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On strongly PI-lifting modules

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Abstract. In this paper, the class of strongly PI-lifting modules is introduced and investigated. The connections between strongly PI-lifting modules and the generalizations of lifting modules are presented. We provide that the class of strongly PI-lifting modules is contained in the class of PI-lifting modules. Moreover, it is proved that for an Abelian ring R, R is PI-lifting as a right R-module if and only if R/I has a projective cover for every right ideal I of R. The structural properties of strongly PI-lifting modules are determined, and examples are provided to exhibit our results.

Keywords: π-extending module, lifting module, projection invariant submodule.

2000 Mathematics subject classification: 16D10, 16D80; Secondary 16D99.

1. Introduction

IIn this paper, all rings are associative with unity and modules are unital right modules. R and M will denote a ring and such an R-module, respectively. Recall that a module M is *extending* (or said to have C_1 condition) [8], if every submodule of M is essential in a direct summand of M. Recall from [5] that a submodule K of M is called *fully (projection) invariant* in M, if $f(K) \subseteq K$ for all (idempotent) endomorphisms

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of M. Observe that every fully invariant submodule is projection invariant, not vice versa. A module M is called *FI-extending (π-extending)* [2], [3] if every fully (projection) invariant submodule of M is essential in a direct summand of M. The notion of FI-extending (π-extending) generalizes the concept of an extending module by asking that only every fully (projection) invariant submodule is essential in a direct summand rather than every submodule.

Recall from [8] that a submodule A of a module M is called *small* in M, if $A + B \neq M$ for any proper submodule B of M. Note that A is a *coessential submodule* of B in M, if B/A is small in M/A. A submodule A of M is said to be *coclosed* if A has no proper coessential submodules in M. A module K is called *hollow* if every proper submodule of K is small. The extending condition dualizes in [8, p.57] as a *lifting module condition* in which for each submodule Y of M, there exists a direct summand X of M such that $X \leq Y$ and Y/X is small in M/X.

In literature, the lifting property has been studied in several module notions: (1) M is called *FI-lifting (PI-lifting)* [7], [1] if for each fully (projection) invariant submodule Y of M, there exists a direct summand X of M such that $X \leq Y$ and Y/X is small in M/X. (2) M is called *strongly FI-lifting* [9] if for each fully invariant submodule Y of M, there exists a fully invariant direct summand X of M such that $X \leq Y$ and Y/X is small in M/X. Note that both FI-lifting and PI-lifting modules are generalizations of lifting modules. It is mentioned in [9] that the class of strongly FI-lifting modules is a subclass of the class of FI-lifting modules. However, strongly FI-lifting modules and lifting modules are incomparable [9].

In this paper, we introduce and investigate the dual counterpart of the concept of a strongly π-extending module defined in [6]. We call a module M_R, *strongly PI-lifting*, for all projection invariant submodule Y of M, there exists a fully invariant direct summand X of M such that $X \leq Y$ and Y/X is small in M/X. We determine the connections between strongly PI-lifting modules and FI-lifting (PI-lifting) modules. To this end, for any module we have the following implications:

$$\begin{array}{ccc} \text{strongly PI-lifting} & \Rightarrow & \text{strongly FI-lifting} \\ \Downarrow & & \Downarrow \\ \text{PI-lifting} & \Rightarrow & \text{FI-lifting} \end{array}$$

Observe that non of above implications are reversible (see, Proposition 3.2). Moreover, we are able to get some characterizations of PI-lifting modules and we obtain module theoretic properties of strongly PI-lifting modules. Furthermore it is shown by examples that lifting modules and strongly PI-lifting modules have different module theoretic notions.

For a right R-module M and $P \subseteq M$, $P \leq M$, $P \leq^{\oplus} M$, $P \ll M$, $P \trianglelefteq_p M$ and $Rad(P)$ mean that P is a submodule of M, P is a direct summand of M, P is a small submodule of M, P is a projection invariant right R-submodule of M, and the Jacobson radical of P, respectively. For further terminology and notation, we refer to [4, 8].

2. Basic Results

In this section, we give some characterizations of PI-lifting modules. The next two results are used implicitly throughout this paper.

Lemma 2.1. *(*[5, Exercise 4], [1, Proposition 3.1]*) Let M be a module. Then*

(i) Let $\{X_i : i \in I\}$ be the family of projection invariant submodules of M. Then $\bigcap_{i\in I} X_i \trianglelefteq_p M$ and $\sum_{i\in I} X_i \trianglelefteq_p M$.

(ii) Let X and Y be submodules of M such that $X \leq Y \leq M$. If $X \trianglelefteq_p Y$ and $Y \trianglelefteq_p M$, then $X \trianglelefteq_p M$.

(iii) Let $M = \bigoplus_{i\in I} M_i$ and $N \trianglelefteq_p M$. Then $N = \bigoplus_{i\in I}(N \cap M_i)$ such that $N \cap M_i \trianglelefteq_p M_i$ for all $i \in I$.

Lemma 2.2. *(*[1, Lemma 4.2]*) M is PI-lifting if and only if for each projection invariant submodule A of M, there is a decomposition $A = X \oplus Y$ where $X \leq^{\oplus} M$ and $Y \ll M$.*

Proposition 2.3. *Assume M is PI-lifting and $Y \trianglelefteq_p M$. If Y is coclosed in M, then Y is PI-lifting.*

Proof. Suppose M is PI-lifting and $Y \trianglelefteq_p M$. Let $X \trianglelefteq_p Y$. Then $X \trianglelefteq_p M$ by Lemma 2.1. Thus there exists $K \leq^{\oplus} M$ such that $K \subseteq X$ and $X/K \ll M/K$. Since $Y \trianglelefteq_p M$, $Y = (Y \cap K) \oplus (Y \cap K')$ by Lemma 2.1, where $M = K \oplus K'$ for some $K' \leq M$. Notice that $Y = K \oplus (Y \cap K')$, so K is a direct summand of Y. Since Y is coclosed in M and $X/K \ll M/K$, [4, 3.9 Lemma] yields that $X/K \ll Y/K$. Therefore Y is PI-lifting. □

Proposition 2.4. *M is PI-lifting if and only if every projection invariant submodule of M has a supplement which is a direct summand of M.*

Proof. Assume M is PI-lifting and $A \trianglelefteq_p M$. Then there exists $K \leq^{\oplus} M$ such that $A = K \oplus S$ and $S \ll M$ by Lemma 2.2. It follows that $A = K \oplus (A \cap K')$ and $A \cap K' \ll M$ where $M = K \oplus K'$ for some $K' \leq M$. Therefore $M = A + K'$ and K' is a direct summand supplement of A in M. Conversely, let $N \trianglelefteq_p M$ and K be a supplement of N which is a direct summand of M. Thus $M = N + K$ and $N \cap K \ll K$. Note that

$M = K \oplus K'$ for some $K' \leq M$. Since $N \trianglelefteq_p M$, $N = (N \cap K) \oplus (N \cap K')$ by Lemma 2.1. Observe that $N \cap K \ll M$. By modular law, it can be seen that $N \cap K' \leq^{\oplus} M$. Therefore M is PI-lifting by Lemma 2.2. □

Proposition 2.5. *Suppose $M = M_1 \oplus M_2$ for some $M_1, M_2 \leq M$. Then M_2 is PI-lifting if and only if for each $N \trianglelefteq_p M_2$, there exists $K \leq^{\oplus} M$ such that $K \subseteq M_2$, $M = K + N$ and $N \cap K \ll M$.*

Proof. Assume that M_2 is PI-lifting and $T \trianglelefteq_p M_2$. Then there exists $A \leq^{\oplus} M_2$ such that $A \subseteq T$, and $T/A \ll M_2/A$. It follows that $T = A \oplus (T \cap A')$ and $T \cap A' \ll A'$ where $M_2 = A \oplus A'$ for some $A' \leq M_2$. Thus $T \cap A' \ll M$ and hence $T + A' = A \oplus (T \cap A') + A' = M$. Conversely, assume $X \trianglelefteq_p M_2$. Thereby, there exists $Q \leq^{\oplus} M$ such that $Q \subseteq M_2$, $M = Q + X$ and $X \cap Q \ll M$. Notice that $M_2 = M_2 \cap (Q + X) = Q + (M_2 \cap X) = Q + X$, and hence $X \cap Q \ll M_2$ which yields that Q is a direct summand supplement of X in M_2. Thus M_2 is PI-lifting by Proposition 2.4. □

Following the idea in [9, 2.12 Theorem], we characterize projective PI-lifting modules in terms of projective covers.

Theorem 2.6. *Let P be a projective module. Then P is PI-lifting if and only if P/N has a projective cover for all projection invariant submodule N of M.*

Proof. Let P be a projective PI-lifting module, and $N \trianglelefteq_p P$. Then $N = X \oplus S$ where $X \leq^{\oplus} P$ and $S \ll P$ by Lemma 2.2. Hence $P = X \oplus K$ for some $K \leq P$. Note that $(X + S)/X \ll P/X$, as $S \ll P$. Thus $g : P/X \to (X + S)/X = P/N$ is a projective cover of P/N. Conversely, assume that P/N has a projective cover for all projection invariant submodule N of M. Let $f : Q \to P/N$ be a projective cover of P/N. Then there exists $\alpha : P \to Q$ such that $f\alpha = \pi$ where $\pi : P \to P/N$ is the canonical map. It can be seen that α is an epimorphism. Since Q is a projective module, there exists $\beta : Q \to P$ such that $\alpha\beta = i_Q$ where $i_Q : Q \to Q$ is the identity map. Hence $P = ker\alpha \oplus \beta(Q)$ where $\beta(Q) \leq^{\oplus} P$. Observe from Lemma 2.1 that $N = (N \cap ker\alpha) \oplus (N \cap \beta(Q))$. Hence $N = ker\alpha \oplus (N \cap \beta(Q))$, as $f\alpha = \pi$. Notice that $N \cap \beta(Q) = \beta(kerf)$ and $kerf \ll Q$. Consequently $\beta(kerf) \ll \beta(Q)$, so $N \cap \beta(Q) \ll P$. Therefore P is PI-lifting by Lemma 2.2. □

Recall that R is an *Abelian* ring if every idempotent of R is central.

Corollary 2.7. *Let R be an Abelian ring. Then R_R is PI-lifting if and only if R/I has a projective cover for every right ideal I of R.*

Proof. Suppose R is Abelian and I is a right ideal of R. Then $eI = Ie \subseteq I$ for all $e^2 = e \in R$. Hence I_R is a projection invariant right ideal of R. Therefore Theorem 2.6 yields the result. □

3. Strongly PI-Lifting Modules

In this section, we deal with the class of strongly PI-lifting modules, and we come by some structural properties for the former class of modules.

Proposition 3.1. *The following conditions are equivalent:*

(i) *M is strongly PI-lifting.*

(ii) *For each projection invariant submodule Y of M, $Y = A \oplus T$ where A is a fully invariant direct summand of M and $T \ll M$.*

(iii) *Every projection invariant submodule of M has a supplement Q which is a direct summand of M such that $M = Q \oplus V$ for some fully invariant submodule V of M.*

Proof. (i) $\Leftrightarrow$ (ii) Let M be strongly PI-lifting and $Y \trianglelefteq_p M$. Thus there exists a fully invariant direct summand A of M such that $A \leq Y$ and $Y/A \ll M/A$. Hence $M = A \oplus A'$ for some $A' \leq M$. Since $Y \trianglelefteq_p M$, $Y = (Y \cap A) \oplus (Y \cap A') = A \oplus (Y \cap A')$ by Lemma 2.1. Note that $Y \cap A' \ll A'$, as $Y/A \ll M/A$. Therefore $Y \cap A' \ll M$. Conversely, assume $X \trianglelefteq_p M$. Then $X = A \oplus T$ where A is a fully invariant direct summand of M and $T \ll M$. Hence $M = A \oplus A'$ for some $A' \leq M$. Thus A' is a supplement of A. Since $T \ll M$, A' is a supplement of X. Therefore $A' \cap X \ll A'$, so $X/A \ll M/A$.

(i) $\Leftrightarrow$ (iii) This part follows the similar arguments in Proposition 2.4. □

Proposition 3.2. *Consider the following conditions:*

(i) *M is strongly PI-lifting.*

(ii) *M is strongly FI-lifting.*

(iii) *M is PI-lifting.*

(iv) *M is FI-lifting.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), but these implications are not reversible.

Proof. All implications hold by the definitions. The following examples show that the aforementioned implications are not reversible.

(iii) $\not\Rightarrow$ (i) and (iv) $\not\Rightarrow$ (ii) Take $M_{\mathbb{Z}} = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ for any prime p. Note that $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^3\mathbb{Z}$ are hollow modules, so $M_{\mathbb{Z}}$ is PI-lifting by [1, Corollary 4.4]. Thus $M_{\mathbb{Z}}$ is FI-lifting. However, $M_{\mathbb{Z}}$ is not strongly FI-lifting by [9, Remarks 3.8(1)]. Therefore $M_{\mathbb{Z}}$ is not strongly PI-lifting.

(iv) $\not\Rightarrow$ (iii) and (ii) $\not\Rightarrow$ (i) Suppose R is a simple domain that is not a division ring. Then the only fully invariant right ideals of R are the trivial ones, so R_R is FI-lifting by [1, p.809]. Note that $Rad(R_R) = 0$, so R_R is strongly FI-lifting by [9, 3.3 Proposition]. Since R is indecomposable, every right ideal of R is projection invariant. However $Rad(R_R) = 0$, so R_R is not PI-lifting. Moreover R_R is not strongly PI-lifting by Proposition 3.3. □

Proposition 3.3. *Suppose $Rad(M_R) = 0$. Then M is strongly PI-lifting if and only if M is PI-lifting.*

Proof. Assume that M is PI-lifting, and let $N \trianglelefteq_p M$. Thus $N = X \oplus Y$, where $X \leq^{\oplus} M$ and $Y \ll M$ by Lemma 2.2. Since $Rad(M_R) = 0$, $Y = 0$. Thus $N = X \leq^{\oplus} M$, so M is strongly PI-lifting. For the converse, Proposition 3.2 proceeds the result. □

Theorem 3.4. *Assume $M = M_1 \oplus M_2$ is strongly PI-lifting for some $M_1, M_2 \leq M$. If $M_1 \trianglelefteq_p M$, then M_1 and M_2 are strongly PI-lifting.*

Proof. Suppose $M = M_1 \oplus M_2$ is strongly PI-lifting and $M_1 \trianglelefteq_p M$. Take $X \trianglelefteq_p M_1$. Thus $X \trianglelefteq_p M$ by Lemma 2.1. Hence there exists a fully invariant direct summand B of M such that $X = B \oplus S$, where $S \ll M$ by Proposition 3.1. Therefore $M = B \oplus B'$ for some $B' \leq M$. Note that $M_1 = B \oplus (B' \cap M_1)$ so $B \leq^{\oplus} M_1$. Moreover $S \ll M_1$, and hence M_1 is strongly PI-lifting by Proposition 3.1.

Now, let $Y \trianglelefteq_p M_2$. Since $M_1 \trianglelefteq_p M$, $M_1 \oplus Y \trianglelefteq_p M$ by [3, Lemma 4.13]. Hence there exists a fully invariant direct summand A of M such that $M_1 \oplus Y = A \oplus T$ where $T \ll M$ by Proposition 3.1. It follows from Lemma 2.1. that $A = (A \cap M_1) \oplus (A \cap M_2)$, where $A \cap M_i \trianglelefteq_p M_i$ for $i = 1, 2$. Hence $A \cap M_2 \leq^{\oplus} M_2$. Consider the projection map $\pi : M \to M_2$. Since $M_1 \oplus Y = A \oplus T$, we obtain $Y = \pi(A) + \pi(T)$. Furthermore, $\pi(T) \ll \pi(M) = M_2$, as $T \ll M$. It follows that M_2 is strongly PI-lifting by Proposition 3.1. □

Corollary 3.5. *Any projection invariant coclosed submodule of a strongly PI-lifting module is strongly PI-lifting.*

Proof. Suppose M is a strongly PI-lifting module and $L \trianglelefteq_p M$ such that L is coclosed in M. Then there exists a fully invariant direct summand X of M such that $X \subseteq L$ and $L/X \ll M/X$. Since L is coclosed in M, $L = X$, so L is a direct summand of M. Thus Theorem 3.4 yields the result. □

Corollary 3.6. *Let M be a strongly PI-lifting module with an Abelian endomorphism ring. Then every direct summand of M is strongly PI-lifting.*

Proof. Assume M has the stated property. Since M has an Abelian endomorphism ring, every direct summand of M is projection invariant. Hence Theorem 3.4 completes the proof. □

Proposition 3.7. *Let M be strongly PI-lifting and N a supplement submodule of M such that $N \trianglelefteq_p M$. If M_R is self-injective, then N is strongly PI-lifting.*

Proof. Let $A \trianglelefteq_p N$. Then $A \trianglelefteq_p M$ by Lemma 2.1. By Proposition 3.1, $A = K \oplus S$ where K is a fully invariant direct summand of M and $S \ll M$. Notice that $K \leq^{\oplus} N$ and $S \ll N$. Observe that any map $f : N \to N$ can be lifted to M, as M is self-injective. Therefore K is a fully invariant in N. Hence N is strongly PI-lifting by Proposition 3.1. □

Let $M_{\mathbb{Z}} = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ for any prime p. Observe from the proof of Proposition 3.2 that $M_{\mathbb{Z}}$ is not strongly PI-lifting, whereas $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^3\mathbb{Z}$ are. Thus, in general, strongly PI-lifting property is not closed under direct sums. In the following result, we give a characterization of the finite direct sums of strongly PI-lifting module.

Theorem 3.8. *Let $\{M_i \,|\, 1 \leq i \leq n\}$ be the family of fully invariant direct summands of M. Then $M = \bigoplus_{i=1}^{n} M_i$ is strongly PI-lifting if and only if M_i is strongly PI-lifting for each $1 \leq i \leq n$.*

Proof. Assume $M = \bigoplus_{i=1}^{n} M_i$ and M_i is strongly PI-lifting, where M_i is fully invariant direct summand of M for all $1 \leq i \leq n$. Let $X \trianglelefteq_p M$. Then $X = \bigoplus_{i=1}^{n}(X \cap M_i)$ where $X \cap M_i \trianglelefteq_p M_i$, for all $1 \leq i \leq n$, by Lemma 2.1. Since M_i is strongly PI-lifting, there exists a fully invariant direct summand K_i of M_i such that $X \cap M_i = K_i \oplus S_i$ where $S_i \ll M_i$. Now, consider $K = \bigoplus_{i=1}^{n} K_i$ and $S = \bigoplus_{i=1}^{n} S_i$. Then $X = K \oplus S$ where K is a fully invariant direct summand of M and $S \ll M$. Therefore, M is strongly PI-lifting by Proposition 3.1. The converse is a consequence of Theorem 3.4. □

The following example explains that strongly PI-lifting modules and lifting modules are different from each other.

Example 3.9. (i) $M_{\mathbb{Z}} = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$ for any prime p. Hence $M_{\mathbb{Z}}$ is a lifting module, but it is not strongly FI-lifting by [9, Remarks 3.8(5)]. Therefore $M_{\mathbb{Z}}$ is not strongly PI-lifting by Proposition 3.2.

(ii) Let R be an incomplete rank one discrete valuation domain with quotient field K. Consider $M_K = K \oplus K$. Then M_K is not lifting by [4, 23.7 Example]. However, K is lifting by [8, Proposition A.7], M_K is PI-lifting by [1, Corollary 4.4]. It follows from Proposition 3.3 that M_K is strongly PI-lifting.

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