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A note on some parameters of domination on the edge neighborhood graph of a graph

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Abstract. The edge neighborhood graph $N_e(G)$ of a simple graph G is the graph with the vertex set $E \cup S$ where S is the set of all open edge neighborhood sets of G and two vertices $u, v \in V(N_e(G))$ adjacent if $u \in E$ and v is an open edge neighborhood set containing u. In this paper, we determine the domination number, the total domination number, the independent domination number and the 2-domination number in the edge neighborhood graph. We also obtain a 2-domination polynomial of the edge neighborhood graph for some certain graphs.

Keywords: Edge neighborhood graph, Domination number, Total domination, Independent domination, 2-domination polynomial.

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1. Introduction

Let $G = (V, E)$ be a simple graph with $|V(G)| = n$ vertices and the size of $|E(G)| = m$ edges. The open neighborhood of a vertex u is denoted by $N_G(u)$ is the set of all vertices adjacent to u in G and the closed neighborhood $N_G[u] = N_G(u) \cup \{u\}$. The number of edges incident to a

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vertex u in G and the minimum degree of vertices of G are denoted by $\deg_G(u)$ and δ_G, respectively [7]. Let e be an edge in G. There are two vertices u and v in $V(G)$ such that $e = uv$. The degree of an edge e is defined to be $\deg(e) = \deg(u) + \deg(v) - 2$. An edge is called an isolated edge if $\deg(e) = 0$ [11].

A fundamental concept in graph theory is domination which has been studied extensively [7, 8]. A set $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. In graph G with no isolated vertex, the set D is a total dominating set of G if every vertex in V is adjacent to some vertex in D. The minimum cardinality of a total dominating set of G is the total domination number $\gamma_t(G)$.

A subset of vertices is called independent if no two vertices in this subset are adjacent. A set $D \subseteq V$ is an independent dominating set if D is the dominating set and the independent set. The independent domination number of G is denoted by $\gamma_i(G)$ which is the minimum size of an independent dominating set of G. A 2-dominating set of G for every vertex of $V \setminus D$ has at least two neighbours in D. The 2-domination number of G, denoted by $\gamma_2(G)$ is the minimum size of the 2-dominating set of G. [7].

Graph polynomials are a useful field for analyzing the properties of graphs. In [2] the domination polynomial of a graph is introduced. Let $D_2(G, i)$ be the number of the 2-dominating sets of a graph G with cardinality i for $i \geq \gamma_2(G)$. The 2-domination polynomial of G is defined as $D_2(G, x) = \sum_{i=\gamma_2(G)}^{|V(G)|} d_2(G, i)x^i$ [12].

Kulli in [10] introduced a neighborhood graph $N(G)$ of graph G and study some properties of this graph. The neighborhood graph $N(G)$ of a graph G is the graph with the vertex set $V(G) \cup S$ where S is the set of all open neighborhood sets of G and two vertices u and v in $N(G)$ are adjacent if $u \in V(G)$ and v is an open neighborhood set containing u. In [1], some domination parameters in the neighborhood graph are computed. These domination parameters are also investigated on the join and the corona of two neighborhood graphs.

Kulli introduced a new concept of graphs as the edge neighborhood graph of a graph that is denoted by $N_e(G)$. $N_e(G)$ of G is the graph with the vertex set $E(G) \cup S$ where S is the set of all open edge neighborhood sets of the edges in G. Two vertices u and v in $N_e(G)$ are adjacent if $u \in E(G)$ and v is an open edge neighborhood set containing u. In Figure 1, graph G and its edge neighborhood graph $N_e(G)$ are shown. For the graph G, $N(e_1) = \{e_2, e_3\}$, $N(e_2) = \{e_1, e_3, e_4\}$, $N(e_3) = \{e_1, e_2, e_4\}$

and $N(e_4) = \{e_2, e_3\}$ are the open edge neighborhood sets of edges of G [11].

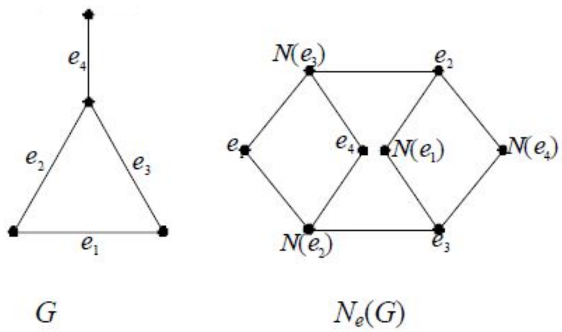

FIGURE 1. The graph G and the edge neighborhood graph of G.

In this paper, we obtain some domination parameters of graph $N_e(G)$ of a graph G. We also determine the 2-domination polynomial of $N_e(G)$ for some certain graphs G.

2. PRELIMINARIES

In this section, we recall some results that establish the domination number, the total domination number, the independent domination number and the 2-domination number for graphs, that are used in this paper.

Lemma 2.1. [11] *For any graph G with n vertices and m edges without isolated edge, $N_e(G)$ is a bipartite graph with $2m$ vertices and the number of edges is equal to*

$$\frac{1}{2}\Bigg[\sum_{e_i \in E(G)} \deg(e_i) + \sum_{e_i \in E(G)} \deg(N(e_i))\Bigg].$$

Lemma 2.2. [11] *If e is an isolated edge of a graph, the $N(e)$ is null set.*

Lemma 2.3. [11] *If P_n is a path with $n \geq 3$,*

$$N_e(P_n) = 2P_{n-1}.$$

Lemma 2.4. [11] *If C_n is a cycle with $n \geq 3$, then*

$$N_e(C_n) = \begin{cases} 2C_n & \text{if } n \text{ is even}, \\ C_{2n} & \text{if } n \text{ is odd}. \end{cases}$$

Lemma 2.5. [11] *$N_e(G) = \bar{K_n}$ if and only if for $n \geq 1$, $G = nK_2$.*

Lemma 2.6. [11] *$N_e(G) = 2nP_2$ if and only if for $n \geq 1$, $G = nP_3$.*

Lemma 2.7. [11] *If a graph G is an r-regular, then $N_e(G)$ is a $2(r-1)$-regular.*

Lemma 2.8. [9] *Let $\gamma(G)$ be the domination number of a graph G.*

(1) *For $n \geq 3$, $\gamma(C_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.*
(2) *For $n \geq 2$, $\gamma(K_n) = 1$.*
(3) *For $n \geq 2$, $\gamma(\bar{K_n}) = n$.*

Lemma 2.9. [9] *Let G be an r-regular graph of order n. Then,*

$$\gamma(G) \geq \frac{n}{r+1}.$$

Lemma 2.10. [3] *Let $\gamma_t(G)$ be the total domination number of G. Then,*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \ (mod\, 4), \\ \frac{n+2}{2} & \text{if } n \equiv 2 \ (mod\, 4), \\ \frac{n+1}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.11. [3] *Let G be a graph of order n with $\delta_G \geq 4$. Then, $\gamma_t(G) \leq \frac{3}{7}n$.*

Lemma 2.12. [4]

(1) *For every $n \geq 4$,*

$$\gamma_2(P_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even}, \\ \frac{n-1}{2} + 1 & \text{if } n \text{ is odd}. \end{cases}$$

(2) *For every $n \geq 4$,*

$$\gamma_2(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Lemma 2.13. [5] *If G is a graph of order n with the minimum degree at least 2, then*

$$\gamma_2(G) \leq \frac{2}{3}n.$$

Lemma 2.14. [6]

(1) *$\gamma_i(P_n) = \gamma_i(C_n) = \lceil \frac{n}{3} \rceil$.*
(2) *If G is a bipartite graph of order n and without isolated vertex, then*

$$\gamma_i(G) \leq \frac{n}{2}.$$

Theorem 2.15. [5] *Let G be a graph without isolated vertex and isolated edge with the minimum degree δ_G. If $\delta_{N_e(G)}$ is the minimum degree of graph $N_e(G)$, then*

$$\delta_G \leq \delta_{N_e(G)}.$$

Proof. Assume that $e \in V(N_e(G))$ with $\deg(e) = \delta_{N_e(G)}$. So, there are two vertices $u, v \in V(G)$ such that $e = uv$. Therefore, we have $\deg(e) = \deg(u) + \deg(v) - 2$. Suppose that $u \in V(G)$ with $\deg(u) = \delta_G$ and $\delta_G \geq 1$.
If $\delta_G = 1$, since G is a graph without isolated edge, $\deg(v) \geq 2$. Therefore $\deg(u) \leq \deg(e)$.
If $\deg(u) \geq 2$ then, $\deg(v) \geq 3$. Therefore, $\deg(u) \leq \deg(e)$. It is shown that the result holds for every $\delta_G \geq 1$. □

Lemma 2.16. [12] *If a graph G consists of m components $G_1, \ldots, G_m$, then*

$$D_2(G, x) = D_2(G_1, x) \ldots D_2(G_m, x).$$

Lemma 2.17. [12] *For every $n \geq 4$,*

(1) $D_2(P_n, x) = \begin{cases} x^{\frac{n}{2}+1}\Big(2(1+x)^{\frac{n}{2}-1} - x^{\frac{n}{2}-1}\Big) & \text{if } n \text{ is even}, \\ x^{\frac{n+1}{2}}\Big((1+x)^{\frac{n-1}{2}}\Big) & \text{if } n \text{ is odd}. \end{cases}$

(2) *Let* $A = \sum_{i=0}^{\frac{n-5}{2}} \Big(\binom{\frac{n-1}{2}}{i} + \binom{\frac{n-3}{2}}{i}\Big)x^i$.

$D_2(C_n, x) = \begin{cases} x^{\frac{n}{2}}\Big(2(1+x)^{\frac{n}{2}} - x^{\frac{n}{2}}\Big) & \text{if } n \text{ is even}, \\ x^{\frac{n+1}{2}}(1 + 2A + nx^{\frac{n-3}{2}} + x^{\frac{n-1}{2}}) & \text{if } n \text{ is odd}. \end{cases}$

3. Main results

In this section, we obtain the main results for computing some domination parameters on an edge neighborhood graph of a graph. Also, the 2-domination polynomial of $N_e(G)$ for some certain graphs are computed.

3.1. Domination.

Theorem 3.1. *Let the edge neighborhood graph of G be $N_e(G)$. Then,*

(1) $\gamma(N_e(P_n)) = 2\lceil \frac{n-1}{3} \rceil$.
(2) $\gamma(N_e(C_n)) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } n \text{ is even}, \\ \lceil \frac{2n}{3} \rceil & \text{if } n \text{ is odd}. \end{cases}$
(3) *for $n \geq 1$, $\gamma(N_e(nK_2)) = n$.*
(4) *for $n \geq 1$, $\gamma(N_e(nP_3)) = 2n$.*

Proof. (1) Using Lemma 2.3, $N_e(P_n)$ is a graph with two components P_{n-1}. Since by Lemma 2.8(1), $\gamma(P_{n-1}) = \lceil \frac{n-1}{3} \rceil$ then,

$$\gamma(N_e(P_n)) = 2\gamma(P_{n-1}) = 2\lceil \frac{n-1}{3} \rceil.$$

(2) If n is even, then using Lemma 2.4, $N_e(C_n) = 2C_n$. So, we determine the domination number of C_n. Therefore,

$$\gamma(N_e(C_n)) = 2\gamma(C_n).$$

Using Lemma 2.8(1). The result is complete.
If n is odd, then since by Lemma 2.4 $N(C_n)$ is equal with C_{2n}, we consider a cycle of order $2n$. For this case, using Lemma 2.8(1), we have

$$\gamma(N_e(C_n)) = \gamma(C_{2n}) = \lceil \frac{2n}{3} \rceil.$$

(3) Let G be a graph with n components as K_2 for $n \geq 1$. By lemma 2.5, $N_e(G) = \bar{K_n}$. Using Lemma 2.8(3), we have $\gamma(\bar{K_n}) = n$. Therefore,

$$\gamma(N_e(nK_2)) = \gamma(\bar{K_n}) = n.$$

(4) Since by Lemma 2.6, $N_e(nP_3) = 2nP_2$ and $\gamma(P_2) = 1$, we have

$$\gamma(N_e(nP_3)) = 2n\gamma(P_2) = 2n.$$

□

Theorem 3.2. *Let G be an r-regular graph with m edges. Then,*

$$\frac{2m}{2r-1} \leq \gamma(N_e(G)) \leq m.$$

Proof. Using Lemma 2.7, since G is an r-regular we have, $N_e(G)$ is a $2(r-1)$-regular graph. Lemma 2.1 implies that if graph G consists m edges then, $N_e(G)$ is a graph with $2m$ vertices. Using Lemma 2.9, we have

$$\gamma(N_e(G)) \geq \frac{2m}{2r-1}.$$

For upper bound, since $\gamma(G) \leq \gamma_i(G)$ then, by lemma 2.14(2) we can obtain

$$\gamma(N_e(G)) \leq \gamma_i(N_e(G)) \leq \frac{2m}{2} = m.$$

□

3.2. **Total domination and Independent domination.**

Theorem 3.3. *Let $\gamma_t(N_e(G))$ be the total domination number of the graph $N_e(G)$. Then*

$$\gamma_t(N(P_n)) = \begin{cases} n-1 & \text{if } n \equiv 1 \ (mod\, 4), \\ n+1 & \text{if } n \equiv 3 \ (mod\, 4), \\ n & \text{otherwise.} \end{cases}$$

Proof. Using Lemma 2.3, $N_e(P_n) = 2P_{n-1}$. Thus, it is sufficient to determine the total domination number for P_{n-1}. Using Lemma 2.10 we consider the following cases.
Case 1: Assume that $n \equiv 1 \pmod 4$. It means that $n-1 \equiv 0 \pmod 4$. Thus, $\gamma_t(P_{n-1}) = \frac{n-1}{2}$. So, $\gamma_t(N_e(P_n)) = 2\gamma_t(P_{n-1}) = n-1$.
Case 2: If $n \equiv 3 \pmod 4$, then $n-1 \equiv 2 \pmod 4$. So, $\gamma_t(P_{n-1}) = \frac{(n-1)+2}{2} = \frac{n+1}{2}$. Therefore, $\gamma_t(N_e(P_n)) = 2\gamma_t(P_{n-1}) = n+1$.
Case 3: If $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$, then $n-1 \equiv 3 \pmod 4$ or $n-1 \equiv 1 \pmod 4$. Using lemma 2.10, $\gamma_t(P_{n-1}) = \frac{(n-1)+1}{2} = \frac{n}{2}$. Therefore,

$$\gamma_t(N_e(P_n)) = 2\gamma_t(P_{n-1}) = 2\left(\frac{n}{2}\right) = n.$$

□

Theorem 3.4. *For any $n \geq 3$,*

(1) *if n is even,* $\gamma_t(N_e(C_n)) = \begin{cases} n & \text{if } n \equiv 0 \ (mod\, 4), \\ n+2 & \text{if } n \equiv 2 \ (mod\, 4), \end{cases}$

(2) *if n is odd,* $\gamma_t(N_e(C_n)) = n+1$.

Proof. (1) If n is even, then using Lemma 2.4, $N_e(C_n) = 2C_n$. So, we consider the total domination number of graph C_n. Because, $\gamma(N_e(C_n)) = 2\gamma(C_n)$.
On the other hand, since n is even then, $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. Using Lemma 2.10,

$$\gamma_t(N_e(C_n)) = 2\gamma_t(C_n) = \begin{cases} n & \text{if } n \equiv 0 \ (mod\, 4), \\ n+2 & \text{if } n \equiv 2 \ (mod\, 4). \end{cases}$$

(2) If n is odd, then $n = 4k+1$ or $n = 4k+3$. Using lemma 2.4 $N_e(C_n) = C_{2n}$. So, we consider a cycle of order $2n$. In this case, $2n \equiv 2 \pmod 4$. Thus, using Lemma 2.10

$$\gamma_t(N_e(C_n)) = \gamma_t(C_{2n}) = \frac{2n+2}{2} = n+1.$$

□

Theorem 3.5. *For $n \geq 1$, $\gamma_t(N_e(nP_3)) = 4n$.*

Proof. Let G be a graph with n components P_3. Using Lemma 2.6, $N_e(nP_3) = 2nP_2$. We consider the total domination number of P_2. Since $\gamma_t(P_2) = 2$, we have

$$\gamma_t(N_e(nP_3)) = 2n\gamma_t(P_2) = 4n.$$

□

Theorem 3.6. *Let G be a graph of order n and size of m with $\delta_G \geq 4$. Then,*

$$\gamma_t(N_e(G)) \leq \frac{6}{7}m.$$

Proof. Using Lemma 2.1, $N_e(G)$ has $2m$ vertices. According to Theorem 2.15, $\delta_G \leq \delta_{N_e(G)}$. Thus, $\delta_{N_e(G)} \geq 4$. Therefore, using Lemma 2.11

$$\gamma_t(N_e(G)) \leq \frac{3}{7}(2m) = \frac{6}{7}m.$$

□

Theorem 3.7. *Let $\gamma_i(G)$ be the independent domination number of G. Then,*

$$\gamma_i(N_e(P_n)) = 2\left\lceil\frac{n-1}{3}\right\rceil.$$

Proof. According to Lemma 2.3 and Lemma 2.14(1), we have

$$\gamma_i(N_e(P_n)) = 2\gamma_i(P_{n-1}) = 2\left\lceil\frac{n-1}{3}\right\rceil.$$

□

Theorem 3.8. *For any $n \geq 3$,*

$$\gamma_i(N_e(C_n)) = \begin{cases} 2\lceil\frac{n}{3}\rceil & \text{if } n \text{ is even,} \\ \lceil\frac{2n}{3}\rceil & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then $N_e(C_n) = 2C_n$. Therefore, $\gamma_i(N_e(C_n)) = 2\gamma_i(C_n)$. Thus, $\gamma_i(N_e(C_n)) = 2\lceil\frac{n}{3}\rceil$.
If n is odd, then $N_e(C_n) = C_{2n}$. On the other hand, by Lemma 2.14(1), we have

$$\gamma_i(N_e(C_n)) = \gamma_i(C_{2n}) = \lceil\frac{2n}{3}\rceil.$$

□

Theorem 3.9. *For $n \geq 1$,*

(1) $\gamma_i(N_e(nK_2)) = n$,
(2) $\gamma_i(N_e(nP_3)) = 2n$.

Proof.
(1) For $n \geq 1$, by Lemma 2.5, if $G = nK_2$ then, $N_e(G) = \bar{K_n}$. Using the definition of the independent dominating set of a graph G, we have

$$\gamma_i(N_e(nK_2)) = \gamma_i(\bar{K_n}) = n.$$

(2) If $G = nP_3$, for any $n \geq 1$, then using Lemma 2.6 and the definition of the independent domination number of G, we can obtain

$$\gamma_i(N_e(nP_3)) = \gamma_i(2nP_3) = 2n\gamma_i(P_3) = 2n.$$

□

Theorem 3.10. *If G be a graph without isolated vertex and isolated edge of order n with m edges, then*

$$\gamma_i(N_e(G)) \leq m.$$

Proof. Using Lemma 2.1, $N_e(G)$ is a bipartite graph of order $2m$. Using Lemma 2.14(2) for any bipartite graph G, $\gamma_i(G) \leq \frac{n}{2}$. For graph $N_e(G)$, we have

$$\gamma_i(N_e(G)) \leq \frac{2m}{2} = m.$$

□

3.3. **2-domination and 2-domination polynomial.**

Theorem 3.11. *For any $n \geq 4$,*

$$\gamma_2(N_e(P_n)) = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. According to Lemma 2.3, for $N_e(P_n)$, we consider the 2-domination number of P_{n-1}. We have the following cases.
Case 1: If n is even, then $n-1$ is odd. Using Lemma 2.12(1), $\gamma_2(P_{n-1}) = \frac{(n-1)-1}{2} + 1 = \frac{n}{2}$. Therefore,

$$\gamma_2(N_e(P_n)) = 2\gamma_2(P_{n-1}) = 2(\frac{n}{2}) = n.$$

Case 2: If n is odd, then $n-1$ is even. According to Lemma 2.12(1), $\gamma_2(P_{n-1}) = \frac{n-1}{2} + 1$. So,

$$\gamma_2(N_e(P_n)) = 2\Big(\frac{n-1}{2} + 1\Big) = n + 1.$$

□

Theorem 3.12. *For any $n \geq 4$,*

$$\gamma_2(N_e(C_n)) = n.$$

Proof. If n is odd, then $N_e(C_n) = 2C_n$. According to Lemma 2.12(2), in this case, $\gamma_2(C_n) = \frac{n}{2}$. So,

$$\gamma_2(N_e(C_n)) = 2\gamma_2(C_n) = 2(\frac{n}{2}) = n.$$

Now, let n be odd. By lemma 2.4, $N_e(C_n)$ is a cycle of order $2n$. So, $2n$ is even and using Lemma 2.12(2) we have

$$\gamma_2(N_e(C_n)) = \gamma_2(C_{2n}) = \frac{2n}{2} = n.$$

So, the result completes. □

Theorem 3.13. *Let G be a graph of order n and size of m with the minimum degree at least 2. Then,*

$$\gamma_2(N_e(G)) \leq \frac{4}{3}m.$$

Proof. Let the minimum degree of G be $\delta_G \geq 2$. Then, by Theorem 2.13, $\delta_{N_e(G)} \geq 2$. Lemma 2.1 implies that $N_e(G)$ is a graph of order $2m$ that m is the number of edges in G. Using Lemma 2.13, we have

$$\gamma_2(N_e(G)) \leq \frac{2}{3}(2m) = \frac{4}{3}m.$$

□

The following theorem is obtained easily from the definition of the 2-domination polynomial from $N_e(G)$.

Theorem 3.14. *For any $n \geq 5$,*

$$D_2(N_e(P_n), x) = \begin{cases} x^n(1+x)^{n-2} & \text{if } n \text{ is even,} \\ x^{n+1}\Big[4(1+x)^{n-3} - 4\big(x(1+x)\big)^{\frac{n-3}{2}} + x^{n-3}\Big] & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The edge neighborhood graph of path P_n is a graph $N_e(P_n)$ with two components P_{n-1}. Using Lemma 2.16, it is sufficient to consider the 2-domination polynomial of P_{n-1}. By Lemma 2.17(1) for path P_{n-1}, if $n-1$ is even, then n is odd. So,

$$D_2(P_{n-1}, x) = x^{\frac{n}{2}}(1+x)^{\frac{n-2}{2}}.$$

Therefore, we have

$$\begin{aligned} D_2(N_e(P_n), x) &= D_2(P_{n-1}, x)D_2(P_{n-1}, x) \\ &= \left(x^{\frac{n}{2}}(1+x)^{\frac{n-2}{2}}\right)^2 \\ &= x^n(1+x)^{n-2}. \end{aligned}$$

If $n-1$ is odd, then n is even. Thus,

$$D_2(N_e(P_n), x) = x^{\frac{n+1}{2}}\left(2(1+x)^{\frac{n-3}{2}} - x^{\frac{n-3}{2}}\right).$$

Therefore,

$$\begin{aligned} D_2(N_e(P_n), x) &= D_2(P_{n-1}, x)D_2(P_{n-1}, x) \\ &= \left(x^{\frac{n+1}{2}}\right)^2\left(2(1+x)^{\frac{n-3}{2}} - x^{\frac{n-3}{2}}\right)^2 \\ &= x^{n+1}\Big[4(1+x)^{n-3} - 4\big(x(1+x)\big)^{\frac{n-3}{2}} + x^{n-3}\Big]. \end{aligned}$$

□

Theorem 3.15. *For any $n \geq 5$,*

$$D_2(N_e(C_n),x) = \begin{cases} 4x^n(1+x)^{\frac{n}{2}}\Big[(1+x)^{\frac{n}{2}} - x^{\frac{n}{2}}\Big] + x^{2n} & \textit{if } n \textit{ is even,} \\ 2(x(1+x))^n - x^{2n} & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. If n is even, then by Lemma 2.4, $N_e(C_n) = 2C_n$. Using Lemma 2.16 and Lemma 2.17(2),

$$\begin{aligned} D_2(N_e(C_n),x) &= D_2(C_n,x)D_2(C_n,x) \\ &= \Big[x^{\frac{n}{2}}\big(2(1+x)^{\frac{n}{2}} - x^{\frac{n}{2}}\big)\Big]^2 \\ &= x^n\Big[2(1+x)^{\frac{n}{2}} - x^{\frac{n}{2}}\Big]^2 \\ &= x^n\Big[4(1+x)^n - 4\big(x(1+x)\big)^{\frac{n}{2}} + x^n\Big] \\ &= 4x^n(1+x)^{\frac{n}{2}}\big[(1+x)^{\frac{n}{2}} - x^{\frac{n}{2}}\big] + x^{2n}. \end{aligned}$$

If n is odd, then by Lemma 2.4, $N_e(C_n) = C_{2n}$. Since $N_e(C_n)$ is a graph of order even, using Lemma 2.16 and Lemma 2.17(2)

$$\begin{aligned} D_2(N_e(C_n),x) &= D_2(C_{2n},x) \\ &= x^{\frac{2n}{2}}\big(2(1+x)^{\frac{2n}{2}} - x^{\frac{2n}{2}}\big) \\ &= x^n\big(2(1+x)^n - x^n\big) \\ &= 2\big(x(1+x)\big)^n - x^{2n}. \end{aligned}$$

□

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