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The induced contractive maps on the covering spaces

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Abstract. Let $(\tilde{X}, p)$ be the universal covering space of a compact metrizable space X, which is compact and locally path connected. In this paper, we show that there exist metrics d and d' for X and $\tilde{X}$, respectively, such that any contractive map $f : X \to X$ induces a contractive map on $\tilde{X}$. As an application, it is obtained that every iterated function system(IFS) on the space X with attractor K, induces an IFS on $\tilde{X}$ with attractor $\tilde{K}$, such that $p(\tilde{K}) = K$.

Keywords: Covering spaces, Iterated function systems, Contractive maps.

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1. Introduction and preliminaries

Let X be a compact metrizable space, and let f be a continuous map of X onto itself. Fix any metric d for X (which is compatible with the topology of X). A mapping $f : X \to X$ is called *contractive* (with a contraction constant λ), provided that there exists a number $\lambda \in [0, 1)$ such that $d(f(u), f(v)) \leq \lambda d(u, v)$ for every $u, v \in X$. We say that $f : X \to X$ is *locally contractive*, if for every $x \in X$ there exist numbers $\lambda \in [0, 1)$ and $\epsilon > 0$, which may depend on x, such that for all $u, v \in$

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$N_\epsilon(x)$, $d(f(u), f(v)) \leq \lambda d(u, v)$. Moreover, a mapping f of X into itself is said to be (ϵ, λ)-*uniformly locally contractive*, if it is locally contractive and both ϵ and λ do not depend on x [11].

The Banach fixed-point theorem states that every contraction mapping on a complete metric space X has a unique fixed point, and that for any $x \in X$ the iterated function sequence $\{f^n(x)\}_{n=0}^{\infty}$ converges to this fixed point. This concept is very useful for iterated function systems where contraction mappings are often used. IFSs firstly introduced and popularized by Hutchinson [6] and Barnsley [1]. According to Hutchinson [6], let (X, d) be a complete metric space and let $f_0, f_1, \ldots, f_N$ be contractive self-maps on X with contraction constants $0 \leq r_i < 1$, then the system $\{X; f_0, f_1, \ldots, f_N\}$ is called an *iterated function system*(IFS) with contractivity factor $r = \max_{1 \leq i \leq N} r_i$. In [6], the author proved that for any IFS $\{X; f_0, f_1, \ldots, f_N\}$, there is a unique compact non-empty set K satisfying $K = f_1(K) \cup \ldots \cup f_N(K)$, which is called the *attractor* of the IFS [6]. Iterated function systems are among the basic methods for constructing fractals; see [2, 6, 10]. On the other hand, Self-similarity is the most important property of the classical fractals. Recently, iterated function systems have proved to be useful tools in data and image compression [3, 7] and in the theory of random dynamical systems [8].

Let X be a topological space, $(\tilde{X}, p)$ is called a *covering space* of X provided that $\tilde{X}$ is path connected, $p : \tilde{X} \to X$ is continuous map and for every $x \in X$, there exists an open neighborhood U of x, such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p. We say that a covering space $\tilde{X}$ of X is *universal* if $\pi_1(\tilde{X}) = 1$. Also, a covering space $\tilde{X}$ of X with fiber F is called *finite-sheeted* if $|F| < \infty$.

Let X be a compact metrizable space and $(\tilde{X}, p)$ be a covering space of X which is compact and locally path connected. This note, proves that every contractive map $f : X \to X$ induces a contractive map on $\tilde{X}$, for some compatible metrics on X and $\tilde{X}$. In particular, we show that any iterated function system on the space X induces an IFS on the covering space $\tilde{X}$. Moreover, the relation between the attractors of these systems is obtained.

2. The contracting maps on the covering spaces

In this section, we show that any contractive map on a compact metrisable space X, induces a contracitive map on the covering space $\tilde{X}$.

Let $(\tilde{X}, p)$ be a covering space of a metrizable space X. By [11, Theorem 3], there exist metrics d on X and d' on $\tilde{X}$ inducing the topologies of X and $\tilde{X}$ respectively, such that the family $\mathcal{S}$ of unit spherical regions in

(X, d) has the following property: for every $S \in \mathcal{S}$, $f^{-1}(S)$ is a union of a family $\mathcal{F}(S)$ consisting of pairwise disjoint open sets in $(\tilde{X}, d')$ each of which is mapped isometrically onto S. In the sequel, we fix the metrics d and d' on the spaces X and $\tilde{X}$, respectively.

Following Ciesielski [11], a metric space X is said to be ϵ-*chainable* if for every $a, b \in X$ there exists an ϵ-*chain* from a to b, that is a finite sequense $\xi = \{a = x_0, x_1, \ldots, x_n = b\}$ such that $d(x_{i-1}, x_i) < \epsilon$, for $i = 1, 2, \ldots, n$. The length of the ϵ-chain ξ is define as $\ell(\xi) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$. It is well known that any connected space is ϵ-chainable for any $\epsilon > 0$. The next lemma shows that in connected spaces a new metric may be defined such that functions locally contractive in original metric become globally contractive in the new one. Also, this metric is topologically equivalent to the former.

Lemma 2.1. [4] *Given $\epsilon > 0$ and assume that (X, d) is connected or, more generally, ϵ-chainable. Consider the metric $D_\epsilon : X \times X \to [0, 1)$ is defined by*

$$D_\epsilon(x, y) = \inf\{\ell(\xi) : \xi \textit{ is an } \epsilon\textit{-chain from } x \textit{ to } y\}.$$

The metric D_ϵ on X is topologically equivalent to d. Moreover,
(i) If (X, d) is complete, then so is (X, D_ϵ),
(ii) If $f : (X, d) \to (X, d)$ is (γ, λ)-uniformly locally contractive for some $\gamma > \epsilon$ and $\lambda \in [0, 1)$, then $f : (X, D_\epsilon) \to (X, D_\epsilon)$ is contractive with constant λ.

Let $(\tilde{X}, p)$ be a covering space of compact metrizable space X. Consider the metric d on X and d' on $\tilde{X}$, as described above and suppose that $\tilde{X}$ is locally path connected. In the following, we show that for any contractive map $f : (X, d) \to (X, d)$, the lifting map $g : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ is contractive, for some metric D'_ϵ topologically equivalent to d'.

Theorem 2.2. *Let (X, d) and $(\tilde{X}, d')$ be compact metric spaces, where $(\tilde{X}, p)$ is a covering space of X and is locally path connected. If the map $f : X \to X$ is contractive and $(f \circ p)_*(\pi_1(\tilde{X}, \tilde{x}_1)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}))$, for some $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}_1 \in p^{-1}(f^{-1}(x))$, then there exists a metric D'_ϵ, for some $\epsilon > 0$, which is topologically equivalent to d' such that the map $g : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ satisfying $p \circ g = f \circ p$ and $g(\tilde{x}_1) = \tilde{x}$, is contractive.*

Proof. By the definition, for the contractive map $f : X \to X$ there is a number $0 \leq \lambda < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \lambda d(x, y)$. Since $\tilde{X}$ is locally path connected and $(f \circ p)_*(\pi_1(\tilde{X}, \tilde{x}_1)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}))$, there is a lifting map $g : \tilde{X} \longrightarrow \tilde{X}$ of $f \circ p$ with $p \circ g = f \circ p$ and $g(\tilde{x}_1) = \tilde{x}$ [9, Theorem 10.13]. For $\tilde{x} \in \tilde{X}$, take a spherical region S_x of the point

$x := p(\tilde{x}) \in X$ with $p^{-1}(S_x) = \cup_j U_j$, where U_j's are pairwise disjoint open sets in $\tilde{X}$ such that $p|_{U_j} : U_j \longrightarrow S_x$ is an isometric. Similarly for $g(\tilde{x}) \in \tilde{X}$, take a spherical region $S_{f(x)}$ of the point $f(x) = f(p(\tilde{x})) = p(g(\tilde{x})) \in X$ with $p^{-1}(S_{f(x)}) = \cup_j V_j$, where V_j's are pairwise disjoint open sets in $\tilde{X}$ such that $p|_{V_j} : V_j \longrightarrow S_{f(x)}$ is an isometric.

Since g is continuous, we can choose an open set $C_{\tilde{x}}$ of $\tilde{x}$ such that $g(C_{\tilde{x}}) \subseteq V_j$. For an open neighborhood U_j of $\tilde{x}$, let $V_{\tilde{x}} := N_{\gamma_{\tilde{x}}}(\tilde{x}) \subseteq C_{\tilde{x}} \cap U_j$, for some $\gamma_{\tilde{x}} > 0$. So for all $\tilde{u}, \tilde{v} \in V_{\tilde{x}}$,

$$\begin{aligned} d'(g(\tilde{u}), g(\tilde{v})) &= d(pg(\tilde{u}), pg(\tilde{v})) \\ &= d(fp(\tilde{u}), fp(\tilde{v})) \leq \lambda d(p(\tilde{u}), p(\tilde{v})) \\ &= \lambda d'(\tilde{u}, \tilde{v}). \end{aligned}$$

Thus the mapping $g|_{V_{\tilde{x}}}$ is contractive. Since $\tilde{X}$ is compact, there exists the Lebesgue number $\gamma > 0$ for open covering $\{V_{\tilde{x}}\}_{\tilde{x} \in \tilde{X}}$. Therefore for all $\tilde{u}, \tilde{v} \in N_\gamma(\tilde{x})$,

$$d'(g(\tilde{u}), g(\tilde{v})) < \lambda d'(\tilde{u}, \tilde{v}),$$

so g is (λ, γ)-uniformly locally contractive map. Since $\tilde{X}$ is connected, for $\tilde{x}, \tilde{y} \in \tilde{X}$, put

$$D'_\epsilon(\tilde{x}, \tilde{y}) = \inf\{\ell'(\xi) : \xi \text{ is an } \epsilon\text{-chain from } \tilde{x} \text{ to } \tilde{y}\},$$

where ℓ' is the length of ϵ-chain ξ. Applying Lemma 2.1 implies that the mapping $g : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ is contractive for some $\epsilon < \gamma$ and has the contraction constant λ. □

The following results are immediately obtained from Theorem 2.2.

Corollary 2.3. *Let (X, d) and $(\tilde{X}, d')$ be compact metric spaces, where $(\tilde{X}, p)$ is a universal covering space of X and is locally path connected. If the map $f : X \to X$ is contractive, then there exists a metric D'_ϵ, for some $\epsilon > 0$, which is topologically equivalent to d' such that the map $g : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ satisfying $p \circ g = f \circ p$, is contractive.*

Corollary 2.4. *Let (X, d) and $(\tilde{X}, d')$ be metric spaces, where X is compact and $(\tilde{X}, p)$ is a universal finite-sheeted covering space of X and is locally path connected. If the map $f : X \to X$ is contractive, then there exists a metric D'_ϵ, for some $\epsilon > 0$, which is topologically equivalent to d' such that the map $g : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ satisfying $p \circ g = f \circ p$, is contractive.*

Example 2.5. For $I = [0, 1]$, the map

$$\begin{aligned} p : I \times I &\longrightarrow S^1 \times S^1 \\ (t, s) &\longmapsto (e^{2\pi i t}, e^{2\pi i s}) \end{aligned}$$

is a universal covering map of the Torus. Using Theorem 2.2, implies that contarctive map $f : S^1 \times S^1 \to S^1 \times S^1$ given by

$$f(e^{2\pi it}, e^{2\pi is}) = (e^{2\pi itr}, e^{2\pi isr})$$

for some $0 < r < 1$, induces a contractive map g on $I \times I$ satisfying $p \circ g = f \circ p$.

Remark 2.6. Let (X, d) and $(\tilde{X}, d')$ be compact metric spaces, where $(\tilde{X}, p)$ is a covering space of X satisfying the assumption of Theorem 2.2 and is convex. If the map $f : X \to X$ is contractive, then the arguments in the proof of the Theorem 2.2, show that the mapping $g : (\tilde{X}, d') \to (\tilde{X}, d')$ is (λ, γ)-uniformly locally contractive and since any uniformly contractive map on the convex space is contractive [5], yields that $g : (\tilde{X}, d') \to (\tilde{X}, d')$ is contractive.

A mapping $f : X \to X$ is said to be *expansive*, if there is an number $\lambda > 1$ such that $d(f(x), f(y)) \geq \lambda d(x, y)$, for any $x, y \in X$. It is called (ϵ, λ)-*uniformly locally expansive*(where $\epsilon > 0$ and $\lambda > 1$) provided that for any $x \in X$, $f|_{N_\epsilon(x)}$ is expansive with the same constant λ.

With the assumptions of Theorem 2.2, the following result is obtained.

Proposition 2.7. *If the map $f : (X, d) \to (X, d)$ is an expansive open map and $(f \circ p)_*(\pi_1(\tilde{X}, \tilde{x}_1)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}))$, for some $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}_1 \in p^{-1}(f^{-1}(x))$, then there exists a metric D'_ϵ, for some $\epsilon > 0$, which is topologically equivalent to d' such that the map $g : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ satisfying $p \circ g = f \circ p$ and $g(\tilde{x}_1) = \tilde{x}$, is expansive.*

Proof. Suppose that f is expansive with constant λ. Let $g : \tilde{X} \longrightarrow \tilde{X}$ be a lifting map of $f \circ p$ satisfying $p \circ g = f \circ p$ and $g(\tilde{x}_1) = \tilde{x}$. For $\tilde{x} \in \tilde{X}$, take a spherical region S_x of the point $x := p(\tilde{x}) \in X$ with $p^{-1}(S_x) = \cup_j U_j$, where U_j's are pairwise disjoint open sets in $\tilde{X}$ such that $p|_{U_j} : U_j \longrightarrow S_x$ is an isometric. Similarly for $g(\tilde{x}) \in \tilde{X}$, take a spherical region $S_{f(x)}$ of the point $f(x) = f(p(\tilde{x})) = p(g(\tilde{x})) \in X$ with $p^{-1}(S_{f(x)}) = \cup_j V_j$, where V_j's are pairwise disjoint open sets in $\tilde{X}$ such that $p|_{V_j} : V_j \longrightarrow S_{f(x)}$ is an isometric. Take an open set $C_{\tilde{x}}$ of $\tilde{x}$ such that $g(C_{\tilde{x}}) \subseteq V_j$. For an open neighborhood U_j of $\tilde{x}$, let $V_{\tilde{x}} := N_{\gamma_{\tilde{x}}}(\tilde{x}) \subseteq C_{\tilde{x}} \cap U_j$, for some $\gamma_{\tilde{x}} > 0$. So for all $\tilde{u}, \tilde{v} \in V_{\tilde{x}}$,

$$\begin{aligned} d'(g(\tilde{u}), g(\tilde{v})) &= d(pg(\tilde{u}), pg(\tilde{v})) \\ &= d(fp(\tilde{u}), fp(\tilde{v})) \geq \lambda d(p(\tilde{u}), p(\tilde{v})) \\ &= \lambda d'(\tilde{u}, \tilde{v}). \end{aligned}$$

These implies that $g|_{V_{\tilde{x}}}$ is expansive and therefore $g^{-1}|_{g(V_{\tilde{x}})}$ is contractive. The same argument as in the proof of the Theorem 2.2, shows that $g^{-1} : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ is contractive for some $\epsilon > 0$. □

3. The attractors of induced iterated function systems

In this section, it is obtained that any iterated function system on the space X induces an iterated function system on the covering space $\tilde{X}$ and the connection between the attractors of these IFSs is studied.

Lemma 3.1. *Let (X, d) and $(\tilde{X}, d')$ be compact metric spaces, where $(\tilde{X}, p)$ is a covering space of X and is locally path connected. If the system $\{X; f_0, f_1, \ldots, f_N\}$ is an IFS with contractivity factor r and $(f_i \circ p)_*(\pi_1(\tilde{X}, \tilde{x})) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}))$, for all $\tilde{x} \in p^{-1}(x)$ and $0 \leq i \leq N$, then there exists a metric D'_ϵ, for some $\epsilon > 0$, such that the system $\{\tilde{X}; g_0, g_1, \ldots, g_N\}$ is an IFS with contractivity factor r(with respect to metric D'_ϵ), with $p \circ g_i = f_i \circ p$.*

Proof. By Theorem 2.2, any $g_i : (\tilde{X}, D'_\epsilon) \to (\tilde{X}, D'_\epsilon)$ is contractive with the same contraction constant of the map f_i. Thus the system $\{\tilde{X}; g_0, g_1, \ldots, g_N\}$ is an IFS and has the contractivity factor r(with respect to metric D'_ϵ). □

Now, we study the the attractor of induced IFS $\{\tilde{X}; g_0, g_1, \ldots, g_N\}$. Let (X, d) be a complete metric space. Denote by $H(X)$ the space of all non-empty compact subset of X. For $x \in X$ and $B \in H(X)$ the distance between the point x and the set B is defined as follows:

$$d(x, B) = \min\{d(x, y) : y \in B\}.$$

Let $A, B \in H(X)$ the distance from the set A to the set B is defined as

$$d(A, B) = \max\{d(x, B) : x \in A\}.$$

Note that $d(A, B) \neq d(B, A)$. Since $d : H(X) \to [0, \infty)$ is not a metric, for $A, B \in H(X)$, the Hausdorff metric $h := h(d)$ on $H(X)$ is defined by

$$h(A, B) = d(A, B) \vee d(B, A), \text{ where } x \vee y = \max\{x, y\}.$$

Suppose that IFS $\{X; f_0, f_1, \ldots, f_N\}$ has a contractivity factor r. Then the *Hutchinson operator* $F : (H(X), h(d)) \to (H(X), h(d))$ defined by $F(B) = \cup_{i=1}^{N} f_i(B)$ for all $B \in H(X)$, is a contractive map with contractivity factor r (see [6]). Therefore, by the Banach fixed-point theorem it has a unique fixed point $K = F(K)$ and is given by $K = \lim_{n\to\infty} F^n(B)$ for any $B \in H(X)$. The fixed point $K \in H(X)$ is called the attractor of IFS $\{X; f_0, f_1, \ldots, f_N\}$.

Proposition 3.2. *Let $(\tilde{X}, p)$ be a covering space of compact space X which is compact and locally path connected. If the IFS $\{X; f_0, f_1, \ldots, f_N\}$ has an attractor K, then the IFS $\{\tilde{X}; g_0, g_1, \ldots, g_N\}$, which is described in Lemma 3.1, has an attractor $\tilde{K}$, with $p(\tilde{K}) = K$.*

Proof. Suppose that IFS $\{X; f_0, f_1, \ldots, f_N\}$ has a contractivity factor r. By Lemma 3.1, there exists a metric D'_ϵ, for some $\epsilon > 0$, such that the transformation $G : (H(\tilde{X}), h(D'_\epsilon)) \to (H(\tilde{X}), h(D'_\epsilon))$ defined by $G(B) = \cup_{i=1}^{N} g_i(B)$ for all $B \in H(\tilde{X})$, is a contractive map with the same factor r. So there exists a compact set $\tilde{K}$, such that $G(\tilde{K}) = \tilde{K}$. Now, we show that $p(\tilde{K}) = K$. Indeed, we have

$$\begin{aligned} p(\tilde{K}) = p(G(\tilde{K})) &= p(\cup_{i=1}^{N} g_i(\tilde{K})) \\ &= \cup_{i=1}^{N} p \circ g_i(\tilde{K}) = \cup_{i=1}^{N} f_i \circ p(\tilde{K}) \\ &= F(p(\tilde{K})), \end{aligned}$$

so $p(\tilde{K})$ is a fixed point of transformation F. Since the mapping p is continuous, the set $p(\tilde{K})$ is compact. But the set K is unique fixed point of transformation F, therefore $p(\tilde{K}) = K$. □

Example 3.3. Let f_0, f_1 be contractive maps on S^1, defined by

$$f_0(e^{2\pi ix}) = e^{2\pi ix/3}, \; f_1(e^{2\pi ix}) = e^{2\pi i(x/4+1/4)}.$$

The IFS $\{S^1; f_0, f_1\}$ has an attractor $K \subset S^1$([6]). Consider the covering space $(I = [0, 1], exp)$ of the circle S^1, where $exp : I \to S^1$ is given by $exp(x) = e^{2\pi ix}$. By Proposition 3.2, The induced IFS $\{I; g_0, g_1\}$ has an attractor $\tilde{K}$ with $exp(\tilde{K}) = K$.

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