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Construction of Closure Operations in a Category of Presheaves

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Abstract. We construct some types of universal closure operations induced by certain collection of morphisms. For this purpose, we use Lawvere-Tierney topologies and universal closure operations that correspond to each other to establish the equivalent conditions over the collection of morphisms. In this way we use multiple sieves instead of principal sieves for constructing results. Examples are also given to illustrate the established results.

Keywords: Class of Morphisms, Lawvere-Tierney Topology, Closure Operation, Presheaf.

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1. Introduction

Closure operations (also closure operators) have been used in broad areas of algebra, Birkhoff [2, 3] and Pierce [13]. Also Kuratowski [11], and Čech [4], studied closure operations intensively in topology. Early appearances of closure operators can be found in logic by Hertz and Tarski, see [7, 14], before Birkhoff's book on lattice theory ([2]) led to more focused investigations on the subject.

Category theory provides a variety of notions which expand on the lattice theoretic concept of closure operation, see [10, 6]. The notions of

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Grothendieck topology and Lawvere-Tierney topology provide standard tools in sheaf and topos theory and are most conveniently described by particular closure operators [9, 12].

A Lawvere-Tierney topology on a topos is a way of saying that something is right locally. Unlike a Grothendieck topology, this is true directly at the logic stage, defining a geometric logic. In fact, it is a generalization of Grothendieck topology in this sense: If C is a small category, then choosing a Grothendieck topology on C and a Lawvere–Tierney topology in the presheaf topos $Set^{C^{op}}$ on C is equivalent.

Throughout this article, let $\mathcal{X}$ be a small category and $\mathcal{M}$ be a set of morphisms of $\mathcal{X}$. The collection, $\mathcal{X}_1/x$, of all the $\mathcal{X}$-morphisms with codomain x is a preordered class by the relation $f \leq g$ if there exists a morphism h such that $f = g \circ h$. The equivalence relation generated by this preorder is $f \sim g$ if $f \leq g$ and $g \leq f$. For a class $\mathcal{M}$ of $\mathcal{X}$-morphisms, we write $f \sim \mathcal{M}$ whenever $f \sim m$ for some $m \in \mathcal{M}$. We say $\mathcal{M}$ is saturated provided that $f \in \mathcal{M}$ whenever $f \sim \mathcal{M}$.

Definition 1.1. ([12]). Given an object x in the category $\mathcal{C}$, a sieve on x is a set S of arrows with codomain x such that $f \in S$ and the composite $f \circ h$ is defined implies $f \circ h \in S$.

Domain and codomain of a morphism f denoted by $d_0 f$ and $d_1 f$ respectively. Recall that ([12]) a sieve in $\mathcal{X}$ generated by a morphism f is called a principal sieve and is denoted by $\langle f \rangle$. Moreover, for a sieve S on x and a morphism f with $d_1 f = x$, $S \cdot f = \{g : f \circ g \in S\}$.

Remark 1.2. Note that, in general, $\mathcal{M}/x$ is a proper class. $\mathcal{X}$ is called $\mathcal{M}$-wellpowered ([5]), if there is a skeleton $\mathcal{M}_0$ of $\mathcal{M}$ such that each class $\mathcal{M}_0/X$ is a set; equivalently, if $\mathcal{M}/x$, up to isomorphism can be labeled by a small set for every object x. We say $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered provided that for each $x \in \mathcal{X}$, $\{\langle f \rangle \mid d_1 f = x, f \in \mathcal{M}\}$ is a set. Obviously, $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if it is $\mathcal{M}$-wellpowered.

For a sieve $S \subseteq \mathcal{X}_1/x$, and a morphism f with codomain x, the class of all the largest elements w in $\mathcal{X}_1/d_0 f$ satisfying $f \circ w \leq s$ for some $s \in S$ is denoted by $(f \Rightarrow S)$, see [8]. Obviously, for a sieve S on x, $(f \Rightarrow S)$ is just the class of maximums of $S \cdot f$. Also the class of all maximal elements of $S \cdot f$ denoted by $[f \Rightarrow S]$.

Definition 1.3. ([8]). A class $\mathcal{M}$ of $\mathcal{X}-$morphisms is said to satisfy the principality property, if for each x, $f \in \mathcal{X}_1/x$ and $m \in \mathcal{M}/x$, $(f \Rightarrow \langle m \rangle) \cap \mathcal{M}/d_0 f \neq \emptyset$.

In fact, class $\mathcal{M}$ has the principality property whenever we can generate any sieve $\langle m \rangle \cdot f$ with one element belonging to the $\mathcal{M}/d_0(f)$.

Let Ω be the subobject classifier of $Set^{\mathcal{X}^{op}}$, see [12]. If $\mathcal{M}$ satisfies the principality and is weakly $\mathcal{M}$-wellpowered, the map $M : \mathcal{X}^{op} \to Set$ with $M(x) = \{\langle f \rangle \mid f \in \mathcal{M}/x\}$ and for $h : y \to x$, $M(h) : M(x) \to M(y)$ the function taking $\langle g \rangle$ to $\langle g \rangle \cdot h$ is a functor and subobject of Ω, see [8]. With the same conditions we can generalize the presheaf $M : \mathcal{X}^{op} \to Set$ and obtain a subobject of Ω by the following lemma:

Lemma 1.4.

(1) *Every class $\mathcal{M}$ of morphisms of $\mathcal{X}$ which satisfies the principality property and is weakly $\mathcal{M}$-wellpowered yields a subobject $M^n : \mathcal{X}^{op} \to Set$ of Ω for every $n \leq card(\mathcal{M})$.*

(2) *Every subobject M^n of Ω yields a class $\mathcal{M}$ of morphisms of $\mathcal{X}$ which is saturated.*

Proof.

(1) for $n \leq card(\mathcal{M})$ define

$$M^n(x) = \{\langle f_1, f_2, \cdots, f_n \rangle \mid f_i \in \mathcal{M}/x, i = 1, \cdots, n\}$$

and for $h : y \to x$, $M^n(h) : M^n(x) \to M^n(y)$ the function taking $\langle g_1, g_2, \cdots, g_n \rangle$ to $\langle g_1, g_2, \cdots, g_n \rangle \cdot h$ is a functor and subobject of Ω. Note that $\langle g_1, g_2, \cdots, g_n \rangle \cdot h = (\langle g_1 \rangle \cdot h) \cup (\langle g_2 \rangle \cdot h) \cup \ldots \cup (\langle g_n \rangle \cdot h)$ and principality property implies that each term in above union is a principal sieve with generators of $\mathcal{M}/y$ and so $\langle g_1, g_2, \cdots, g_n \rangle \cdot h$ has at most n generators which is in $\mathcal{M}/y$.

(2) Let $M^n : \mathcal{X}^{op} \to Set$ be a subobject of Ω. Then for each object $x \in \mathcal{X}$ we can define $\mathcal{M}/x$ consists of all generators belongs to some members of $M^n(x)$. If $f \in \mathcal{M}$ and $g \sim f$, then there exists sieve S on $d_1 f$ such that f is a maximal element (generator) of S. Since $f \sim g$ so $\langle f \rangle = \langle g \rangle$ and thus g also is a maximal element (generator) of S. Therefore $\mathcal{M}$ is saturated.

□

Definition 1.5. Let $\mathcal{M}$ be a set of $\mathcal{X}$-morphisms. $\mathcal{M}$ is said to have:

(1) enough retractions, if for all objects x in $\mathcal{X}$, $\mathcal{M}/x$ has a retraction.

(2) almost enough retractions, if for all objects x in $\mathcal{X}$, $\mathcal{M}/x = \emptyset$ or $\mathcal{M}/x$ has a retraction.

(3) the n-identity property if for all objects x in $\mathcal{X}$ and for all sieves S on x whenever $\mathcal{M}_S = \{f \in \mathcal{X}_1/x \mid card([f \Rightarrow S] \cap \mathcal{M}/d_0 f) \leq n\}$ has at the most n maximal elements which are in $\mathcal{M}/x$, then $1_x \in \mathcal{M}_S$.

(4) the n-maximal property if for all objects x in $\mathcal{X}$ and for all sieves S on x, whenever $S \cap \mathcal{M}/x \neq \emptyset$, then S has at the most n maximal elements which are in $\mathcal{M}/x$ and not less than one.

(5) the n-quasi meet property if for all objects x in $\mathcal{X}$ and $m_1, ..., m_k \in \mathcal{M}/x$ and $n_1, ..., n_l \in \mathcal{M}/x$ such that $k, l \leq n$, there exists maximum elements $h_i \in [m_i \Rightarrow \langle n_1, ..., n_l \rangle]$ such that $m_i \circ h_i \sim \mathcal{M}/x$ for $i = 1, ..., k$.

Let $T : \mathcal{X}^{op} \to Set$ be the terminal object defined by $T(x) = \{T_x\}$, where T_x is the maximal sieve on x.

Remark 1.6. In the previous definition enough retractions implies almost enough retractions. Also n-maximal property implies almost enough retractions, because total sieve T_x that contains all morphisms with codomain x, intersect $\mathcal{M}/x$ and so contains at most n maximal elements which are in $\mathcal{M}/x$. Since $1_x \in T_x$ so at least has a retraction.

Definition 1.7. Consider functor category $Set^{\mathcal{X}^{op}}$.

(a) For two subobjects $A, B : \mathcal{X}^{op} \to Set$ of Ω in $Set^{\mathcal{X}^{op}}$, we write $A \leq B$ provided that for all objects $x \in \mathcal{X}$, $A(x) \subseteq B(x)$ and we write $A \wedge B$ for pointwise intersection, i.e. $(A \wedge B)(x) = A(x) \cap B(x)$;

(b) For parallel maps $\Phi, \Psi : A \to \Omega$ in $Set^{\mathcal{X}^{op}}$, define $\Phi \preceq \Psi$ if $\Phi_x(s) \subseteq \Psi_x(s)$ for all $x \in \mathcal{X}$ and $s \in A(x)$.

Remark 1.8. In the above lemma presheaf $M^n(x)$ consists of all sieves with at most n generators which belong to $\mathcal{M}/x$. We can choose all or some of the n generators equally and that means $M^{n-1} \leq M^n$ for each $n > 1$ (Note that $M^1 = M$).

Corollary 1.9. *Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms that satisfies the principality property and is weakly $\mathcal{M}$-wellpowered. For each n the induced presheaves satisfy in the following inequality:*

$$M \leq M^2 \leq \cdots \leq M^n \leq M^{n+1} \leq \cdots \leq \Omega(x).$$

Proof. Straightforward. □

Definition 1.10. ([12]). A subobject $A : \mathcal{X}^{op} \to Set$ of Ω in $Set^{\mathcal{X}^{op}}$ is said to be:

(1) a filter provided that for each object x in $\mathcal{X}$, $A(x)$ is a filter, (i.e. for two sieves S_1, S_2 on x, if $S_1 \subseteq S_2$ and $S_1 \in A(x)$ then $S_2 \in A(x)$).

(2) closed under binary intersection provided that for each object x in $\mathcal{X}$, $A(x)$ is closed under binary intersection, (i.e. for two sieves S_1, S_2 on x, if $S_1, S_2 \in A(x)$ then $S_1 \cap S_2 \in A(x)$).

Given a subobject A of Ω, each sieve S on an object x yields a sieve S_A on x given by $S_A = \{f \mid d_1 f = x, S \cdot f \in A(d_0 f)\}$. Since $S_A \cdot f = (S \cdot f)_A$,

$\hat{A}$ defined to take each object x to $\hat{A}(x) = \{S_A \mid S \text{ is a sieve on } x\}$ and each morphism $g : x \to y$ to $\hat{A}(g) : \hat{A}(y) \to \hat{A}(x)$ taking S_A to $S_A \cdot g$, is easily seen to be a subobject of Ω, see [8].

Theorem 1.11. *If $\mathcal{M}$ satisfies the principality property and M^n is the associated presheaf, or if M^n is a subobject of Ω and $\mathcal{M}$ is the associated saturated class, then:*

(1) *$T \le M^n$ iff $\mathcal{M}$ has enough retractions;*
(2) *$\widehat{M^n} \wedge T \le M^n$ iff $\mathcal{M}$ has almost enough retractions;*
(3) *$\widehat{M^n} \wedge M^n \le T$ iff $\mathcal{M}$ has the n-identity property;*
(4) *M^n is a filter iff $\mathcal{M}$ has the n-maximal property;*
(5) *M^n is closed under binary intersection iff $\mathcal{M}$ has the n-quasi meet property.*

Proof. (1) If $T_x \subseteq M^n(x)$, then $1_x \in M^n(x)$ and so one of generators of $M^n(x)$ generates 1_x and this implies the generator is retraction. Converse is obvious.

(2) For given $x \in \mathcal{X}$, $\widehat{M^n}(x) \cap T(x) \subseteq M^n(x)$ if and only if $\widehat{M^n}(x) \cap T_x \neq \emptyset$ or $T_x \in M^n(x)$ if and only if $\mathcal{M}/x = \emptyset$ or $\mathcal{M}/x$ has a retraction if and only if $\mathcal{M}$ has almost enough retractions.

(3) For given $x \in \mathcal{X}$, $\widehat{M^n}(x) \cap M^n(x) \subseteq T_x$ if and only if $S_M \in \widehat{M^n}(x) \cap M^n(x)$ implies that $S_{M^n} = T_x$ if and only if S_{M^n} has some maximal in $\mathcal{M}/x$ less than n implies that $1_x \in S_{M^n}$ if and only if $\mathcal{M}$ has the n-identity property.

(4) M^n is a filter if and only if for each $x \in \mathcal{X}$, $S \in M^n(x)$ and $S \subseteq S'$ implies $S' \in M^n(x)$ if and only if $S' = \langle k_1, k_2, \dots, k_n\rangle$ for some $k_1, k_2, \dots, k_n \in \mathcal{M}$ if and only if $\mathcal{M}$ has the n-maximal property.

(5) M^n is closed under binary intersection if and only if for each $x \in \mathcal{X}$, $\langle f_1, f_2, \cdots, f_n\rangle, \langle g_1, g_2, \dots, g_n\rangle \in M^n(x)$ implies that $\langle f_1, f_2, \dots, f_n\rangle \cap \langle g_1, g_2, \dots, g_n\rangle \in M^n(x)$ if and only if $f_1, f_2, \dots f_n$ and $g_1, g_2, \dots, g_n \in \mathcal{M}/x$ implies that there exists $f_i \in [m_i \Rightarrow \langle n_i\rangle]$ for $i = 1, \dots, n$, such that $m_i \circ f_i \sim \mathcal{M}/x$ if and only if $\mathcal{M}$ has the n-quasi-meet property.

□

2. Main result

According to the construction stated in [8], each class $\mathcal{M}$ with principality property which is satisfy in certain conditions induces a Lawvere-Tierney topology $j : \Omega \to \Omega$ and then each Lawvere-Tierney topology induces an universal closure operation c. Now we consider subpresheaves of Ω which is obtained by k-sieves (sieve with minimum k generators) where $k \leq n$, namely M^n, and study universal closure operation induced by k-sieves, $k \leq n$, namely c^n.

Definition 2.1. In a category $\mathcal{C}$ with finite limits, a subobject classifier is a monomorphism, $t : 1 \to \Omega$ such that to every monomorphism $S \rightarrowtail X$ in $\mathcal{C}$ there is a unique arrow Φ which, with the given monomorphism, forms a pullback square

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & p.b. & \downarrow t \\ X & \xrightarrow[\Phi]{} & \Omega \end{array}$$

In other words, every subobject is uniquely a pullback of a universal monomorphism t.

With this motivation, see [12], the proposed subobject classifier Ω for the topos $Sets^{\mathcal{X}^{op}}$ is defined on objects by

$$\Omega(C) = \{S \mid S \,\text{is a sieve on}\; x \;\text{in}\, \mathcal{X}\}$$

and on arrows $g : y \to x$ by

$$(-) \cdot g : \Omega(x) \to \Omega(y) \qquad \text{and} \qquad S \cdot g = \{h \mid g \circ h \in S\}.$$

We know subobject $i : M^n \rightarrowtail \Omega$ in $Set^{\mathcal{X}^{op}}$ associated to map $j^n : \Omega \to \Omega$ via the following pullback square, see [12],

$$\begin{array}{ccc} M^n & \xrightarrow{!_{M^n}} & 1 \\ i \downarrow & p.b. & \downarrow t \\ \Omega & \xrightarrow[j^n]{} & \Omega \end{array}$$

Note that for a given M^n, j^n is defined by the maps j^n_x that take each sieve S on x to set $S_{M^n} = \{f \mid d_1 f = x, S \cdot f \in M^n(d_0 f)\}$ and for a given j, M is defined by $M(x) = \{S : j_x(S) = T_x\}$. With M and j corresponding to each other, we obviously have $j_x(S) = T_x$ iff $S \in M^n(x)$.

Definition 2.2. ([12]). Let Ω be its subobject classifier of the topos $Sets^{\mathcal{X}^{op}}$. A Lawvere-Tierney topology is a map $j : \Omega \to \Omega$ in $Sets^{\mathcal{X}^{op}}$ with the following three properties:

(1) $j \circ t = t$;
(2) $j \circ j = j$;
(3) $j \circ \wedge = \wedge \circ (j \times j)$.

Corollary 2.3. *Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms that satisfies the principality property. For each n the induced Lawvere-Tierney topologies satisfy in the following inequality:*

$$j \preceq j^2 \preceq \cdots \preceq j^n \preceq j^{n+1} \preceq \cdots .$$

Proof. Straightforward. □

Theorem 2.4. *With $j^n : \Omega \longrightarrow \Omega$ and subobject M^n of Ω corresponding to each other, we have:*

(1) $j^n \circ t = t$ *iff* $T \leq M^n$*;*
(2) $j^n \preceq j^n \circ j^n$ *iff* $\widehat{M^n} \wedge T \leq M^n$*;*
(3) $j^n \circ j^n \preceq j^n$ *iff* $\widehat{M^n} \wedge M^n \leq T$*;*
(4) $j^n \circ \wedge \preceq \wedge \circ (j^n \times j^n)$ *iff* M^n *is a filter;*
(e) $\wedge \circ (j^n \times j^n) \preceq j^n \circ \wedge$ *iff* M^n *is closed under binary intersection.*

Proof. (1) Let $j^n \circ t = t$. If $x \in \mathcal{X}$, then ${j^n}_x(T_x) = T_x$ and so $T_x \in M^n(x)$. Let $T \leq M^n$. We just show that ${j^n}_x(T_x) = T_x$ for all $x \in \mathcal{X}$. The result follow by the fact that $T_x \in M^n(x)$.

(2) Let $j^n \preceq j^n \circ j^n$. For $x \in \mathcal{X}$ if $S_{M^n} = T_x$ for a sieve S on x, then $S \in M^n(x)$. Hence ${j^n}_x(S) = T_x$ and so ${j^n}_x({j^n}_x)(S) = T_x$. Thus ${j^n}_x(S) \in M^n(x)$ and the result follows.

Let $\widehat{M^n} \wedge T \leq M^n$. For $x \in \mathcal{X}$ and $S \in \Omega(x)$, if $f \in {j^n}_x(S)$ then $S \cdot f \in M^n(d_0 f)$ and so ${j^n}_{d_0 f}(S \cdot f) = T_{d_0 f}$. Thus by naturallity of j^n, ${j^n}_x(S) \cdot f \in M^n(d_0 f)$ and therefore $f \in {j^n}_x({j^n}_x)(S)$.

(3) The proof is similar to the part (2).

(4) Let $j^n \circ \wedge \preceq \wedge \circ (j^n \times j^n)$. If $x \in \mathcal{X}$ and $S_1, S_2 \in \Omega(x)$. $S_1 \subseteq S_2$, $S_1 \in M^n(x)$ implies that ${j^n}_x(S_1 \cap S_2) = {j^n}_x(S_1) = T_x$. Hence by assumption $S_2 \in M^n(x)$.

Let M^n be a filter. If $x \in \mathcal{X}$ and $S_1, S_2 \in \Omega(x)$, $f \in {j^n}_x(S_1 \cap S_2)$ implies that $(S_1 \cap S_2) \cdot f \in M^n(d_0 f)$ and so $(S_1 \cdot f) \cap (S_2 \cdot f) \in M^n(d_0 f)$. Since M^n is a filter, $S_1 \cdot f \in M^n(d_0 f)$. Similarly $S_2 \cdot f \in M^n(d_0 f)$. Therefore $f \in {j^n}_x(S_1) \cap {j^n}_x(S_2)$.

(5) The proof is similar to the part (4).

□

Corollary 2.5. *If saturated $\mathcal{M}$ satisfies the principality property and the induced map $j^n : \Omega \to \Omega$ by M^n which M^n is a subobjects of Ω, then:*

(1) $j^n \circ t = t$ *iff* $\mathcal{M}$ *has enough retractions;*
(2) $j^n \preceq j^n \circ j^n$ *iff* $\mathcal{M}$ *has almost enough retractions;*
(3) $j^n \circ j^n \preceq j^n$ *iff* $\mathcal{M}$ *has the n-identity property;*
(4) $j^n \circ \wedge \preceq \wedge \circ (j^n \times j^n)$ *iff* $\mathcal{M}$ *has the n-maximal property;*
(5) $\wedge \circ (j^n \times j^n) \preceq j^n \circ \wedge$ *iff* $\mathcal{M}$ *has the n-quasi meet property.*

Proof. Follows from theorems 1.11 and 2.4. □

Corollary 2.6. *Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms that satisfies the principality property. The induced map $j^n : \Omega \to \Omega$ is a Lawvere-Tierney topology iff $\mathcal{M}$ satisfies (1), (3), (4) and (5) of Definition 1.5.*

Definition 2.7. ([5]). An operation $c : \mathcal{M} \longrightarrow \mathcal{N}$ between saturated classes $\mathcal{M}$ and $\mathcal{N}$ that have $\mathcal{X}-$pullbacks, is a collection $(c_x : \mathcal{M}/x \longrightarrow \mathcal{N}/x)_{x\in\mathcal{X}}$ of functions, that preserve the relation "$\sim$". Such an operation is called:

(1) extensive, provided that for all $x \in \mathcal{X}$ and $f \in \mathcal{M}/x$, $f \leq c_x(f)$.
(2) monotone, provided that for all $x \in \mathcal{X}$, c_x preserves the preorder "$\leq$".
(3) semi-universal, provided that for all $f : y \to x$ in $\mathcal{X}$ and $m \in \mathcal{M}/x$, $c_y(f^{-1}(m)) \leq f^{-1}(c_x(m))$.
(4) universal, provided that for all $f : y \to x$ in $\mathcal{X}$ and $m \in \mathcal{M}/x$, $c_y(f^{-1}(m)) = f^{-1}(c_x(m))$.

An universal operation is an universal operation that satisfies extensive and monotone properties.

Let $\alpha : A \rightarrowtail X$ be a subobject of X in $Set^{\mathcal{X}^{op}}$. First we classify α by subobject classifier t and get $\hat{\alpha} : X \to \Omega$. Next consider pullback of t throughout $j \circ \hat{\alpha}$ and get $c(\alpha) : c(A) \to \Omega$. The following pullbacks explain these construction:

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ {\scriptstyle\alpha}\downarrow & p.b. & \downarrow{\scriptstyle t} \\ X & \xrightarrow[\hat{\alpha}]{} & \Omega \end{array} \qquad\qquad \begin{array}{ccc} c(A) & \xrightarrow{!_{c(A)}} & 1 \\ {\scriptstyle c(\alpha)}\downarrow & p.b. & \downarrow{\scriptstyle t} \\ X & \xrightarrow[j\circ\hat{\alpha}]{} & \Omega \end{array}$$

By the above construction we have a universal closure operation $c_X : Sub(X) \to Sub(X)$ such that for each subobject $\alpha : A \rightarrowtail X$ of X, $c(\alpha) : c(A) \rightarrowtail \Omega$ is closure of α.

If we start by collection $\mathcal{M}$ of morphisms of category $\mathcal{X}$ which satisfies in principality property, then we can obtain subobject of M^n of Ω, and next Lawvere-Tierney topology j^n, then finally introduce universal closure operation c^n by topology j^n.

Lemma 2.8. *Let j_1, j_2 be two Lawvere-Tierney topologies such that $j_1 \preceq j_2$. If $\alpha : A \rightarrowtail X$ is a subobject of X, then $c_1(\alpha) \leq c_2(\alpha)$.*

Proof. We can see for two Lawvere-Tierney topologies $j_1 \preceq j_2$, if $\alpha : A \rightarrowtail X$ is a subobject of X, then $j_1 \circ \hat{\alpha} \preceq j_2 \circ \hat{\alpha}$. Set c_1, c_2 induced closure operation by j_1, j_2, respectively. Hence $c_1(\alpha) \leq c_2(\alpha)$. □

Lemma 2.9. *([8]). Let "$-$" be an universal operation.*

(1) *Let $X \in Set^{\mathcal{X}^{op}}$ and $\alpha, \beta \in Sub(X)$. Then $\alpha \leq \beta$ if and only if $\widehat{\alpha} \preceq \widehat{\beta}$;*

(2) *Let* $X \in Set^{\mathcal{X}^{op}}$. *For all* $\alpha, \beta \in Sub(X)$, *(*$\alpha \leq \beta \Rightarrow \bar{\alpha} \leq \bar{\beta}$*) if and only if for all* $\alpha, \beta \in Sub(X)$, $\overline{\alpha \wedge \beta} \leq \bar{\alpha} \wedge \bar{\beta}$.

Theorem 2.10. *Let the Lawvere-Tierney topology* $j^n : \Omega \to \Omega$ *and the universal operation* c^n *correspond to each other. We have*

(1) *For all* X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha \in Sub(X)$, $\alpha \leq c^n(\alpha)$ *if and only if* $j^n \circ t = t$;
(2) *For all* X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha \in Sub(X)$, $\alpha \leq c^n(c^n(\alpha))$ *if and only if* $j^n \preceq j^n \circ j^n$;
(3) *For all* X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha \in Sub(X)$, $c^n(c^n(\alpha)) \leq \alpha$ *if and only if* $j^n \circ j^n \preceq j^n$;
(4) *For all* X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha, \beta \in Sub(X)$, $c^n(\alpha \wedge \beta) \leq c^n(\alpha) \wedge c^n(\beta)$ *if and only if* $j^n \circ \wedge \preceq \wedge \circ (j^n \times j^n)$;
(5) *For all* X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha, \beta \in Sub(X)$, $c^n(\alpha) \wedge c^n(\beta) \leq c^n(\alpha \wedge \beta)$ *if and only if* $\wedge \circ (j^n \times j^n) \preceq j^n \circ \wedge$.

Proof. (1) Choose $\alpha = t$, so $t \leq c^n(t)$. Since $j^n \circ c^n(t) = t \circ !_{c^n(1)}$ so $j^n \circ t = t \circ 1 = t$.

Suppose that $j^n \circ t = t$. For given subobject $\alpha : A \to X$ of X we have $\hat{\alpha} \circ \alpha = t \circ !_A$, so $j^n \circ \hat{\alpha} \circ \alpha = j^n \circ t \circ !_A = t \circ !_A$. Since $j^n \circ \hat{\alpha}$ is the classifying map of $c^n(\alpha)$, there exists an unique γ such that $c^n(\alpha) \circ \gamma = \alpha$. Thus $\alpha \leq c^n(\alpha)$.

(2) Choose $\alpha = t$, so $c^n(t) \leq c^n(c^n(t))$. Since $\widehat{c^n(t)} = j^n$ and $\widehat{c^n(c^n(t))} = j^n \circ j^n$, the result follows.

Suppose that $j^n \preceq j^n \circ j^n$. Let X be an object in $Set^{\mathcal{X}^{op}}$ and α be in $Sub(X)$. For each x in $\mathcal{X}$ and $S \in \Omega(x)$, $\widehat{\alpha}_x(S) \in \Omega(x)$ and so $j^n_x(\widehat{\alpha}_x(S)) \subseteq (j^n_x \circ j^n_x)(\widehat{\alpha}_x(S))$. Therefore $j^n \circ \widehat{\alpha} \preceq j^n \circ j^n \circ \widehat{\alpha}$. Hence $c^n(\alpha) \leq c^n(c^n(\alpha))$.

(3) The proof is similar to the part (2).

(4) Suppose that for all X in $Set^{\mathcal{X}^{op}}$ and α, β be in $Sub(X)$, $c^n(\alpha \wedge \beta) \leq c^n(\alpha) \wedge c^n(\beta)$. We know $\langle t, t \rangle = \langle t \circ !_\Omega, 1 \rangle \wedge \langle 1, t \circ !_\Omega \rangle$. Let $\alpha = \langle t \circ !_\Omega, 1 \rangle$ and $\beta = \langle 1, t \circ !_\Omega \rangle$. One can verify that $\widehat{\alpha} = \pi_1$ and $\widehat{\beta} = \pi_2$ are the projections, and $\alpha \wedge \beta = \langle t, t \rangle$. Since $\langle \widehat{\alpha}, \widehat{\beta} \rangle$ is the identity, the result follows.

Suppose that $j^n \circ \wedge \preceq \wedge \circ (j^n \times j^n)$. Let X be an object in $Set^{\mathcal{X}^{op}}$ and $\alpha, \beta \in Sub(X)$. We have $j^n \circ \wedge \circ \langle \widehat{\alpha}, \hat{\beta} \rangle \preceq \wedge \circ (j^n \circ j^n) \circ \langle \widehat{\alpha}, \hat{\beta} \rangle$. Therefore $c^n(\alpha \wedge \beta) \leq c^n(\alpha) \wedge c^n(\beta)$.

(5) The proof is similar to the part (4).

□

Corollary 2.11. *Let* $\mathcal{M}$ *be a class of morphisms of the category* $\mathcal{X}$ *such that satisfies the principality property. The induced universal operation* c^n *is an universal closure operation if and only if* $\mathcal{M}$ *satisfies (1) and*

(4) of Definition 1.5. In addition, c^n is idempotent if and only if $\mathcal{M}$ satisfies condition (3) as well.

Proof. Follows from theorems 1.11, 2.4 and 2.10. □

Example 2.12. Let $\mathcal{X}$ be a category and $\mathcal{M}$ be a pullback stable subset of all monomorphisms (epimorphisms) of $\mathcal{X}$. The principality property holds by pullback stability. If furthermore $\mathcal{M}$ has enough retractions and is weakly closed under composition (i.e. closed under composition up to relation $\sim$) which both hold for class of all monomorphisms (epimorphisms), then the n-identity property as well as the n-quasi-meet property hold. Thus, under the above conditions, the induced functor M^n, map j^n, and universal operation c^n satisfy (1), (2), (3), and (5) of theorems 1.11, 2.4 and 2.10, respectively. As a special case let $\mathcal{X}$ be the full subcategory of Top (category of topological spaces and continuous functions) consisting of finite ordinal topological spaces and consider $\mathcal{X}$ as all monomorphisms (epimorphisms).

Example 2.13. Let $(X, \leq)$ be a preordered set and $\mathcal{X} = C(X, \leq)$ be the category it induces (see [1]). We know in case $x \leq y$, $Hom(x, y)$ has a unique morphism, which we denote by (x, y). It is not hard to see that $\langle (a, x) \rangle \cdot (b, x) = \{(c, b) : c \leq b \quad and \quad c \leq a\}$ and that $((b, x) \Rightarrow \langle (a, x) \rangle) \neq \emptyset$ if and only if a meet $a \wedge b$ exists, in which case $(a \wedge b, b) \in ((b, x) \Rightarrow \langle (a, x) \rangle)$ or equivalently $\langle (a, x) \rangle \cdot (b, x) = \langle (a \wedge b, b) \rangle$.

Let $\mathcal{M}$ be a class of morphisms of $\mathcal{X}$. $\mathcal{M}$ satisfies the principality property if and only if for each $(a, x) \in \mathcal{M}/x$ and $(b, x) \in \mathcal{X}_1/x$ a meet $a \wedge b$ exists and $(a \wedge b, b) \in \mathcal{M}/b$; $\mathcal{M}$ has enough retractions (almost enough retractions) if and only if for each x, $1_x \in \mathcal{M}/x$ ($1_x \in \mathcal{M}/x$ or $\mathcal{M}/x \neq \emptyset$); $\mathcal{M}$ has the n-identity property if and only if for all x and for all sieves S on x, if the set $\mathcal{M}_S$ has at most n maximal elements in $\mathcal{M}$, then it contains 1_x; $\mathcal{M}$ has the n-maximal property if and only if for all x, every nonempty subset of $(\mathcal{M}/x, \leq^{op})$ has at most n maximal elements and also for all x and $(a, x) \in \mathcal{X}_1/x$, either there is (b, x) in $\mathcal{M}/x$ such that $b \cong a$ or for all $(b, x) \in \mathcal{M}/x$, $b \geq a$; and finally $\mathcal{M}$ has the n-quasi-meet property if and only if $\mathcal{M}$ has local binary meet (i.e. for all objects x, $\mathcal{M}/x$ has binary meet).

In case $(X, \leq)$ is a partially ordered set, every maximum or meet that exists is unique and if $(X, \leq)$ is a lattice then every binary meet exists and is unique.

Let $(X, \leq)$ be any partially ordered set such that every nonempty subset of X has a maximum ($\leq^{op}$ is then indeed a total order and $(X, \leq^{op}$) is well-ordered). Obviously every sieve on an object $x \in \mathcal{X}$ is principal. Now suppose for all x, $\mathcal{M}/x \neq \emptyset$ and for $a \leq b \leq x$, $(a, x) \in \mathcal{M}$ if and only if $(a, b) \in \mathcal{M}$ and $(b, x) \in \mathcal{M}$. One can then verify that $\mathcal{M}$ satisfies the principality as well as all the properties listed in Definition 1.5.

So by theorems 1.11, 2.4, 2.10, the induced presheaf M is a Grothendieck topology; the induced j is a Lawvere-Tierney topology; and the induced universal operation " $-$ " is an idempotent universal closure operation.

References

[1] J. Adamek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories. Wiley, New York, 1990.

[2] G. Birkhoff, A meaning of completeness, *Ann. Math.* **38***(1937), 57-60.*

[3] G. Birkhoff, Lattice Theory. American Mathematical Society Colloquium Publications (providence 1940). 3rd edition 1967.

[4] E. Čech, On bicompact spaces, *Ann. Math.* **38***(1937), 823-844.*

[5] D. Dikranjan, W. Tholen, Categorical Structure of Closure Operations. Kluwer, Netherlands 1995.

[6] H. Herrlich, Topologische Reflexionen and Coreflexionen. Lecture Notes in Math., **78**, Springer, Berlin 1968.

[7] P. Hertz, Über axiomensysteme für beliebige satzsysteme. Teil I, *Math. Ann. 87(1922), 246-269.*

[8] S. N. Hosseini, M. Nodehi, Morphism classes producing (weak) Grothendieck topologies, (weak) Lawvere-Tierney topologies, and universal closure operations, *Turk. J. Math.* **37***(2013), 818-829.*

[9] P. Johnstone, Topos Theory, L. M. S. Mathematical Monographs. **10**, Academic Press, New York 1977.

[10] J. F. Kennison, Reflexive functors in general topology and elsewhere, *Trans. Amer. Math. Soc.* **118***(1965), 303-315.*

[11] K. Kuratowski, Sur l'operation Ā de l'analysis situs, *Fund. Math.* **3***(1922), 182-199.*

[12] S. Maclane, I. Moerdijk, Sheaves in Geometry and Logic: A First Introduction to Topos Theory. Springer, Berlin Heidelberg, New York 1992.

[13] R. Pierce, Closure spaces with application to ring theory, in: Lectures on Rings and Modules, Lecture Notes in Mathematics. **246**, Springer, Berlin, Heidelberg (1972), 516-616.

[14] A. Tarski, Remarques sur les notions fondamentales de la médthodologie des mathématiques, *Ann. Soc. Math. Polon.* **7***(1929), 270-272.*