)+\rho(R^P(d\Psi(\frac{\partial}{\partial t}),d\Psi(e_i))d\Psi(e_i),\\
&2\alpha(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\tau(\psi_t))\Bigg)\mid_{t=0}d\upsilon_g\\
&=\int_M\rho\Bigg(V,-\Delta^\psi[2\alpha(1+\mid\tau(\psi)\mid^2)^{\alpha-1}\tau(\psi)]\\
&-R^P(d\psi(e_i),2\alpha(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\tau(\psi_t)))d\psi(e_i)\Bigg)d\upsilon_g. \qquad (3.10)
\end{aligned}
$$

By (3.3) and (3.10), it can be seen that

$$\frac{d}{dt}E_{\alpha,2}(\psi_t)\mid_{t=0}=\int_M\rho(\tau_{\alpha,2}(\psi),V)d\upsilon_g. \qquad (3.11)$$

This completes the proof. □

According to the theorem 3.2, the corresponding Euler-Lagrange equation of α -Sacks-Uhlenbeck Bienergy functional, $E_{\alpha,2}$, is given by

$$
\begin{aligned}
\tau_{\alpha,2}(\psi)=&-\Delta^\psi[2\alpha(1+\mid\tau(\psi)\mid^2)^{\alpha-1}\tau(\psi)]\\
&-trace_gR^P(d\psi,2\alpha(1+\mid\tau(\psi)\mid^2)^{\alpha-1}\tau(\psi))d\psi=0 \qquad (3.12)
\end{aligned}
$$

$\tau_{\alpha,2}(\psi)$ is called the *α-Sacks-Uhlenbeck bitension field* of ψ. In terms of the Euler-Lagrange equation, a map ψ is called α-Sacks-Uhlenbeck biharmonic map if $\tau_{\alpha,2}(\psi)=0$. The map ψ is called non-trivial α-Sacks-Uhlenbeck biharmonic if ψ is an α-Sacks-Uhlenbeck biharmonic map but $\tau(\psi)\neq 0$.

Theorem 3.3. *(the second variation formula of α-Sacks-Uhlenbeck bienergy functional) Let $\psi : (M, g) \longrightarrow (P, \rho)$ be a critical point of Sacks-Uhlenbeck bienergy functional for $\alpha > 1$. Moreover, let $\{\psi_t\}$ be an arbitrary smooth variation of ψ such that $\psi_0 = \psi$. Then, the second variation formula of $E_{\alpha,2}(\psi_t)$ is given as follows:*

$$\begin{aligned}
&\frac{d^2}{dt^2} \mid_{t=0} E_{\alpha,2}(\psi_t) \\
&= \int_M 2\alpha(1+ \mid \tau(\psi) \mid^2)^{\alpha-1} l_\psi(V) + \rho\bigg(V, S_{\alpha,\psi}(V)\tau_2(\psi) \\
&-2\alpha\Delta(1+ \mid \tau(\psi) \mid^2)^{\alpha-1} J^\psi(V) - \Delta S_{\alpha,\psi}(V)\tau(\psi) \\
&+2\nabla^\psi_{grad\ \ (S_{\alpha,\psi}(V))}\tau(\psi) + 2\nabla^\psi_{2\alpha\ grad\ (1+|\tau(\psi)|^2)^{\alpha-1}} J^\psi(V) \\
&-4\alpha R^P(d\psi(grad\ \ (1+ \mid \tau(\psi) \mid^2)^{\alpha-1}), V)\tau(\psi)\bigg) d\upsilon_g, \qquad (3.13)
\end{aligned}$$

where $V = \frac{d\psi_t}{dt} \mid_{t=0}$, $S_{\alpha,\psi}(V) := 2\alpha \frac{d}{dt} \mid_{t=0} (1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1}$ and $l_\psi(V)$ is defined by (2.9).

Proof. According to the notations used in the proof of theorem 3.2 and considering the equations (3.2) and (3.12), together with the assumption that ψ is a critical point of Sacks-Uhlenbeck bienergy functional for $\alpha > 1$, we have

$$\begin{aligned}
&\frac{d^2}{dt^2} E_{\alpha,2}(\psi_t)\bigg|_{t=0} \\
&= 2\alpha \int_M \bigg\langle d\Psi(\frac{\partial}{\partial t}), \nabla^\Psi_{\frac{\partial}{\partial t}} \bigg\{ \nabla^\Psi_{e_k} \nabla^\Psi_{e_k} [(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} ({}_{e_i} d\Psi)(e_i)] \\
&- \nabla^\Psi_{\tilde{\nabla}_{e_k} e_k} [(1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} (\tilde{\nabla}_{e_i} d\Psi)(e_i)] \\
&- R^P(d\Psi(e_k), (1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1} (\tilde{\nabla}_{e_i} d\Psi)(e_i)) d\Psi(e_k) \bigg\} \bigg\rangle\bigg|_{t=0} d\upsilon_g, \qquad (3.14)
\end{aligned}$$

where $<,>$ is the inner product on $\Psi^{-1}TP$, with respect to g and ρ. By calculating the right hand side of (3.14) , we obtain:

$$\begin{aligned}
&\frac{d^2}{dt^2}E_{\alpha,2}(\psi_t)\Big|_{t=0}\\
=&2\alpha\int_M \Big\langle d\Psi(\frac{\partial}{\partial t}), \Big(e_k(e_k((1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}))\nabla^{\Psi}_{\frac{\partial}{\partial t}}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&-(\bar{\nabla}_{\bar{\nabla}_{e_k}e_k}(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1})\nabla^{\Psi}_{\frac{\partial}{\partial t}}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\Big)\\
&+2e_k(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\nabla^{\Psi}_{\frac{\partial}{\partial t}}\nabla^{\Psi}_{e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&+2e_k(\frac{\partial(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}}{\partial t})\nabla^{\Psi}_{e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&+\Big(e_k(e_k(\frac{\partial(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}}{\partial t}))[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&-(\bar{\nabla}_{\bar{\nabla}_{e_k}e_k}(\frac{\partial(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}}{\partial t}))[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\Big)\\
&+\frac{\partial(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}}{\partial t}\Big(\nabla^{\Psi}_{e_k}\nabla^{\Psi}_{e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&-\nabla^{\Psi}_{\bar{\nabla}_{e_k}e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]-R^P(d\Psi(e_k),(\tilde{\nabla}_{e_i}d\Psi)(e_i))d\Psi(e_k)\Big)\\
&+(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\nabla^{\Psi}_{\frac{\partial}{\partial t}}\Big(\nabla^{\Psi}_{e_k}\nabla^{\Psi}_{e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]-\nabla^{\Psi}_{\bar{\nabla}_{e_k}e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&-R^P(d\Psi(e_k),(\tilde{\nabla}_{e_i}d\Psi)(e_i))d\Psi(e_k))\Big)\Big\rangle\Big|_{t=0}dv_g. \qquad (3.15)
\end{aligned}$$

Noting that

$$\nabla^{\Psi}_{\frac{\partial}{\partial t}}d\Psi(e_i)-\nabla^{\Psi}_{e_i}d\Psi(\frac{\partial}{\partial t})=d\Psi([\frac{\partial}{\partial t},e_i])=0. \qquad (3.16)$$

By ([5], Eq. 23, pp. 214) and (3.16) we have

$$\begin{aligned}
\nabla^{\Psi}_{\frac{\partial}{\partial t}}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\mid_{t=0}=&\sum_i\Big\{(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}d\Psi)(\frac{\partial}{\partial t})-(\tilde{\nabla}_{\bar{\nabla}_{e_i}e_i}d\Psi)(\frac{\partial}{\partial t})\\
&-R^P(d\Psi(e_i),d\Psi(\frac{\partial}{\partial t}))d\Psi(e_i)\Big\}\Big|_{t=0}\\
=&J^{\psi}(V). \qquad (3.17)
\end{aligned}$$

Furthermore

$$
\begin{aligned}
& e_k[(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}]\nabla^{\Psi}_{\frac{\partial}{\partial t}}\nabla^{\Psi}_{e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\Big|_{t=0}\\
=& e_k[(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}]\Big\{R^P(d\Psi(\frac{\partial}{\partial t}),d\Psi(e_k))[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&+\nabla^{\Psi}_{e_k}\nabla^{\Psi}_{\frac{\partial}{\partial t}}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]+\nabla^{\Psi}_{[\frac{\partial}{\partial t},e_k]}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\Big\}\Big|_{t=0}\\
=& R^P(V,d\psi(grad\ ((1+\mid\tau(\psi)\mid^2)^{\alpha-1})))\tau(\psi)\\
&+\nabla^{\psi}_{grad((1+|\tau(\psi)|^2)^{\alpha-1})}J^{\psi}(V).
\end{aligned}
\tag{3.18}
$$

By (2.9) and (3.17), the latter term of (3.15) can be obtained as follows

$$
\begin{aligned}
&\int_M(1+\mid\tau(\psi_t)\mid^2)^{\alpha-1}\Big\langle d\psi(\frac{\partial}{\partial t}),\nabla^{\Psi}_{\frac{\partial}{\partial t}}\Big(\nabla^{\Psi}_{e_k}\nabla^{\Psi}_{e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&-\nabla^{\Psi}_{\tilde{\nabla}_{e_k}e_k}[(\tilde{\nabla}_{e_i}d\Psi)(e_i)]\\
&-R^P(d\Psi(e_k),[(\tilde{\nabla}_{e_i}d\Psi)(e_i)])d\Psi(e_k)\Big)\Big\rangle\Big|_{t=0}d\upsilon_g\\
=&\int_M(1+\mid\tau(\psi)\mid^2)^{\alpha-1}l_{\psi}(V)d\upsilon_g.
\end{aligned}
\tag{3.19}
$$

Substituting formulas (3.17)-(3.19) into (3.15), we get (3.13). This completes the proof of the theorem. $\square$

Definition 3.4. Let $\psi:(M,g)\longrightarrow(P,\rho)$ be an α- Sacks-Uhlenbeck biharmonic map and $\{\psi_t\}$ be a smooth variation of ψ such that $\psi_0=\psi$. Set

$$
I^{\psi}_{\alpha,2}(V,V):=\frac{d^2}{dt^2}\mid_{t=0}E_{\alpha,2}(\psi_t),\tag{3.20}
$$

where $V=\frac{d\psi_t}{dt}\mid_{t=0}$. The map ψ is called stable α-Sacks-Uhlenbeck biharmonic map if $I^{\psi}_{\alpha,2}(V,V)\geq 0$, for every variational vector fields V along ψ.

Remark 3.5. Let $Id:(P,\rho)\longrightarrow(P,\rho)$ be an identity map. By (2.9) and (3.13) and noting that $\tau(Id)=0$, it can be concluded that Id is stable α-Sacks-Uhlenbeck biharmonic. In other words, Id is an absolute minimum of the α-Sacks-Uhlenbeck bienergy functional.

Theorem 3.6. *Let $\psi:(M,g)\longrightarrow(P,\rho)$ be a non-trivial α-Sacks-Uhlenbeck biharmonic map satisfying (2.3), and let (P,ρ) be a Riemannian manifold with a constant positive sectional curvature K. Assume that ψ is biharmonic. Then, ψ is unstable α-Sacks-Uhlenbeck biharmonic map.*

Proof. By (2.9) and (3.13), and considering that P is a constant sectional curvature manifold, i.e., $\nabla^P R^P = 0$, it can be obtained that

$$
\begin{aligned}
&\frac{d^2}{dt^2}\bigg|_{t=0} E_{\alpha,2}(\psi_t)\\
=&2\alpha \int_M (1+ \mid \tau(\psi) \mid^2)^{\alpha-1} \left| -\Delta^{\psi} V - trace_g R^P(d\psi, V)d\psi \right|^2 dv_g\\
&-2\alpha \int_M (1+ \mid \tau(\psi) \mid^2)^{\alpha-1} \Big(\rho(V, R^P(\tau(\psi), V)\tau(\psi))\\
&+2\; trace_g\; R^P(d\psi, V)\nabla^{\psi}\tau(\psi) + 2\; trace_g\; R^P(d\psi, \tau(\psi))\nabla^{\psi} V \Big) dv_g\\
&+ \int_M \rho \Big(V, S_{\alpha,\psi}(V)\tau_2(\psi) - 2\alpha\Delta(1+ \mid \tau(\psi) \mid^2)^{\alpha-1} J^{\psi}(V) - \Delta S_{\alpha,\psi}(V)\tau(\psi)\\
&+2\nabla^{\psi}_{grad\ \ (S_{\alpha,\psi}(V))}\tau(\psi) + 4\alpha\nabla_{grad\ \ [(1+|\tau(\psi)|^2)^{\alpha-1}]} J^{\psi}(V)\\
&-4\alpha R^P(d\psi(grad\ \ (1+ \mid \tau(\psi) \mid^2)^{\alpha-1}), V)\tau(\psi) \Big) dv_g.
\end{aligned}
\tag{3.21}
$$

According to the definition of $S_{\alpha,\psi}(V)$, and considering the following equation

$$
\begin{aligned}
&R^P(d\Psi(\frac{\partial}{\partial t}), d\Psi(e_i))d\Psi(e_i)\\
&= \nabla^{\Psi}_{\frac{\partial}{\partial t}} \nabla^{\Psi}_{e_i} d\Psi(e_i) - \nabla^{\Psi}_{e_i}\nabla^{\Psi}_{\frac{\partial}{\partial t}} d\Psi(e_i) - \nabla^{\Psi}_{[\frac{\partial}{\partial t}, e_i]} d\Psi(e_i)
\end{aligned}
\tag{3.22}
$$

we get

$$
\begin{aligned}
&S_{\alpha,\psi}(V)\\
&= 2\alpha \frac{d}{dt} \mid_{t=0} (1+ \mid \tau(\psi_t) \mid^2)^{\alpha-1}\\
&= 4\alpha(\alpha-1)(1+ \mid \tau(\psi) \mid^2)^{\alpha-2} \rho(J^{\psi}(V), \tau(\psi)).
\end{aligned}
\tag{3.23}
$$

Setting $V = \tau(\psi)$ in (3.23) together with the assumption that ψ is biharmonic, i.e., $\tau_2(\psi) = -J^{\psi}(\tau(\psi)) = 0$, it can be obtained that

$$S_{\alpha,\psi}(\tau(\psi)) = 0. \tag{3.24}$$

Furthermore, by (2.3) we get

$$\rho(d\psi(e_i), \nabla^{\psi}_{e_i}\tau(\psi)) = - \mid \tau(\psi) \mid^2 . \tag{3.25}$$

According to the assumption, $\tau_2(\psi) = 0$, and using (2.3) and (3.24)-(3.25), the right hand side of (3.21) can be rewritten as follows:

$$\begin{aligned} I^{\psi}_{\alpha,2}(\tau(\psi),\tau(\psi)) &= \int_M \rho(\tau(\psi), 4trace_g\ R^P(d\psi,\tau(\psi))\nabla^{\psi}\tau(\psi))d\upsilon_g \\ &= 4K\int_M \Big\{\rho(d\psi(e_i),\nabla^{\psi}_{e_i}\tau(\psi))\rho(\tau(\psi),\tau(\psi)) \\ &- \rho(d\psi(e_k),\tau(\psi))\rho(\tau(\psi),\nabla^{\psi}_{e_k}\tau(\psi))\Big\}d\upsilon_g \\ &= -4K\int_M \mid \tau(\psi)\mid^4 d\upsilon_g \leq 0. \end{aligned} \tag{3.26}$$

By (3.26), It follows that

$$I^{\psi}_{\alpha,2}(\tau(\psi),\tau(\psi)) = 0 \Longleftrightarrow \tau(\psi) = 0. \tag{3.27}$$

Due to the fact that ψ is a non-trivial α-Sacks-Uhlenbeck biharmonic map and using (3.26)- (3.27), it can be concluded that

$$I^{\psi}_{\alpha,2}(\tau(\psi),\tau(\psi)) < 0.$$

Thus, the map ψ is unstable α-Sacks-Uhlenbeck biharmonic map. This completes the proof. □

By theorem 3.3 and formula (3.23), we get

Corollary 3.7. *Let $\psi : (M,g) \longrightarrow (P,\rho)$ be an α-Sacks-Uhlenbeck biharmonic map. Moreover, let ψ is the strongly stable biharmonic map satisfying*

$$grad\ (1+\mid\tau(\psi)\mid^2)^{\alpha-1} = grad\ (\rho(J^{\psi}(V),\tau(\psi))) = 0, \tag{3.28}$$

for any variational vector field V along ψ. Then, ψ is stable α-Sacks-Uhlenbeck biharmonic map.

Acknowledgement

The authors will acknowledge that this paper was formerly, in its concise and abridged form, submitted, accepted and presented at the second congress on mathematics and statistics which was held at Gonbad Kavous University of Iran on February 2020. This full length paper was decided to submit to the Caspian Journal of Mathematical Sciences as the congress executive and scientific committee had already expressed its consent for doing so.

References

[1] A. Balmus, S. Montaldoa, C. Oniciuc, Biharmonic maps between warped product manifolds, Journal of Geometry and Physics. 57.2 (2007), 449-466.

[2] J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109-160.

[3] Y. Han, S. Feng, Some results of F-biharmonic maps, Acta Mathematica Universitatis Comenianae. 83.1 (2014), 47-66.

[4] S. Herr, T. Lamm, T. Schmid, R. Schnaubelt, Biharmonic wave maps: local wellposedness in high regularity, Nonlinearity, 33.5 (2020), 2270.

[5] G.Y. Jiang, . 2-harmonic maps and their first and second variational formulas. Chinese Ann. Math. Ser. A. 7 (1986), 389-402.

[6] J. Jost, L. Liu, M. Zhu, The qualitative behavior at the free boundary for approximate harmonic maps from surfaces, Mathematische annalen. 374.1-2(2019), 133-177.

[7] J. Jost, C. Zhou, M. Zhu, Vanishing Pohozaev constant and removability of singularities, Journal of Differential Geometry. 111.1 (2019), 91-144.

[8] S. M. Kazemi Torbaghan, M. M. Rezaii, Stability of F-biharmonic maps, Bulletin of the Iranian Mathematical Society. 43.6 (2017), 1657-1669.

[9] Z. Pirbodaghi, M. Mirmohammad Rezaii, S. M. Kazemi Torbaghan, Stability of λ-Harmonic Maps, Mathematics. 6.6 (2018), 94.

[10] M. Rupflin, P. M. Topping, Teichmüller harmonic map flow into nonpositively curved targets, Journal of Differential Geometry. 108.1 (2018), 135-184.

[11] J. Sacks, K. Uhlenbeck, The existence of minimal immersions of 2-spheres, Annals of mathematics.(1981), 1-24.

[12] S. Vigdergauz, I. Elishakoff, Stress minimization around a hole with a stochastically simulated micro-rough edge in a loaded elastic plate, Journal of Mechanics of Materials and Structures. 15.2 (2020), 277-289.