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Concerning strongly divisible, strongly fixed, and strongly z-ideals in $\mathcal{R}L$

Mostafa Abedi [1]
Esfarayen University of Technology, Esfarayen, North Khorasan, Iran

Abstract. As usual, the ring of continuous real-valued functions on a frame L is denoted by $\mathcal{R}L$. We determine the relation among strongly z-ideals, strongly divisible ideals and uniformly closed ideals in the ring $\mathcal{R}L$. We characterize Lindelöf frames based on strongly fixed ideals in $\mathcal{R}L$. We observe that a weakly spatial frame L is Lindelöf if and only if every strongly divisible ideal in $\mathcal{R}L$ is strongly fixed; if and only if every closed ideal in $\mathcal{R}L$ is strongly fixed.

Keywords: Ring of all real-valued continuous functions on a frame, Strongly divisible ideal, Zero set in pointfree topology, Closed ideal, Lindelöf frame.

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1. Introduction

A *frame* is a complete lattice for which finite meets distribute over arbitrary joins. A *frame homomorphism* is a map between frames that preserves finite meets (including the empty meet, which is the top element) and all joins (including the empty join, which is the bottom element). Recall from [7] that the frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators. Now,

[1] ms_abedi@yahoo.com, abedi@esfarayen.ac.ir
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for any frame L, *the real-valued continuous functions* on L are the homomorphisms $\mathcal{L}(\mathbb{R}) \to L$. The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [7]. Let $\mathfrak{O}(\mathbb{R})$ be the frame of open subsets of the topological space $\mathbb{R}$. We know that the frames $\mathcal{L}(\mathbb{R})$ and $\mathfrak{O}(\mathbb{R})$ are isomorphic. So under this framework, $\mathcal{R}L$ (or $C(L)$ which is notation used in [6]) is equal to the set of all frame homomorphisms from $\mathfrak{O}(\mathbb{R})$ to L. We use the notations of [7].

A decisive connection between L and $\mathcal{R}L$ is given by the *cozero map*, $\operatorname{coz} : \mathcal{R}L \to L$. A *cozero element* of L is an element of the form $\operatorname{coz}\varphi$ for some $\varphi \in \mathcal{R}L$. For more details about cozero map and its properties see [7].

An ideal Q in $\mathcal{R}L$ is called a *z-ideal* if, for any $\varphi, \psi \in \mathcal{R}L$, $\operatorname{coz}\varphi = \operatorname{coz}\psi$ and $\varphi \in Q$ imply $\psi \in Q$. In the paper [3], authors have examined the relation among z-ideals, strongly divisible ideals and uniformly closed ideals in the ring $\mathcal{R}L$. Uniformly closed ideals in rings $\mathcal{R}L$ have a lucid characterization in terms of z-ideals and cozero elements which often reduce calculations with these types of ideals (for more details see [3]).

The main aim of the present paper is to generalize the characterization of uniformly closed ideals of $\mathcal{R}L$ and Lindelöf frames as found in [1, 3], based on zero sets in pointfree topology and strongly z-ideals and strongly fixed ideals in $\mathcal{R}L$. The concept of strongly z-ideals is used in this paper is different from that concept which is used in [20] and [22].

Here, we collect a few definitions and results that will be relevant for our discussion. For undefined terms and notation our references are [7] for pointfree function rings, [16] for $C(X)$, and [21] for frames.

Throughout this context, L will denote a completly regular frame. We write 0 and 1 for the top element and the bottom element of L, respectively. An element $p \in L$ is called *prime* if $p < 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. We denoted the set of all points of L by ΣL. Usually one considers as the spectrum functor the contravariant *functor* Σ from the category **Frm** of frames to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL with $\Sigma_a = \{p \in \Sigma L : a \not\leq p\}$ $(a \in L)$ as its open sets and we have

$$\Sigma_0 = \emptyset \quad , \quad \Sigma_1 = \Sigma L$$
$$\Sigma_a \cap \Sigma_b = \Sigma_{a\wedge b} \quad , \quad \bigcup_{i\in I} \Sigma_{a_i} = \Sigma_{\bigvee_{i\in I} a_i}.$$

Also, for a frame map $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_* : M \to L$ is the *right adjoint* of h characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$.

For every $\varphi \in \mathcal{R}L$, we define $\varphi[p] = \widetilde{p}(\varphi)$ where $\widetilde{p}(\varphi)$ is a Dedekind cut for a real number. If p is a prime element of frame L, then $\widetilde{p} : \mathcal{R}(L) \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism. Also, $\widetilde{p}$ is a linear map with $\widetilde{p}(\mathbf{1}) = 1$ (for more details see [10]).

Let $\varphi \in \mathcal{R}L$. We define $Z(\varphi) = \{p \in \Sigma L : \varphi[p] = 0\}$. Such a set is said to be a *zero-set* in L. For $A \subseteq \mathcal{R}L$, we write $Z[A]$ to designate the family of zero sets $\{Z(\varphi) : \varphi \in A\}$. The family $Z[\mathcal{R}L]$ of all zero sets in L will also be denoted, for simplicity, by $Z[L]$ (for more details see [12]). The properties of the zero-sets that we shall frequently use are:

(1) For every $n \in \mathbb{N}$, $Z(\varphi) = Z(|\varphi|) = Z(\varphi^n)$.
(2) $Z(\varphi) \cap Z(\psi) = Z(|\varphi| + |\psi|) = Z(\varphi^2 + \psi^2)$.
(3) $Z(\varphi) \cup Z(\psi) = Z(\varphi\psi)$.
(4) If φ is a unit of $C(L)$, then $Z(\varphi) = \emptyset$. The converse hold whenever L is a weakly spatial frame.
(5) $Z[L]$ is closed under countable intersection.

2. Strongly divisible ideals

An ideal I in a commutative ring A is said to be *divisible* if for two members x_1 and x_2 in I there exist $x \in I$ and $a_1, a_2 \in A$ such that $x_1 = xa_1$ and $x_2 = xa_2$ (see [5]). For example, every principal ideal is a divisible ideal. Let Q be any ideal in $\mathcal{R}L$. Recall from [13] that if $\bigcap Z[Q]$ is nonempty, I is called a *strongly fixed ideal* (s-fixed); if $\bigcap Z[Q] = \emptyset$, then Q is a strongly free ideal (s-free).

We recall that a frame L is called *weakly spatial*, if $\Sigma_a = \Sigma L$, then a is the top element of L. In Theorem 4.16 of [13], it is shown that a weakly spatial L is compact if and only if every proper ideal in $\mathcal{R}L$ is strongly fixed if and only if every maximal ideal in $\mathcal{R}L$ is strongly fixed. In the following theorem, we give a characterization of weakly spatial compact frames based on the divisible ideals in $\mathcal{R}L$.

Theorem 2.1. *A weakly spatial frame L is compact if and only if every divisible ideal in $\mathcal{R}L$ is strongly fixed.*

Proof. ($\Rightarrow$). By [13, Theorem 4.16] is true.

($\Leftarrow$). We show that every maximal ideal in $\mathcal{R}L$ is strongly fixed. Let M be a maximal ideal of $\mathcal{R}L$. By the present hypothesis, it is enough to show that M is a divisible ideal. Assume $\phi_1, \phi_2 \in M$. Let $\psi_1 = \frac{\phi_1}{\mathbf{1}+|\phi_1|}$, $\psi_2 = \frac{\phi_2}{\mathbf{1}+|\phi_2|}$ and $\gamma = (|\psi_1|+|\psi_2|)^{1/2}$. Since $Z(\psi_1) = Z(|\psi_1|)$ and $Z(\psi_2) = Z(|\psi_2|)$, we can conclude, by [12, Proposition 5.5.], $|\psi_1|, |\psi_2| \in M$, and so, γ belong to M. Since ψ_1 and ψ_2 are in $\mathcal{R}^*L$, and $\mathcal{R}^*L$ is isomorphic to a $C(X)$ via an f-ring isomorphism, by the proof of Theorem 2 in [17], we infer that $\psi_1 = \frac{\phi_1}{\mathbf{1}+|\phi_1|}$ and $\psi_2 = \frac{\phi_2}{\mathbf{1}+|\phi_2|}$ are multiples of γ. This implies ϕ_1 and ϕ_2 are a multiple of γ. Therefore, M is a divisible ideal and the proof is complete. $\square$

The main tool to be used in this paper is the concept of strongly divisible ideals, which was defined by Azarpanah in [5], and used there to characterize Lindelöf spaces (also, see [23]).

Definition 2.2. A proper ideal I of a ring A is said to be *strongly divisible* if for every countable subset $\{x_n \mid n \in \mathbb{N}\}$ of I there are $a \in I$ and a subset $\{a_n \mid n \in \mathbb{N}\}$ of A such that for each $n \in \mathbb{N}$, $aa_n = x_n$.

Recall from [12] that an ideal Q in $\mathcal{R}L$ is called *strongly z-ideal* (briefly, sz-ideal) if $Z(\varphi) \in Z[Q]$ implies that $\varphi \in Q$, that is to say, $Q = Z^{\leftarrow}[Z[Q]]$. The relation between sums of strongly z-ideals and prime ideals in $\mathcal{R}L$ is studied in [15].

The relation between strongly z-ideals and strongly divisible ideals is given in the following proposition. This proposition plays an important role in this paper. In order to state the proposition, we need some background. Suppose that (ϕ_n) is a sequence of positive elements of $\mathcal{R}L$. The set

$$\big\{(\phi_1 \wedge \mathbf{2}^{-1}) + (\phi_2 \wedge \mathbf{2}^{-2}) + \ldots + (\phi_n \wedge \mathbf{2}^{-n}) \mid n \in \mathbb{N}\big\}$$

has a supremum in the poset $\mathcal{R}L$ (see [8, Section 6] and [24, Lemma 4]). This supremum is denoted by

$$\sum_{n=1}^{\infty} (\phi_n \wedge \mathbf{2}^{-n}).$$

Lemma 2.3. [18] *Let $\phi, \psi \in \mathcal{R}L$. If $|\phi| \leq |\psi|^p$ for some $1 < p \in \mathbb{Q}$, then ϕ is a multiple of ψ.*

Let $n \in \mathbb{N}$ be an odd number. By definition of the frame of the reals $\mathcal{L}(\mathbb{R})$ there is a frame map $\rho : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ such that for every $p, q \in \mathbb{Q}$, $\rho(p, q) = (p^n, q^n)$. Next, let $\varphi \in \mathcal{R}L$. The frame map $\sqrt[n]{\varphi} : \mathcal{L}(\mathbb{R}) \to L$ given by $\sqrt[n]{\varphi} = \varphi \circ \rho$ is an n^{th} root of φ (see Proposition 3.1 in [2] and Lemma 3.13 in [14]).

Proposition 2.4. *Let L be a weakly spatial frame. The following statements hold for an ideal Q of the ring $\mathcal{R}L$.*

1. *If Q is a strongly z-ideal of $\mathcal{R}L$ such that $Z[Q]$ is closed under countable intersection, then Q is strongly divisible.*
2. *If Q is strongly divisible, then $Z[Q]$ is closed under countable intersection.*

Proof. (1). Let $\phi_n \in Q$ for every $n \in \mathbb{N}$. Then, we have

$$\psi = \sum_{n=1}^{\infty} \mathbf{2}^{-n} \phi_n^{\frac{2}{3}} (\mathbf{1} + \phi_n^{\frac{2}{3}})^{-1} \in \mathcal{R}L$$

and it is clear that $\operatorname{coz}\psi = \bigvee_{n=1}^{\infty} \operatorname{coz}\phi_n$, and hence,

$$\begin{aligned} Z(\psi) &= \Sigma L - \Sigma_{\operatorname{coz}(\psi)} \quad \text{by [12, Lemma 3.2]} \\ &= \Sigma L - \Sigma_{\bigvee_{n=1}^{\infty} \operatorname{coz}\phi_n} \\ &= \Sigma L - \bigcup_{n=1}^{\infty} \Sigma_{\operatorname{coz}\phi_n} \\ &= \bigcap_{n=1}^{\infty} (\Sigma L - \operatorname{coz}\phi_n) \\ &= \bigcap_{n=1}^{\infty} Z(\phi_n). \end{aligned}$$

Since $Z[Q]$ is closed under countable join, $Z(\psi) \in Z[Q]$, implying that ψ belongs to the strongly z-ideal Q. But for each n, $\mathbf{2}^{-n} \frac{\phi_n^{\frac{2}{3}}}{\mathbf{1}+\phi_n^{\frac{2}{3}}} \leq \psi$, and so $|\phi_n| \leq |\psi \mathbf{2}^n (\mathbf{1} + \phi_n^{\frac{2}{3}})|^{\frac{3}{2}}$. Lemma 2.3 implies that each ϕ_n is a multiple of $\psi \mathbf{2}^n(\mathbf{1} + \phi_n^{\frac{2}{3}})$, which implies that ϕ_n is a multiple of ψ. Therefore, Q is strongly divisible.

(2). Let $\phi_n \in Q$ for every $n \in \mathbb{N}$. Then $\{Z(\phi_n) : n \in \mathbb{N}\}$ is a countable subset of $Z[Q]$. Since Q is strongly divisible, choose $\psi \in Q$ such that ψ divides ϕ_n, for each n. This shows that for each n, $Z(\psi) \subseteq Z(\phi_n)$, implying that $Z(\psi) \subseteq \bigcap_{n=1}^{\infty} Z(\phi_n)$. But, by [12, Proposition 4.2], $Z[Q]$ is a z-filter containing $Z(\psi)$, therefore $\bigcap_{n=1}^{\infty} Z(\phi_n)$ belongs to $Z[Q]$. $\square$

An element $\phi \in \mathcal{R}L$ is called *bounded* if $\phi(p, q) = 1$ for some $p, q \in \mathbb{Q}$, and L is called *pseudocompact* if $\mathcal{R}L = \mathcal{R}^*L$, where $\mathcal{R}^*L$ denotes the subring of $\mathcal{R}L$ consisting of its bounded elements. In what follows, we aim to give a characterization of weakly spatial pseudocompact frames in terms of strongly divisible ideals. Before this characterization is presented, we need some background. We know that, for any maximal ideal M, the field $\frac{\mathcal{R}L}{M}$ always contains a canonical copy of the field $\mathbb{R}$ of real numbers: the set of images of the constant functions under the canonical homomorphism. Now, M is said to be a *real ideal* when the canonical copy of the field $\mathbb{R}$ is the entire field $\frac{\mathcal{R}L}{M}$ (see [9], also see [16, Chapter 5] for more details). In [9], Dube has shown that a frame L is pseudocompact if and only if every maximal ideal of $\mathcal{R}L$ is real. Also, he has proved that a maximal ideal M of $\mathcal{R}L$ is real if and only if $\operatorname{coz}[M]$ is closed under countable join. In light of the basic properties of zero sets, we have the following proposition, the proof of which is routine.

Proposition 2.5. *Let L be a weakly spatial frame. A maximal ideal M of $\mathcal{R}L$ is real if and only if $Z[M]$ is closed under countable intersection..*

The combination of the foregoing proposition and Proposition 2.4 with the above facts imply that the next result.

Corollary 2.6. *A weakly spatial frame L is pseudocompact if and only if every ideal is contained in a strongly divisible sz-ideal.*

A *cover* of a frame L is a subset S of L with $\bigvee S = 1$, then say that a frame L is called *Lindelöf* if every cover of L has a countable subset. We are going to characterize Lindelöf frames based on strongly fixed ideals in $\mathcal{R}L$. Let us recall that a space X is Lindelöf if every open cover of X admits a countable subfamily whose union is X. For a weakly spatial frame, it is clear that if ΣL is a Lindelöf space, then L is a Lindelöf frame. The next lemma show that the converse is also true when L is weakly spatial.

Lemma 2.7. *A weakly spatial frame L is Lindelöf if and only if ΣL is a Lindelöf space.*

Proof. To prove the non-trivial part of the lemma, let $\Sigma L = \bigcup_{\lambda\in\Lambda} \Sigma_{x_\lambda}$. Then $\Sigma L = \Sigma_{\bigvee_{\lambda\in\Lambda} x_\lambda}$, and hence, $\bigvee_{\lambda\in\Lambda} x_\lambda = 1$ since L is weakly spatial. By the hypothesis there exists a countable subset S of Λ such that $\bigvee_{\lambda\in S} x_\lambda = 1$, consequently $\bigcup_{\lambda\in S} \Sigma_{x_\lambda} = \Sigma L$ and the proof is complete. $\square$

Before proving the next result, let us notice the following about zero sets. Let $f : \Sigma L \to \mathbb{R}$ be a continuous function. For every $p, q \in \mathbb{Q}$, define $\widehat{f}(p, q) = \bigvee\{a \in L : f(\Sigma_a) \subseteq \langle p, q\rangle\}$, where $\langle p, q\rangle = \{x \in \mathbb{R} : p < x < q\}$. In [19, Lemma 4.5], it is shown that $\widehat{f} : \mathcal{L}(\mathbb{R}) \to L$ is a frame map and $Z(\widehat{f}) = Z(f)$. Also, The map $\Phi : C(\Sigma L) \to \mathcal{R}L$ given by $\Phi(f) = \widehat{f}$ is an f-ring homomorphism which is a monomorphism.

We will use the following characterizations of Lindelöf spaces to prove the theorem number. A space X is Lindelöf provided that every family of closed sets with the countable intersection property has nonempty intersection (see [16]).

Theorem 2.8. *A weakly spatial frame L is Lindelöf if and only if every strongly divisible ideal in $\mathcal{R}L$ is strongly fixed.*

Proof. Necessity. Assume that Q is a strongly divisible ideal in $\mathcal{R}L$. Then, by Proposition 2.4, $Z[Q]$ is closed under countable intersection. Now suppose, by way of contradiction, that $\bigcap Z[Q] = \emptyset$, and so, we can infer that $\bigvee \mathrm{coz}[Q] = 1$. Since the frame L is Lindelöf, there is a countable $I \subseteq Q$ with $\bigvee \mathrm{coz}[I] = \bigvee \mathrm{coz}[Q] = 1$. This shows that $\bigcap Z[I] = \bigcap Z[Q] = \emptyset$. Since the ideal Q is strongly divisible, there exists $\varphi \in Q$ such that $\langle I\rangle \subseteq \langle\varphi\rangle$. It follows that $Z(\varphi) \subseteq \bigcap Z[I] = \bigcap Z[Q] = \emptyset$. Therefore, $Z(\varphi) = \emptyset$ and this implies that φ is a unit element of $\mathcal{R}L$, which is a contradiction. Therefore, $\bigcap Z[Q] = \emptyset$, this means that Q is strongly fixed.

Sufficiency. In light of the forgoing lemma, it is enough to show that ΣL is a Lindelöf space. Let $\mathbb{A}$ be a collection of closed sets in ΣL which is closed under countable intersection. We show that $\bigcap_{A\in\mathbb{A}} A \neq \emptyset$. If we denote by $\mathbb{A}_z$ the collection of all zero-sets containing a member of $\mathbb{A}$, then clearly $\mathbb{A}_z$ is a z- filter such that it is closed under countable intersection since the family $\mathbb{A}$ is closed under countable intersection. Thus, $\mathbb{A}_z^{-1} = \{f \in C(\Sigma L) : Z(f) \in \mathbb{A}_z\}$ is a z-ideal.

In consequence, $\widehat{\mathbb{A}_z^{-1}} = \{\widehat{f} : f \in \mathbb{A}_z^{-1}\}$ is a strongly z-ideal and $\widehat{\mathbb{A}_z} = \{\widehat{Z} : Z \in \mathbb{A}_z\} = \{Z(\widehat{f}) : Z(f) \in \mathbb{A}_z\}$ is closed under countable intersection. By Proposition 2.4, $\widehat{\mathbb{A}_z^{-1}}$ is strongly divisible and hence it is strongly fixed by the current hypothesis. Since the family $Z[\Sigma L]$ of all zero sets in ΣL is a base for closed sets in ΣL, we can infer that

$$\bigcap_{A\in\mathbb{A}} A \supseteq \bigcap_{Z\in\mathbb{A}_z} Z = \bigcap_{Z\in\mathbb{A}_z} \widehat{Z} \neq \emptyset,$$

which implies that ΣL is a Lindelöf space. $\square$

3. CLOSE IDEALS

Let $\varphi \in \mathcal{R}L$. For each $r \in \mathbb{Q}^+$, let

$$B(\varphi, r) = \{\psi \in \mathcal{R}L \mid |\psi - \varphi| \leq \mathbf{r}\}, \text{ and } B_\varphi = \{B(\varphi, r) \mid r \in \mathbb{Q}^+\}.$$

Then there is a unique topology on $\mathcal{R}L$ for which for any $\varphi \in \mathcal{R}L$, the family $\{B(\varphi, r) \mid r \in \mathbb{Q}^+\}$ forms a base for the neighborhood system of φ. This topology is called the uniform topology on $\mathcal{R}L$ (briefly u-topology on $\mathcal{R}L$, for more details see [4] and [7]).

We are going to determine the relation between strongly divisible ideals and closed ideals in the uniform topology on $\mathcal{R}L$. Before, we need some background.

There is a homeomorphism $\tau : \Sigma\mathcal{L}(\mathbb{R}) \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r, s) \not\leq p$ for all prime elements p of $\mathcal{L}(\mathbb{R})$ and all $r, s \in \mathbb{Q}$ (see Proposition 1 of [7], page 12).

Let $\varphi \in \mathcal{R}L$. Define $\overline{\varphi} : \Sigma L \to \mathbb{R}$ by $\overline{\varphi}(p) = \widetilde{p}(\varphi)$. Then, there is an f-ring homomorphism $\Psi : \mathcal{R}L \to C(\Sigma L)$ given by $\Psi(\varphi) = \tau \circ \Sigma\varphi$, by [10, Proposition 3.9]. Moreover, if $p \leq q$, then $\Psi(\varphi)(p) = \widetilde{q}(\varphi)$ for every $\varphi \in \mathcal{R}L$. In particular, $(\tau \circ \Sigma\varphi)(p) = \Psi(\varphi)(p) = \widetilde{p}(\varphi)$ for every $p \in \Sigma L$. Therefore, $\overline{\varphi} = \tau \circ \Sigma\varphi$, and so $\overline{\varphi}$ is continuous. Note that

$$\begin{aligned} Z(\overline{\varphi}) = \{p \in \Sigma L : \varphi[p] = 0\} &= \{p \in \Sigma L : \widetilde{p}(\varphi) = 0\} \\ &= \{p \in \Sigma L : \overline{\varphi} = 0\} = Z(\varphi). \end{aligned}$$

Recall from [11] that a frame L is called *coz-dense* if $\Sigma_{\mathrm{coz}(\varphi)} = \emptyset$ implies $\varphi = \mathbf{0}$.

Lemma 3.1. *Let L be a coz-dense frame. Suppose $\varphi, \psi \in \mathcal{R}L$ such that $Z(\varphi) \subseteq int_{\Sigma L} Z(\psi)$. Then ψ is a multiple of φ.*

Proof. Since $Z(\overline{\varphi}) = Z(\varphi) \subseteq int_{\Sigma L} Z(\psi) = int_{\Sigma L} Z(\overline{\psi})$, by [16, 1D. 1], we have $\overline{\psi}$ is a multiple of $\overline{\varphi}$. Now, Proposition 4.7 of [19] shows that ψ is a multiple of φ. □

We notice that if Q is a strongly z-ideal of $\mathcal{R}L$ and $\psi \in \overline{Q}$, then $|\psi| \in \overline{Q}$. To see this, for every positive integer n, take $\psi_n \in Q$ such that $|\psi - \psi_n| \leq \frac{\mathbf{1}}{\mathbf{n}}$. It follows that $||\psi| - |\psi_n|| \leq |\psi - \psi_n| \leq \frac{\mathbf{1}}{\mathbf{n}}$ for every $n \in \mathbb{N}$. Since Q is a strongly z-ideal, $|\psi_n| \in Q$ for every $n \in \mathbb{N}$. Therefore $|\psi| \in \overline{Q}$.

Proposition 3.2. *Let L be a coz-dense frame and let Q be an ideal of $\mathcal{R}L$. Then Q is closed in the uniform topology on $\mathcal{R}L$ if and only if Q is a strongly divisible sz-ideal.*

Proof. Assume first that Q is a closed ideal of $\mathcal{R}L$. We first show that Q is a strongly z-ideal. Take $\varphi \in \mathcal{R}L$ and $\psi \in Q$ such that $Z(\psi) = Z(\varphi)$. For every positive integer n, let

$$\delta_n = \Big[(\varphi - \frac{\mathbf{1}}{\mathbf{n}}) \vee \mathbf{0}\Big] + \Big[(\varphi + \frac{\mathbf{1}}{\mathbf{n}}) \wedge \mathbf{0}\Big].$$

Then for each n, $Z(\delta_n) = Z(\overline{\delta}_n) = (\overline{\varphi})^{\leftarrow}[\frac{-1}{n}, \frac{1}{n}]$, so

$$Z(\psi) = Z(\varphi) = Z(\overline{\varphi}) \subseteq int_{\Sigma L} Z(\overline{\delta}_n) = int_{\Sigma L} Z(\delta_n).$$

Now, Lemma 3.1 implies that each δ_n is a multiple of ψ, thus each δ_n belongs to the ideal Q. But $|\varphi - \delta_n| \leq \frac{\mathbf{2}}{\mathbf{n}} \to \mathbf{0}$ as $n \to \infty$, therefore δ_n converges to φ in the uniform topology. This implies that $\varphi \in Q$, showing that Q is a strongly z-ideal. It remains to show that Q is strongly divisible. Let (φ_n) be a countable subset of Q. In light of the first part on Proposition 2.4, to prove strongly divisibility it suffices to show that $\bigcap_{n=1}^{\infty} Z(\varphi_n) \in Z[Q]$. Putting $\rho_n = \sum_{i=1}^{n}(|\varphi_i| \wedge \mathbf{2}^{-i})$ for each $n \in \mathbb{N}$, it is clear that the sequence ρ_n converges to $\rho = \sum_{n=1}^{\infty}(|\rho_n| \wedge \mathbf{2}^{-n})$ in the uniform topology. But for each n, $Z(\rho_n) = \bigcap_{i=1}^{n} Z(\varphi_i) \in Z[Q]$. This shows that each $\rho_n \in Q$ because Q is a strongly z-ideal. In consequence, $\rho \in Q$ since Q is a closed ideal. Thus

$$\bigcap_{n=1}^{\infty} Z(\varphi_n) = Z(\rho) \in Z[Q].$$

Conversely, suppose the stated conditions hold. Consider $\varphi \in \overline{Q}$. Then, for every positive integer n, take $\psi_n \in Q$ such that $|\varphi - \psi_n| \leq \frac{\mathbf{1}}{\mathbf{n}}$. It follows that $|\overline{\varphi} - \overline{\psi}_n| \leq \frac{\mathbf{1}}{\mathbf{n}}$ for every $n \in \mathbb{N}$ since $\widetilde{p} : \mathcal{R}L \to \mathbb{R}$ preserves

$+, ., \wedge, \vee$ for every $p \in \Sigma L$. This implies that $\bigcap_{n=1}^{\infty} Z(\overline{\psi_n}) \subseteq Z(\overline{\varphi})$ since

$$\begin{aligned} x \in \textstyle\bigcap_{n=1}^{\infty} Z(\overline{\psi_n}) \quad & \Rightarrow \overline{\psi_n}(x) = 0 && \text{for every } n \in \mathbb{N} \\ & \Rightarrow |\overline{\varphi}(x) - 0| \leq \tfrac{\mathbf{1}}{\mathbf{n}} && \text{for every } n \in \mathbb{N} \\ & \Rightarrow |\overline{\varphi}(x)| = 0 \\ & \Rightarrow x \in Z(\overline{\varphi}). \end{aligned}$$

And because, for every $\alpha \in \mathcal{R}L$, $Z(\alpha) = Z(\overline{\alpha})$, we conclude that

$$\bigcap_{n=1}^{\infty} Z(\psi_n) \subseteq Z(\varphi).$$

By Proposition 2.4, we have $\bigcap_{n=1}^{\infty} Z(\psi_n) \in Z[Q]$ and so $Z(\varphi) \in Z[Q]$. But Q is a strongly z-ideal, and hence $\varphi \in Q$. Therefore $\overline{Q} = Q$, which implies that Q is closed. $\square$

Combination Proposition 2.4 with the foregoing proposition yields the following corollary.

Corollary 3.3. *An ideal Q of $\mathcal{R}L$ is closed if and only if it is a strongly z-ideal such that $Z[Q]$ is closed under countable intersection.*

For any $p \in \Sigma L$, we write

$$\begin{aligned} M_p \quad & = \{\varphi \in \mathcal{R}L : p \in Z(\varphi)\} \\ & = \{\varphi \in \mathcal{R}L : \varphi[p] = 0\} \\ & = \{\varphi \in \mathcal{R}L : \widetilde{p}(\varphi) = 0\} \end{aligned}$$

and

$$O_p = \{\varphi \in \mathcal{R}L : p \in int_{\Sigma L} Z(\varphi)\}$$

which are two strongly z-ideals in $\mathcal{R}L$. In Lemma 3.2 of [12], it is proved that $\varphi[p] = 0$ if and only if $\operatorname{coz}\varphi \leq p$. Therefore, $M_p = \{\varphi \in \mathcal{R}L : \operatorname{coz}\varphi \leq p\}$.

Proposition 3.4. *If $p \in \Sigma L$, then M_p is a closed ideal in $\mathcal{R}L$ and $\overline{O_p} = M_p$.*

Proof. First, we show that M_p is a closed ideal in $\mathcal{R}L$. We must prove that $\overline{M_p} = M_p$. Since $M_p \subseteq \overline{M_p}$, it is enough to show that $\overline{M_p} \subseteq M_p$. Suppose $\psi \in \overline{M_p}$. Take $n \in \mathbb{N}$. Then there is an element $\varphi_n \in M_p$ such that

$$|\psi - \varphi_n| \leq \frac{\mathbf{1}}{\mathbf{n}},$$

and so $|\overline{\psi} - \overline{\varphi_n}| \leq \frac{\mathbf{1}}{\mathbf{n}}$. This shows that $|\overline{\psi}(p) - \overline{\varphi_n}(p)| \leq \frac{\mathbf{1}}{\mathbf{n}}(p)$ which implies that $|\overline{\psi}(p) - 0| \leq \frac{1}{n}$ since $\varphi_n \in M_p$. Thus, for every $n \in \mathbb{N}$, $|\overline{\psi}(p) \leq \frac{1}{n}$ which implies $p \in Z(\overline{\psi}) = Z(\psi)$. It follows that $\psi \in M_p$ and so $\overline{M_p} \subseteq M_p$ which implies $\overline{M_p} = M_p$.

It remains to show that $\overline{O_p} = M_p$. Since M_p is a closed ideal in $\mathcal{R}L$ and $O_p \subseteq M_p$, it is enough to show that $M_p \subseteq \overline{O_p}$. Let $\varphi \in M_p$ and $r \in \mathbb{Q}^+$. Take

$$\delta_r = \Big[(\varphi - \frac{\mathbf{r}}{\mathbf{2}}) \vee \mathbf{0}\Big] + \Big[(\varphi + \frac{\mathbf{r}}{\mathbf{2}}) \wedge \mathbf{0}\Big].$$

Then, $Z(\delta_r) = Z(\overline{\delta}_r) = (\overline{\varphi})^{\leftarrow}[\frac{-r}{2}, \frac{r}{2}]$, so

$$Z(\varphi) = Z(\overline{\varphi}) \subseteq int_{\Sigma L} Z(\overline{\delta}_r) = int_{\Sigma L} Z(\delta_r).$$

By definition of O_p, $\delta_r \in O_p$. But $|\varphi - \delta_r| \leq \mathbf{r}$ for every $r \in \mathbb{Q}^+$, therefore $\varphi \in \overline{O_p}$ and the proof is complete.

□

The next result characterizes Lindelöf frames in terms of closed ideals of $\mathcal{R}L$. We note that every weakly spatial frame is a coz-dense frame (see [13, Lemma 3.5])

Theorem 3.5. *A weakly spatial frame L is Lindelöf if and if every closed ideal in $\mathcal{R}L$ is strongly fixed.*

Proof. Necessity. By Proposition 3.2 and Theorem 2.8, it is obvious.

Sufficiency. By Theorem 2.8, it is enough to show that every strongly divisible ideal in $\mathcal{R}L$ is strongly fixed. Let Q be a strongly divisible ideal of $\mathcal{R}L$. Then $Q \subseteq M = Z^{\leftarrow}[Z[Q]]$. Since M is a strongly divisible sz-ideal, Proposition 3.2 implies that it is closed. Thus the present hypothesis shows that $\bigcap Z[M] \neq \emptyset$, implying that $\bigcap Z[Q] \neq \emptyset$, since $\bigcap Z[M] \subseteq \bigcap Z[Q]$. Therefore Q is strongly fixed. □

Combination Theorem 2.8 with the previous theorem yields the following corollary.

Corollary 3.6. *For a weakly spatial frame L, every closed ideal in $\mathcal{R}L$ is strongly fixed if and only if every strongly divisible ideal in $\mathcal{R}L$ is strongly fixed.*

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