Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611
CJMS. **9** (2) (2020), 210-223

An introduction to topological hyperrings

Mehdi Nodehi, Morteza Norouzi [1] and Omid Reza Dehghan
Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P. O. Box 1339, Bojnord 94531, Iran
m.nodehi@ub.ac.ir, m.norouzi65@yahoo.com, dehghan@ub.ac.ir

Abstract. In this paper, we define topological hyperrings and study their basic concepts which supported by illustrative examples. We show some differences between topological rings and topological hyperrings. Also, by the fundamental relation Γ^*, we indicate the role of complete parts (saturated subsets) and complete hyperrings in topological hyperrings and specially we show that if every (closed) open subset is a complete part in a topological complete hyperring then its fundamental ring is a topological ring. Finally, we study the quotient topology induced by Γ^*-relation on an associated Krasner hyperring obtained by a ring and show that it is isomorphic to a quotient space of the ring by its ideals.

Keywords: Topological hyperring; Γ^*-relation; complete part; complete hyperring.
2010 Mathematics subject classification: 20N20, 22A30.

1. Introduction

In 1956, M. Krasner [14] introduced the notion of hyperrings, as a field of hyperstructures theory defined by Marty [17] in 1934. The hyperrings are algebraic hyperstructures endowed with two (hyper)operations where both are hyperoperations (general hyperrings ([24])), or just one is a hyperoperation and the other is an operation. In Krasner hyperrings

[1]Corresponding author: m.norouzi65@yahoo.com, m.norouzi@ub.ac.ir
Received: 12 June 2020
Revised: 17 July 2020
Accepted: 18 July 2020

[15] the addition is a hyperoperation such that the additive hyperstructure is a canonical hypergroup, while the multiplication structure is a semigroup. A good review of studies on hyperrings can be seen in the book "Hyperrings Theory and Applications" ([7]) written by Davvaz and Leoreanu-Fotea (also can see [18], [20] and [26]).

The concept of topology on hypergroups was presented by R. Ameri in [3]. He introduced the concept of a (pseudo, strong pseudo) topological (transposition) hypergroup and studied the relationships between pseudo topological polygroups and topological polygroups. Later on, this concept was studied by Heidari et al. [8] as a generalization of topological groups. By fundamental relations on a hypergroup, they show that if every open subset of a topological hypergroup is a complete part, then its fundamental group is a topological group. Moreover, topological isomorphism theorems on polygroups were investigated by them in [9]. Salehi Shadkami et al. [22, 23] established connections of complete parts and open sets to obtain some new results in topological polygroups. S. Hoskova-Mayerova [10] introduced the concept of topological hypergroupoids, β-topological hypergroupoids, τ_L-topological hypergroupoids and $\tau_{\aleph}$-topological hypergroupoids by using concepts of the pseudocontinuity and the strong pseudocontinuity. Also, an over review of topological hypergroupoid can be found in [2]. Paratopological polygroups versus topological polygroups were studied in [12]. Moreover, I. Cristea and S. Hoskova- Mayerova defined the concept of fuzzy pseudotopological hypergroupoids in [6].

Topological hyperrings are a generalization of topological rings and also topological hypergroups. In this paper, we give some basic concepts to construct a topology on hyperrings. Then, we define the notion of a topological hyperring which supported by illustrative examples. Also, we mention some differences of topological rings and topological hyperrings. By considering the fundamental relation Γ^*, we study the role of complete parts (saturated subsets) and complete hyperrings in topological hyperrings which shows some difference between topological hypergroups and topological hyperrings. We show that if every (closed) open subset is a complete part in a topological complete hyperring then its fundamental ring is a topological ring. Moreover, we investigate relationships between a topological ring and a associated topological Krasner hyperring and indicate that the topological fundamental quotient of the associated topological Krasner hyperring is isomorphic to a quotient space of the related topological ring by its ideals.

2. Basic concepts of topology on hyperstructures

Let $(R,+,\cdot)$ be a ring and (R,τ) be a topological space. Then $(R,+,\cdot,\tau)$ is a topological ring whenever operations "$+$" and "$\cdot$" from $R\times R$ to R are continuous. Note that we consider the product topology on $R\times R$.

In algebraic hyperstructures theory, we use "$+$" $: R\times R \longrightarrow P^*(R)$, where $P^*(R)$ is the set of all nonempty subsets of R. The map "$+$" is called a *hyperoperation* on R. In this case, we define $A+B=\bigcup_{a\in A, b\in B} a+b$, $x+A=\{x\}+A$ and $A+x=A+\{x\}$, for $x\in R$ and $A,B\in P^*(R)$. We say that $(R,+)$ is a *semihypergroup* if for all x,y,z of R, we have $(x+y)+z=x+(y+z)$. A semihypergroup $(R,+)$ is called a *hypergroup* if $x+R=R+x=R$, for all $x\in R$. A nonempty subset K of a hypergroup $(R,+)$ is called *subhypergroup*, if for all $k\in K$, we have $k+K=K+k=K$.

Definition 2.1. ([7]) A commutative hypergroup $(R,+)$ is called *canonical* if

- (i) there exists $0\in R$, such that $0+x=\{x\}$, for every $x\in R$;
- (ii) for all $x\in R$ there exists a unique $-x\in R$, such that $0\in x+(-x)$;
- (iii) $x\in y+z$ implies $y\in x+(-z)$.

Definition 2.2. ([7]) An algebraic system $(R,+,\cdot)$ is said to be a *(general) hyperring*, if $(R,+)$ is a hypergroup, $(R,\cdot)$ is a semihypergroup, and "$\cdot$" is distributive with respect to "$+$".

A hyperring $(R,+,\cdot)$ is called a *Krasner hyperring* [15], if $(R,+)$ is a canonical hypergroup and $(R,\cdot)$ is a semigroup such that for all $x\in R$, we have $x\cdot 0=0=0\cdot x$, where 0 is a zero element of $(R,+)$.

In a hyperoperation $+: R\times R\to P^*(R)$, assume that R has a topology τ, then we can define a topology on $P^*(R)$ by the way of Hošková [10]. She defined two topologies on $P^*(R)$ which were called upper and lower topologies. In this paper, we choose an upper topology on $P^*(R)$ and consider the collection $\{S_V\}_{V\in\tau}$ as its basis, where $S_V=\{U\in P^*(R): U\subseteq V\}$.

Theorem 2.3. [16] *Let $f: X\to Y$ be a map between topological spaces X and Y. Then the following conditions are equvalent:*

1. *$f: X\to Y$ is continuous;*
2. *for all open subset V of Y, $f^{-1}(V)$ is open in X;*
3. *for all $x\in X$ and all open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $f(U)\subseteq V$.*

Remark 2.4. The above theorem is also true whenever use term of basis elements instead of open sets. Now by previous theorem, hyperoperation

$+ : R \times R \to P^*(R)$ is continuous, if for all basis element S_U, $U \in \tau$, the set $+^{-1}(S_U) = \{V \times W \ : \ +(V, W) \in S_U\}$ is open in $R \times R$. Also we can use the other equivalent condition for continuity of " $+$ ".

In the following we present some properties of hyperstructures and topological preliminaries that we shall use in later. Consider $+ : R\times R \to P^*(R)$ where R endowed with a topology τ and $P^*(R)$ endowed with topology generated by the basis $\{S_V\}_{V\in\tau}$.

Proposition 2.5. *For a hyperoperation $+ : R \times R \to P^*(R)$ the following conditions are equivalent:*

(1) *for any open subset V of R, the set $\{(x, y) \in R\times R : x+y \subseteq V\}$ is open in $R \times R$;*

(2) *for all $x, y \in R$ and any open subset V of R containing $x + y$, there exist open subsets U_x and U_y of R containing x and y, respectively, such that $a + b \subseteq V$, for every $a \in U_x$ and $b \in U_y$.*

Proof. Let $x, y \in R$ and V be an open subset of R such that $x + y \subseteq V$. By (1) there exists basis element $U_x \times U_y$ in $R \times R$ such that $(x, y) \in U_x \times U_y \subseteq \{(x, y) \in R \times R : x + y \subseteq V\}$, $x \in U_x$ and $y \in U_y$, where U_x and U_y are open subsets of R. Now if $(a, b) \in U_x \times U_y$, then $(a, b) \in \{(x, y) \in R \times R : x + y \subseteq V\}$. Hence $a + b \subseteq V$.

Conversely, let V be an open subset of R and $(x_0, y_0) \in \{(x, y) \in R \times R : x + y \subseteq V\}$. Then by (2) there exist open subsets U_{x_0} and U_{y_0} of R containing x_0 and y_0, respectively, such that $a + b \subseteq V$, for every $a \in U_{x_0}$ and $b \in U_{y_0}$. Thus $(x_0, y_0) \in U_{x_0} \times U_{y_0} \subseteq \{(x, y) \in R \times R : x + y \subseteq V\}$. □

Lemma 2.6. *Consider $+ : R \times R \to P^*(R)$ and let U, V and O are open subsets in R. Then the following conditions are equivalent:*

(1) *$+(U \times V) \subseteq S_O$, where $+(U \times V) = \{x + y : x \in U, y \in V\}$;*

(2) *$U + V \subseteq O$, where $U + V = \cup\{x + y : x \in U, y \in V\}$.*

Proof. Let $+(U \times V) \subseteq S_O = \{W \in P^*(R) : W \subseteq O\}$. Then for all $x \in U$ and $y \in V$ we have $x + y \in S_O$, and so $x + y \subseteq O$. Thus $\cup\{x + y : x \in U, y \in V\} \subseteq O$. Hence $U + V \subseteq O$.

Conversely, if $U + V \subseteq O$, then $\cup\{x + y : x \in U, y \in V\} \subseteq O$. Thus for all $x \in U$ and $y \in V$, $\{x + y : x \in U, y \in V\} \subseteq S_O$. □

Theorem 2.7. *Let $+ : R \times R \to P^*(R)$ be a hyperoperation. Then the following conditions are equivalent:*

(1) *for all open subset V of R, the set $\{(x, y) \in R \times R : x + y \subseteq V\}$ is open in $R \times R$;*

(2) *for all $x, y \in R$ and all open subset V of R containing $x + y$, there exist open subsets U_x and U_y of R containing x and y, respectively, such that $U_x + U_y \subseteq V$.*

Proof. It follows from Proposition 2.5 and Lemma 2.6. □

Similar to what happens in Lemma 2.6, for the hyperoperation "/" : $R \times R \to P^*(R)$ defined by $x/y = \{z \in R : x \in z + y\}$ and all open subset U, V and O of R, we have $/(U \times V) = \{x/y : x \in U, y \in V\} \subseteq S_O$ if and only if $U/V = \bigcup\{x/y : x \in U, y \in V\} \subseteq O$.

3. Topological Hyperrings

Definition 3.1. Let $(R, +, \cdot)$ be a hyperring and (R, τ) be a topological space. $(R, +, \cdot, \tau)$ is said to be a topological hyperring if three hyperoperations "$+$", "$\cdot$" and "/" are continuous.

Lemma 3.2. *For every hyperring $(R, +, \cdot)$ with a topology τ, the hyperoperation "$+$" (resp., "$\cdot$" and "/") is continuous if and only if for every $x, y \in R$ and $U \in \tau$ such that $x + y \subseteq U$ (resp., $x \cdot y \subseteq U$ and $x/y \subseteq U$), there exist $V, W \in \tau$ with $x \in V$, $y \in W$ and $V + W \subseteq U$ (resp., $V \cdot W \subseteq U$ and $V/W \subseteq U$).*

Proof. It is straightforward by Theorem 2.7. □

Example 3.3. Every topological ring is a topological hyperring by trivial hyperoperations.

Example 3.4. For any non-empty set R, $(R, +, \cdot)$ is a topological hyperring with every arbitrary topology on R, where $x + y = x \cdot y = R$.

Example 3.5. If R is a Hausdorff topological space, then $(R, +, \cdot)$ is a topological hyperring, where $x + y = x \cdot y = \{x, y\}$.

Example 3.6. If $(R, +, \cdot)$ is a topological ring, then $(R, \oplus, \odot)$ is a topological hyperring, where $x \oplus y = \{x, y\}$ and $x \odot y = \{x \cdot y\}$.

Example 3.7. By considering $B = \{\{a, -a\} : a \in \mathbb{N} \cup \{0\}\}$ as a basis for a topology on $\mathbb{Z}$, $(\mathbb{Z}, \oplus, \odot)$ is a topological hyperring, where $x \oplus y = \{x, y, -x, -y\}$, $x \odot y = \{xy, -xy\}$. Note that every subset A of $\mathbb{Z}$ is an open subset of $\mathbb{Z}$ if and only if A consists of all its opposite elements.

Some properties in topological rings are not valid in topological hyperrings. For instant, if R is a topological ring and U is an open subset of R, then $a + R$ is open in R for every $a \in R$, while it does not hold in topological hyperrings. For example consider $R = \mathbb{R}$ with the standard topology in Example 3.6, then $2 \oplus (0, 1) = \{2\} \cup (0, 1)$ is not open in $\mathbb{R}$ ($(0, 1)$ is the unite open interval). On the other hand, in Example 3.7, $a \oplus V = \bigcup\limits_{y \in V} a \oplus y = \bigcup\limits_{y \in V} \{a, -a, y, -y\} = B_a \cup (\bigcup\limits_{y \in V} B_y)$ is an open subset of $\mathbb{Z}$, for all $a \in \mathbb{Z}$.

Let $\sim$ be an equivalence relation on a topological space X. Then U is open in the quotient space $X/\sim$ if and only if $p^{-1}(U)$ is an open subset of X, where $p : X \to X/\sim$ is the natural projection map. Also, the saturation of $A \subseteq X$ with respect to $\sim$ is the set $\widehat{A} = \{x \in X : \exists a \in A, x \sim a\}$. If $\widehat{A} = A$, then A is called saturated.

Now, consider the Γ-relation ([25]) on the hyperring $(R, +, \cdot)$ which is defined as:

$$x\Gamma y \Leftrightarrow \exists n \in \mathbb{N}, \exists k_i \in \mathbb{N}, \exists (z_{i1}, \ldots, z_{ik_i}) \in R^{k_i}, 1 \leq i \leq n; \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij}.$$

Let Γ^* be the transitive closure of Γ. If $(R, +, \cdot, \tau)$ is a topological hyperring, then $(R/\Gamma^*, \overline{\tau})$ is a topological space, where $\overline{\tau}$ is the quotient topology induced by natural mapping $\pi : R \to R/\Gamma^*$, and $A \subseteq R/\Gamma^*$ is open iff $\pi^{-1}(A) \subseteq R$ is open.

Complete parts were introduced and studied for the first time by Koskas [13]. Afterward, this topic was analyzed by Corsini and Sureau [5] mostly in hypergroups. Also, complete parts and their generalizations were studied on hyperrings theory, for example see [1, 4, 19, 21]. Recall that a subset $A \subseteq R$ is said to be a complete part of R if $A \cap \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij} \neq \emptyset$ implies that $\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij} \subseteq A$, for $n \in \mathbb{N}$, $k_i \in \mathbb{N}$, $(z_{i1}, \ldots, z_{ik_i}) \in R^{k_i}$ and $1 \leq i \leq n$.

Lemma 3.8. *Let $(R, +, \cdot)$ be a hyperring. Then every saturated subset of R is a complete part.*

Proof. Let A be a saturated subset of R such that $A \cap \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij} \neq \emptyset$, for $n, k_i \in \mathbb{N}$ and $(z_{i_1}, \ldots, z_{ik_i}) \in R^{k_i}$, $1 \leq i \leq n$. Then there exists $a \in A$ such that $a \in \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij}$. Thus for all $x \in \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij}$, $x\Gamma^* a$ and so $x \in \widehat{A} = A$, since A is saturated. Hence A is a complete part. $\square$

Recall that a hyperring $(R, +, \cdot)$ is said to be a complete hyperring ([7]) if $\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij} = \Gamma(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij})$, for all $z_{ij} \in R$. As a characterization, $(R, +, \cdot)$ is a complete hyperring if and only if $\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} z_{ij}$ is a complete part of R, for all $z_{ij} \in R$. Hence we have the following:

Corollary 3.9. *In every complete hyperring, a non-empty subset A of R is a complete part if and only if it is saturated.*

Proof. Let $x \in \widehat{A}$. Then there exists $a \in A$ such that $a\Gamma^\star x$. This means that $a = a_1\Gamma a_2\Gamma \cdots \Gamma a_{n-1}\Gamma a_n = x$, for $a_1, \ldots, a_n \in R$. Thus $\{a_t, a_{t+1}\} \subseteq \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_{t+1})_{ij}$, for $1 \leq t \leq n-1$. Since R is a complete hyperring, $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_2)_{ij} = \cdots = \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_n)_{ij}$. Also $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_2)_{ij}) \cap A \neq \emptyset$, which implies that $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_n)_{ij}) \cap A \neq \emptyset$. So $x \in \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_n)_{ij} \subseteq A$, since A is a complete part. Therefor $\widehat{A} = A$, that is A is saturated. $\square$

We recall that a hyperring $(R, +, \cdot)$ is said to be a hyperfield if $(R, \cdot)$ is a hypergroup. Then, similar to what happen for hypergroups ([8]) we have:

Corollary 3.10. *In every hyperfield any non-empty subset is a complete part if and only if it is saturated.*

Proof. In any hyperfield, $\Gamma^* = \Gamma$ (Γ is transitive [4]). $\square$

Proposition 3.11. *Let $(R, +, \cdot)$ be a complete hyperring with topology τ such that every open subset of R is a complete part. If $V \in \tau$, then $\Gamma^*(V) \in \tau$.*

Proof. Let $x \in \Gamma^*(V)$. Then there exists $v \in V$ such that $x\Gamma^* v$. Thus there exists an open subset U such that $v \in U \subseteq V$. On the other hand $x = a_1\Gamma a_2\Gamma \cdots \Gamma a_{n-1}\Gamma a_n = v$, which means that $\{a_t, a_{t+1}\} \subseteq \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_{t+1})_{ij}$, for $1 \leq t \leq n-1$. By $v \in U \cap (\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_n)_{ij})$ we have $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_n)_{ij} \subseteq U$. Moreover R is a complete hyperring and so $x \in \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_1)_{ij} = \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i}(z_n)_{ij} \subseteq U$, thus $x \in U$. Clearly $x \in U \subseteq \Gamma^*(U) \subseteq \Gamma^*(V)$. Hence $\Gamma^*(V)$ is an open subset of R. $\square$

Corollary 3.12. *Let $(R, +, \cdot)$ be a hyperfield with topology τ such that every open subset of R is a complete part. If $V \in \tau$, then $\Gamma^*(V) \in \tau$.*

Proof. It is straightforward by $\Gamma^* = \Gamma$. $\square$

The above results imply that the natural mapping $\pi : R \to R/\Gamma^*$ is an open mapping in a topological complete hyperring or topological hyperfield, where any open subset is a complete part.

Note that in any (topological) hyperring, complement of every complete part (saturated subset) is a complete part (saturated subset). Indeed, let $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} z_{ij})\cap A^c \neq \emptyset$. If $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} z_{ij})\cap A \neq \emptyset$, then $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} z_{ij} \subseteq A$ (A is a complete part), and so $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} z_{ij}) \cap A^c = \emptyset$, which is a contradiction. Hence $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} z_{ij}) \cap A = \emptyset$, i.e. $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} z_{ij} \subseteq A^c$. Also, let A be a saturated subset of R, $x \in A^c$ and $x\Gamma^* y$. If $y \notin A^c$, then $y \in A = \widehat{A}$. Thus $x \in \widehat{A} = A$, which is a contradiction. Hence $y \in A^c$ and so A^c is saturated.

Moreover, it is well-known that Γ^* is the smallest equivalence relation on a hyperring $(R, +, \cdot)$ such that $(R/\Gamma^*, \oplus, \odot)$ is a classical ring, where $\Gamma^*(x) \oplus \Gamma^*(y) = \{\Gamma^*(z) \mid z \in \Gamma^*(x) + \Gamma^*(y)\}$ and $\Gamma^*(a) \odot \Gamma^*(b) = \{\Gamma^*(d) \mid d \in \Gamma^*(a) \cdot \Gamma^*(b)\}$. Thus, $\oplus$ and $\odot$ can be seen as $\Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z)$, for all $z \in \Gamma^*(x) + \Gamma^*(y)$ and $\Gamma^*(a) \odot \Gamma^*(b) = \Gamma^*(d)$, for all $d \in \Gamma^*(a) \cdot \Gamma^*(b)$. Hence, $(R/\Gamma^*, \oplus, \odot)$ is called the fundamental ring obtained by the Γ^*-relation. Hence, we have:

Theorem 3.13. *Let $(R, +, \cdot, \tau)$ be a topological complete hyperring such that every (closed) open subset of R is a complete part. Then $(R/\Gamma^*, \oplus, \odot, \tau^*)$ is a topological ring, where τ^* is the quotient topology on R/Γ^*.*

Proof. The proof is similar to what happened in topological hypergroups ([8]). □

Lemma 3.14. *Let R be a topological hyperring such that every saturated subset of R is open. Then $\pi : R \to R/\Gamma^*$ is an open mapping.*

Proof. For every open subset V of R, $\widehat{V} = \pi^{-1}(\pi(V))$ is a saturated and so it is open, by the hypothesis. This completes the proof. □

Corollary 3.15. *If R is a topological hyperring such that every saturated subset of R is open, then $(R/\Gamma^*, \oplus, \odot, \tau^*)$ is a topological ring.*

Proof. It is concluded by Lemma 3.14. □

We know that a non-empty subset I of a hyperring $(R, +, \cdot)$ is a *hyperideal*, if $x, y \in I$ implies $x + y \subseteq I$ and for every $r \in R$ we have $r \cdot x \cup x \cdot r \subseteq I$. Let $(R, +, \cdot)$ be a topological hyperring and I be a hyperideal of R. Then $R/I = \{x + I : x \in R\}$ is a hyperring with two hyperoperations $(x + I) \oplus (y + I) = \{z + I \mid z \in (x + I) + (y + I)\}$ and $(x + I) \odot (y + I) = \{t + I \mid t \in (x + I) \cdot (y + I)\}$. Also, R/I is a topological hyperring with respect to the quotient topology induced by $\pi : R \to R/I$. Note that every open subset of R/I is in the form $\{u + I : u \in U\}$ for some open subset U of R ([8]).

Now let $(R, +, \cdot)$ be a commutative ring with identity and put $\overline{R} = \{\overline{x} = \{x, -x\} : x \in R\}$. Define $\overline{x} \oplus \overline{y} = \{\overline{x+y}, \overline{x-y}\}$ and $\overline{x} \odot \overline{y} = \overline{x \cdot y}$ on $\overline{R}$. Then $(\overline{R}, \oplus, \odot)$ is a Krasner hyperring by [7], $\overline{R}$ is said to be the associated Krasner hyperring with respect to the ring R. Hence we have:

Lemma 3.16. *A non-empty subset I of R is an ideal of R if and only if $\overline{I}$ is a hyperideal of $\overline{R}$, where $\overline{I} = \{\overline{x} : x \in I\}$.*

Proof. Let $\overline{x}, \overline{y} \in \overline{I}$. Then $\overline{x} \oplus \overline{y} = \{\overline{x+y}, \overline{x-y}\} \subseteq \overline{I}$, since $x+y, x-y \in I$ (note that $\overline{x} \in \overline{I}$ implies that there exists $t \in I$ such that $\overline{x} = \bar{t}$. Then $\{x, -x\} = \{t, -t\}$ and so $x = t$ or $x = -t$. Hence, $x + y, x - y \in I$). Similarly, $\overline{r} \odot \overline{x} = \overline{r \cdot x} \subseteq \overline{I}$. Thus $\overline{I}$ is a hyperideal of $\overline{R}$ by [7, Lemma 3.2.3].

Conversely, let $x, y \in I$. Then

$$\begin{array}{rcl} \overline{x} \oplus \overline{y} \subseteq \overline{I} & \Rightarrow & \{\overline{x+y}, \overline{x-y}\} \subseteq \overline{I} \\ & \Rightarrow & \overline{x+y}, \overline{x-y} \in \overline{I} \\ & \Rightarrow & \overline{x+y} = \bar{t}, \overline{x-y} = \bar{s}, \text{ for some } \bar{t}, \bar{s} \in \overline{I} \\ & \Rightarrow & x+y, x-y \in I. \end{array}$$

Similarly, $r \cdot x \in I$, for all $r \in R$. $\square$

Since $\overline{R}/\overline{I}$ is a Krasner hyperring endowed with $(\overline{x} \oplus \overline{I}) \boxplus (\overline{y} \oplus \overline{I}) = \{\overline{z} \oplus \overline{I} : \overline{z} \in (\overline{x} \oplus \overline{I}) \oplus (\overline{y} \oplus \overline{I})\}$ and $(\overline{x} \oplus \overline{I}) \boxdot (\overline{y} \oplus \overline{I}) = \overline{x} \odot \overline{y} \oplus \overline{I} = \overline{x \cdot y} \oplus \overline{I}$, in the following we show that the fundamental ring of $\overline{R}/\overline{I}$ is isomorphic to a quotient of R:

Consider $\Gamma^*(\overline{R}/\overline{I})$ and define $L_x = \sum\limits_{\overline{y} \oplus \overline{I} \in \Gamma^*(\overline{x} \oplus \overline{I})} \langle \{\prod\limits_{j=1}^{k_i} t_{ij};\ 1 \leq i \leq n\} \rangle +$ I such that $\{\overline{x} \oplus \overline{I}, \overline{y} \oplus \overline{I}\} \subseteq \boxplus_{i=1}^{n} \boxdot_{j=1}^{k_i} (\overline{t_{ij}} \oplus \overline{I})$. If there exists $z_{ij} \in R$ such that $\{\overline{x} \oplus \overline{I}, \overline{y} \oplus \overline{I}\} \subseteq \boxplus_{i=1}^{n'} \boxdot_{j=1}^{k'_i} (\overline{z_{ij}} \oplus \overline{I})$, then we consider $\langle \{\prod\limits_{j=1}^{k_i} t_{ij};\ 1 \leq i \leq n\} \rangle \cap \langle \{\prod\limits_{j=1}^{k'_i} z_{ij};\ 1 \leq i \leq n'\} \rangle$. Now, put $L = \sum\limits_{x \in R} L_x$. Clearly, L is an ideal of R.

Theorem 3.17. *Let $(R, +, \cdot)$ be a commutative ring and $\overline{I}$ be a hyperideal of $(\overline{R}, \oplus, \odot)$. Then $\overline{R}/\overline{I}/\Gamma^* \cong R/L$, for some ideal L of R.*

Proof. Define a mapping $f : \overline{R}/\overline{I}/\Gamma^* \to R/L$ by $f(\Gamma^*(\overline{x} \oplus \overline{I})) = x + L$, where $L = \sum\limits_{x \in R} L_x$ (which presented above). At first we show that f is well-defined. Let $\Gamma^*(\overline{x} \oplus \overline{I}) = \Gamma^*(\overline{y} \oplus \overline{I})$. Then there exist $z_1, \ldots, z_n \in R$

such that $\overline{x}\oplus\overline{I}=\overline{z}_1\oplus\overline{I}\ \Gamma\ \overline{z}_2\oplus\overline{I}\ \Gamma\ \cdots\ \Gamma\overline{z}_{n-1}\oplus\overline{I}\ \Gamma\overline{z}_n\oplus\overline{I}=\overline{y}\oplus\overline{I}$. By $\overline{z}_1\oplus\overline{I}\ \Gamma\ \overline{z}_2\oplus\overline{I}$ we have $\{\overline{z}_1\oplus\overline{I},\overline{z}_2\oplus\overline{I}\}\subseteq\boxplus_{i=1}^{n}\boxdot_{j=1}^{k_i}(\overline{t}_{ij}\oplus\overline{I})$, for $t_{ij}\in R$. Thus $z_1-z_2\in\langle t_{11}\dots t_{1k_1}\rangle+\dots+\langle t_{n1}\dots t_{nk_n}\rangle+I\subseteq L$, which implies that $z_1+L=z_2+L$, i.e. $f(\Gamma^*(\overline{z}_1\oplus\overline{I}))=f(\Gamma^*(\overline{z}_2\oplus\overline{I}))$. Similarly, it follows that $f(\Gamma^*(\overline{z}_2\oplus\overline{I}))=f(\Gamma^*(\overline{z}_3\oplus\overline{I}))=\dots=f(\Gamma^*(\overline{z}_n\oplus\overline{I}))$, and so $f(\Gamma^*(\overline{x}\oplus\overline{I}))=f(\Gamma^*(\overline{y}\oplus\overline{I}))$. Clearly, f is an epimorphism. Now let $\Gamma^*(\overline{x}\oplus\overline{I})\in\ker f$. Then $f(\Gamma^*(\overline{x}\oplus\overline{I}))=L$, which implies that $x\in L$. Hence $x\in\sum\limits_{i=1}^{n}r_i(t_{i1}\dots t_{ik_i})+I$, for $t_{i1}\dots t_{ik_i}\in R$, where $\{\overline{x}\oplus\overline{I},\overline{y}\oplus\overline{I}\}\subseteq\boxplus_{i=1}^{n}\boxdot_{j=1}^{k_i}(\overline{t}_{ij}\oplus\overline{I})$, for $\overline{y}\oplus\overline{I}\in\Gamma^*(\overline{x}\oplus\overline{I})$. Thus $\{\overline{x}\oplus\overline{I},\overline{I}\}\subseteq\boxplus_{i=1}^{n}\boxdot_{j=1}^{k_i}(\overline{q}_{ij}\oplus\overline{I})$, such that $q_{i1}=r_it_{i1}$ and $q_{ij}=t_{ij}$, $1\le i\le n$, $2\le j\le k_i$. Therefore $\Gamma^*(\overline{x}\oplus\overline{I})=\Gamma^*(\overline{I})=0_{\overline{R}/\overline{I}/\Gamma^*}$, and so $\ker f=\{0_{\overline{R}/\overline{I}/\Gamma^*}\}$, i.e. f is one-to-one. Consequently, f is an isomorphism and the proof is completed. □

For instance, consider the ring $(\mathbb{Z}_4,+,\cdot)$. Then $\overline{\mathbb{Z}}_4=\{\overline{0},\overline{1},\overline{2}\}$ which $\overline{I}=\{\overline{0},\overline{2}\}$ is a hyperideal of $\mathbb{Z}_4$. Also, $\overline{\mathbb{Z}}_4/\overline{I}=\{\overline{I},\overline{1}\oplus\overline{I}\}$ and $\overline{\mathbb{Z}}_4/\overline{I}/\Gamma^*=\{\Gamma^*(\overline{I}),\Gamma^*(\overline{1}\oplus\overline{I})\}$. Moreover, $L=L_0=L_1=I$ and so $\mathbb{Z}_4/L=\{I,1+I\}$.

We started from a commutative ring $(R,+,\cdot)$ and defined the associated Krasner hyperring $(\overline{R},\oplus,\odot)$. Now if we have a topological commutative ring $(R,+,\cdot,\tau)$, then it can be induced a topology on $\overline{R}$ from R such that both hyperoperations $\oplus$ and $\odot$ are continuous.

Define $\overline{\tau}=\{\overline{U}:U\in\tau\}$ where $\overline{U}=\{\overline{x}=\{x,-x\}:x\in U\}$. Then $\overline{\tau}$ is a topology on $\overline{R}$. Now we show that $(\overline{R},\oplus,\odot,\overline{\tau})$ is a topological hyperring, that is $\oplus$, $\odot$ are continuous. Consider $\{\overline{U}\times\overline{V}:\overline{U},\overline{V}\in\overline{\tau}\}$ and $\{\overline{R}_{\overline{W}}\}_{\overline{W}\in\overline{\tau}}$ as the basis of $\overline{R}\times\overline{R}$ and $P^*(\overline{R})$, respectively. Then $\oplus$ is continuous if and only if $\{(\overline{x},\overline{y}):\overline{x}\oplus\overline{y}\subseteq\overline{V}\}$ is an open subset of $\overline{R}\times\overline{R}$ for all $\overline{V}\in\overline{\tau}$. Let $(\overline{x},\overline{y})\in\{(\overline{x},\overline{y}):\overline{x}\oplus\overline{y}\subseteq\overline{V}\}$. Then $\overline{x}\oplus\overline{y}\subseteq\overline{V}$ and so $\{\overline{x+y},\overline{x-y}\}\subseteq\overline{V}$. Thus $\overline{x+y}\in\overline{V}$ and $\overline{x-y}\in\overline{V}$. Hence $(x+y\in V$ or $-x-y\in V)$ and $(x-y\in V$ or $y-x\in V)$. Now we have 4 cases:

(1) $x+y\in V$ and $x-y\in V$;
(2) $x+y\in V$ and $y-x\in V$;
(3) $-x-y\in V$ and $x-y\in V$;
(4) $-x-y\in V$ and $y-x\in V$.

Case 1: $x+y\in V$ and $x-y\in V$. Since " + " is continuous, so there exist open subsets W_1,W_2 containing x,y, respectively, such that $x+y\in W_1+W_2\subseteq V$. Also " − " is continuous, then there exist open subsets U_1,U_2 containing x,y, respectively, such that $x-y\in U_1-U_2\subseteq V$. Thus $x\in W_1\cap U_1\subseteq R$ and $y\in W_2\cap U_2\subseteq R$ and so $(\overline{x},\overline{y})\in\big(\overline{W_1\cap U_1}\big)\times\big(\overline{W_2\cap U_2}\big)$. Furthermore, $\overline{W_1\cap U_1}\times\overline{W_2\cap U_2}\subseteq\{(\overline{x},\overline{y}):\overline{x}\oplus\overline{y}\subseteq\overline{V}\}$, because of $(\overline{a},\overline{b})\in\overline{W_1\cap U_1}\times\overline{W_2\cap U_2}$, so ($a$ or $-a\in W_1\cap U_1$) and (b or

$-b \in W_2 \cap U_2$). If $a \in W_1 \cap U_1$ and $b \in W_2 \cap U_2$, then $a+b \in W_1+W_2 \subseteq V$, so $\overline{a+b} \in \overline{V}$, also $a-b \in U_1-U_2 \subseteq V$. Thus $\overline{a-b} \in \overline{V}$. Hence $\overline{a} \oplus \overline{b} \subseteq \overline{V}$. Similarly, if ($a \in W_1 \cap U_1$ and $-b \in W_2 \cap U_2$) or ($-a \in W_1 \cap U_1$ and $b \in W_2 \cap U_2$) or ($-a \in W_1 \cap U_1$ and $-b \in W_2 \cap U_2$), then $\overline{a} \oplus \overline{b} \subseteq \overline{V}$. Therefore in this case $\{(\overline{x}, \overline{y}) : \overline{x} \oplus \overline{y} \subseteq \overline{V}\}$ is an open subset of $\overline{R} \times \overline{R}$ for all $\overline{V} \in \overline{\tau}$. In a similar way, the other cases are proved.

Now we show that the operation $\odot$ defined by $\overline{x} \odot \overline{y} = \overline{xy}$ is continuous. Let $\overline{U}$ be an open subset of $\overline{R}$. Consider $\odot^{-1}(\overline{U}) = \{(\overline{x}, \overline{y}) : \overline{x} \odot \overline{y} \in \overline{U}\}$. For $(\overline{x}, \overline{y}) \in \odot^{-1}(\overline{U})$ we have $\overline{x} \odot \overline{y} \in \overline{U}$, so $\overline{xy} \in U$. Thus $xy \in U$ or $-xy \in U$. If $xy \in U$, then continuity of "$\cdot$" in R implies that there exist open subsets W_1, W_2 of R containing x, y, respectively, such that $xy \in W_1 \cdot W_2 \subseteq U$. Thus $(\overline{x}, \overline{y}) \in \overline{W_1} \times \overline{W_2}$ and $\overline{W_1} \times \overline{W_2} \subseteq \{(\overline{x}, \overline{y}) : \overline{x} \odot \overline{y} \in \overline{U}\}$. If $(\overline{a}, \overline{b}) \in \overline{W_1} \times \overline{W_2}$, then ($a$ or $-a \in W_1$) and (b or $-b \in W_2$). It is easy to verify that in any cases $\overline{a} \odot \overline{b} \in \overline{U}$. If $-xy \in U$, then the same result is similarly obtained. Therefore $(\overline{x}, \overline{y}) \in \overline{W_1} \times \overline{W_2} \subseteq \odot^{-1}(\overline{U})$ and it means that $\odot^{-1}(\overline{U})$ is open and so $\odot$ is continuous.

Theorem 3.18. *Consider the associated Krasner hyperring $(\overline{R}, \oplus, \odot)$ and the hyperideal $\overline{I}$ of $\overline{R}$. Then $\pi : \overline{R}/\overline{I} \to \overline{R}/\overline{I}/\Gamma^*$ is an open map.*

Proof. Let A be an open subset of $\overline{R}/\overline{I}$, i.e. $A = \{\overline{u} \oplus \overline{I} : \overline{u} \in \overline{U}\}$, for some open subset $\overline{U}$ of $\overline{R}$. We must show that $\pi(A)$ is open in $\overline{R}/\overline{I}/\Gamma^*$. For this, we show that $\pi^{-1}(\pi(A))$ is open in $\overline{R}/\overline{I}$, more precisely, $\pi^{-1}(\pi(A)) = B$, where $B = \{\overline{n} \oplus (\overline{u} \oplus \overline{I}) : \overline{u} \in \overline{U}, \overline{n} \in \overline{L}\}$. Suppose $\overline{t} \oplus \overline{I} \in \pi^{-1}(\pi(A))$. Then

$$\begin{aligned}
\pi(\overline{t} \oplus \overline{I}) \in \pi(A) \quad &\Rightarrow \quad \exists \overline{u} \in \overline{U};\ \ \pi(\overline{t} \oplus \overline{I}) = \pi(\overline{u} \oplus \overline{I}) \\
&\Rightarrow \quad \Gamma^*(\overline{t} \oplus \overline{I}) = \Gamma^*(\overline{u} \oplus \overline{I}) \\
&\Rightarrow \quad \{\overline{t} \oplus \overline{I}, \overline{u} \oplus \overline{I}\} \subseteq \boxplus_{i=1}^{n} \boxdot_{j=1}^{k_i} (\overline{z_{ij}} \oplus \overline{I}).
\end{aligned}$$

Similar to the proof of Theorem 3.17, it follows that:

$$\begin{aligned}
t - u \in L \quad &\Rightarrow \quad t \in u + L \\
&\Rightarrow \quad \overline{t} \in \overline{u+L} = \overline{u} \oplus \overline{L} \\
&\Rightarrow \quad \overline{t} \oplus \overline{I} \in \overline{u+L} \oplus \overline{I} = \overline{u} \oplus \overline{I} \oplus \overline{L} \\
&\Rightarrow \quad \exists \overline{n} \in \overline{L};\ \ \overline{t} \oplus \overline{I} = \overline{u} \oplus \overline{I} \oplus \overline{n} \in \{\overline{n} \oplus (\overline{u} \oplus \overline{I}) : \overline{u} \in \overline{U}, \overline{n} \in \overline{L}\}.
\end{aligned}$$

Thus $\pi^{-1}(\pi(A)) \subseteq B$. Conversely, let $\overline{n} \oplus (\overline{u} \oplus \overline{I}) \in B$. Then $\overline{n} \in \overline{L}$ and so $n \in L$. Thus $n \in \sum_{i=1}^{n} r_i(t_{i1} \ldots t_{ik_i}) + I$, where $t_{i1} \ldots t_{ik_i} \in R$. Hence $\overline{n} \oplus \overline{I}\ \Gamma^* \overline{I}$ and so $\overline{n} \oplus (\overline{u} \oplus \overline{I})\ \Gamma^*\ \overline{u} \oplus \overline{I}$. This means that $\overline{n} \oplus (\overline{u} \oplus \overline{I}) \in \pi^{-1}(\pi(\{\overline{u} \oplus \overline{I} : \overline{u} \in \overline{U}\})) = \pi^{-1}(\pi(A))$. Therefore $\pi(A)$ is open in $\overline{R}/\overline{I}/\Gamma^*$. ◻

Proposition 3.19. *Let $(R,+,\cdot,\tau)$ be a commutative topological ring and $(\overline{R},\oplus,\odot,\overline{\tau})$ be the induced topological Krasner hyperring. Then $(\overline{R}/\overline{I}/\Gamma^*,\boxplus_{\Gamma^*},\boxdot_{\Gamma^*})$ is a topological ring, for every ideal I of R.*

Proof. We know $(\overline{R}/\overline{I}/\Gamma^*,\boxplus_{\Gamma^*},\boxdot_{\Gamma^*})$ is a ring. Hence we prove that the mappings $\boxplus_{\Gamma^*}$ and $\boxdot_{\Gamma^*}$ are continuous. Suppose that A is an open subset of $\overline{R}/\overline{I}/\Gamma^*$ such that $\pi(\overline{x}\oplus\overline{I})\boxplus_{\Gamma^*}\pi(\overline{y}\oplus\overline{I})\in A$. We know $\pi:\overline{R}/\overline{I}\longrightarrow\overline{R}/\overline{I}/\Gamma^*$ is a continuous map, because of $\overline{R}/\overline{I}/\Gamma^*$ has quotient topology induced from $\overline{R}/\overline{I}$. Thus $\pi^{-1}(A)$ is an open subset of $\overline{R}/\overline{I}$ containing $\overline{x}\oplus\overline{y}\oplus\overline{I}$. So there exists an open subset U of R such that $\pi^{-1}(A)=\{\overline{z}\oplus\overline{I}:\overline{z}\in\overline{U}\}$. If $\bar{t}\in\bar{x}\oplus\bar{y}$, then $\bar{t}\oplus\bar{I}\subseteq\bar{x}\oplus\bar{y}\oplus\bar{I}$. Hence, $\pi(\bar{t}\oplus\bar{I})=\Gamma^*(\bar{t}\oplus\bar{I})=\Gamma^*(\bar{x}\oplus\bar{y}\oplus\bar{I})=\pi(\bar{x}\oplus\bar{y}\oplus\bar{I})\in A$. So, $\bar{t}\oplus\bar{I}\in\pi^{-1}(A)$ and we have $\bar{t}\oplus\bar{I}=\bar{z}\oplus\bar{I}$ such that $\bar{z}\in\bar{U}$. Now, $\bar{t}\in\bar{t}\oplus\bar{I}=\bar{z}\oplus\bar{I}\subseteq\bar{U}\oplus\bar{I}$. Then $\bar{x}\oplus\bar{y}\subseteq\bar{U}\oplus\bar{I}$. Since $\bar{U}\oplus\bar{I}$ is open, then there exist open subsets $\bar{V}$ and $\bar{W}$ containing $\bar{x}$ and $\bar{y}$ such that $\bar{V}\oplus\bar{W}\subseteq\bar{U}\oplus\bar{I}$. Now, by Theorem 3.18, π is an open map, thus $\pi(\{\overline{v}\oplus\overline{I}:\overline{v}\in\overline{V}\})$ and $\pi(\{\overline{w}\oplus\overline{I}:\overline{w}\in\overline{W}\})$ are open in $\overline{R}/\overline{I}/\Gamma^*$ containing $\Gamma^*(\overline{x}\oplus\overline{I})$ and $\Gamma^*(\overline{y}\oplus\overline{I})$, respectively. Hence

$$\begin{aligned}\pi(\{\overline{v}\oplus\overline{I}:\overline{v}\in\overline{V}\})\boxplus_{\Gamma^*}\pi(\{\overline{w}\oplus\overline{I}:\overline{w}\in\overline{W}\}) &= \{\pi(\overline{v}\oplus\overline{w}\oplus\overline{I}):\overline{v}\in\overline{V},\overline{w}\in\overline{W}\}\\ &\subseteq \{\pi(\overline{z}\oplus\overline{I}):\overline{z}\in\overline{U}\}\\ &\subseteq \pi(\pi^{-1}(A))\\ &\subseteq A.\end{aligned}$$

Therefore the mapping $\boxplus_{\Gamma^*}$ is continuous. Similarly we can prove that the mapping $\boxdot_{\Gamma^*}$ is continuous. $\square$

Proposition 3.20. *Let $(R,+,\cdot,\tau)$ be a topological commutative ring and I be an ideal of R. Then there exists an ideal L of R such that the topological rings R/L and $\overline{R}/\overline{I}/\Gamma^*$ are homeomorphic.*

Proof. Define the map $f:\overline{R}/\overline{I}/\Gamma^*\longrightarrow R/L$ where L is the defined ideal in Theorem 3.17. It is enough to show that f is open and continuous. Suppose that A is open in R/L. Thus $A=\{z+L:z\in U\}$, where U is open in R. It is sufficient to show that $\pi^{-1}(f^{-1}(A))$ is open in $\overline{R}/\overline{I}$. Let $\overline{x}\oplus\overline{I}\in\pi^{-1}(f^{-1}(A))$. Then $\pi(\overline{x}\oplus\overline{I})\in f^{-1}(A)$, and so $f(\pi(\overline{x}\oplus\overline{I}))\in A$. Thus $x+L\in A$. This means that there exists $z\in U$ such that $x+L=z+L$. So there exists $t\in L$ such that $x+t=z$. Because of "$+$" is continuous in R, there exist open subsets V,W of R containing x,t, respectively, such that $V+W\subseteq U$. We claim that $B=\{\overline{v}\oplus\overline{I}:\overline{v}\in\overline{V}\}\subseteq\pi^{-1}(f^{-1}(A))$. If $\overline{v}\oplus\overline{I}\in B$, then $f(\pi(\overline{v}\oplus\overline{I}))=v+L=(v+t)+L\in A$, so $\overline{v}\oplus\overline{I}\in\pi^{-1}(f^{-1}(A))$. Hence B is an open subset of $\overline{R}/\overline{I}$ contains $\overline{v}\oplus\overline{I}$ $(v\in V)$ such that $B\subseteq\pi^{-1}(f^{-1}(A))$. So $\overline{v}\oplus\overline{I}$ is an interior point of $\pi^{-1}(f^{-1}(A))$. Thus $\pi^{-1}(f^{-1}(A))$ is open in $\overline{R}/\overline{I}$. It is concluded that $f^{-1}(A)$ is open in $\overline{R}/\overline{I}/\Gamma^*$. Therefore f is continuous. Now suppose that A is an open subset of $\overline{R}/\overline{I}/\Gamma^*$ and $x+L\in f(A)$. Then $\Gamma^*(\overline{x}\oplus\overline{I})\in A$. Thus there exists an open subset B of $\overline{R}/\overline{I}/\Gamma^*$ such that $\Gamma^*(\overline{x}\oplus\overline{I})\in B\subseteq A$. Hence there exists an open subset U of R such that $\pi^{-1}(B)=\{\overline{z}\oplus\overline{I}:\overline{z}\in\overline{U}\}$. We claim that $\{z+L:z\in U\}\subseteq f(A)$. If $z\in U$, then $z+L=f(\Gamma^*(\overline{x}\oplus\overline{I}))\in f(B)\subseteq f(A)$. So $f(A)$ is open in R/L. Hence $f^{-1}(A)$ is open. Therefore f is open and the proof is completed. □

References

[1] A. Adineh Zadeh, M. Norouzi and I. Cristea, A comparison of complete parts on m-idempotent hyperrings, *Symmetry,* **12** *(2020) 554.*

[2] M. Al Tahan, S. Hoskova-Mayerova and B. Davvaz, An overview of topological hypergroupoids, *J. Intell. Fuzzy Syst.,* **34** *(3) (2018) 1907-1916.*

[3] R. Ameri, Topological (transposition) hypergroups, *Ital. J. Pure Appl. Math.,* **13** *(2003) 171–176.*

[4] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, Transitivity of Γ-relation on hyperfields, *Bull. Math. de la Soci. des Sci. Math. Roumanie,* **51** *(3) (2008) 233-243.*

[5] P. Corsini, Prolegomena of hypergroup theory, Aviani Editore, Italy, (1993).

[6] I. Cristea and S. Hoskova-Mayerova, Fuzzy pseudotopological hypergroupoids, *Iran. J. Fuzzy Syst.,* **6** *(4) (2009) 11–19.*

[7] B. Davvaz and V. Leoreanu-Fotea, Hyperring theory and applications, International Academic Press, USA, (2007).

[8] D. Heidari, B. Davvaz and S.M.S. Modarres, Topological hypergroups in the sense of Marty, *Commun. Algebra,* **42** *(2014) 4712–4721.*

[9] D. Heidari, B. Davvaz and S.M.S. Modarres, Topological polygroups, *Bull. Malays. Math. Sci. Soc.,* **39** *(2016) 707-721.*

[10] S. Hoskova-Mayerova, Topological hypergroupoids, *Comput. Math. Appl.,* **64** *(2012) 2845–2849.*

[11] S. Hoskova-Mayerova, An Overview of topological and fuzzy topological hypergroupoids, *Ratio Math.,* **33** *(2017) 21-38.*

[12] J. Jamalzadeh, Paratopological polygroups versus topological polygroups, *Filomat,* **32** *(8) (2018) 2755-2761.*

[13] M. Koskas, Groupoids, demi-hypergroupes et hypergroupes, *J. Math. Pures Appl.,* **49** *(9) (1970) 155-192.*

[14] M. Krasner, Approximation des corps values complets de caracterislque $p\neq 0$ par ceux de caracteristique 0, In Colloque dAlgebre Superleure (Bruxelles), (1956).

[15] M. Krasner, A class of hyperrings and hyperfields, *Int. J. Math. Math. Sci.,* **6** *(2) (1983) 307-311.*

[16] J. R. Munkres, Topology, A first course, Prentice Hall of India, Private Limited, (1996).

[17] F. Marty, Sur une generalization de la notion de groupe, 8^{iem} *congres des Mathematiciens Scandinaves, Stockholm (1934) 45-49.*

[18] Ch. G. Massouros, On the theory of hyperrings and hyperfields, *Algebra Logika,* **24** *(1985) 728–742.*

[19] S. Mirvakili, S. M. Anvariyeh and B. Davvaz, On α-relation and transitivity conditions of α, *Commun. Algebra,* **36** *(5) (2008) 1695-1703.*

[20] A. Nakassis, Recent results in hyperring and hyperfield theory, *Int. J. Math. Math. Sci.,* **11** *(1988) 209-220.*

[21] M. Norouzi and I. Cristea, Transitivity of the ε_m-relation on (m-idempotent) hyperrings, *Open. Math.,* **16** *(2018) 1012-1021.*

[22] M. Salehi Shadkami, M.R. Ahmadi Zand and B. Davvaz, Left big subsets of topological polygroups, *Filomat,* **30** *(12) (2016) 3139-3147.*

[23] M. Salehi Shadkami, M.R. Ahmadi Zand and B. Davvaz, The role of complete parts in topological polygroups, *Int. J. Anal. Appl.,* **11** *(1) (2016) 54–60.*

[24] S. Spartalis, A class of hyperrings, *Riv. Mat. Pura Appl.,* **4** *(1989) 55–64.*

[25] T. Vougiouklis, The fundamental relation in hyperrings. The general hyperfield, *Algebraic hyperstructures and applications (Xanthi, 1990), World Sci. Publ., Teaneck, NJ, 1991, 203-211.*

[26] T. Vougiouklis, Hyperstructures and their Representations, Hadronic Press Inc., (1994).