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Linear System of Equations with Doubly Stochastic Interval Coefficient Matrix

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Abstract. In this paper, we first give an overview of doubly stochastic interval matrices. Then, we present some theories about the interval linear system whose coefficient matrix is doubly stochastic interval matrix. Also, we give an outer estimation for the solution set of these systems.

Keywords: Stochastic matrix, Interval matrix, Doubly stochastic interval matrix, interval linear system.

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1. Introduction

A real nonnegative matrix A is said to be r-doubly stochastic matrix if each of its row and column sums is r. The set of $n\times n$ r-doubly stochastic matrices is denoted by Γ_n^r[1]. These matrices have wide applications in engineering, elevator, robotic problems, etc.

But the elements of a matrix, occurring in practice are usually obtained from experiments, hence they may appear with uncertainties. We represent the uncertain elements in interval forms. Therefore, we generalize the definition of r-doubly stochastic matrix to interval matrices. A nonnegative $n \times n$ interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is said to be $[\alpha, \beta]$-doubly stochastic interval matrix ($[\alpha, \beta]$-D.S.I matrix) and denoted by $\mathbf{A}_{[\alpha,\beta]}$

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if $\underline{A}$ and $\overline{A}$ are α-doubly stochastic and β-doubly stochastic matrices, respectively.

A nonnegative interval matrix $\mathbf{A}$ is an $[\alpha,\beta]$-doubly stochastic interval matrix if and only if

$$\mathbf{A}J_n = J_n\mathbf{A} = ([\alpha,\beta])_{n\times n},$$

where J_n is the $n\times n$ matrix whose entries are 1 and $([\alpha,\beta])_{n\times n}$ is the $n\times n$ interval matrix whose entries are $[\alpha,\beta]$. The product of two doubly stochastic interval matrices is a doubly stochastic interval matrix. The eigenvalue set of an $[\alpha,\beta]$-D.S.I matrix lie in the interval $[-\beta,+\beta]$ and $e=(1,1,\cdots,1)^T$ is a real eigenvector. For more details about interval doubly stochastic matrices, we refer to [7].

In this paper, we use notations $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ as the field of real numbers and the vector space of $m\times n$ real matrices, respectively. We denote any orthant of $\mathbb{R}^n$ by O and the set of all $m\times n$ interval matrices by $\mathbb{IR}^{m\times n}$. We assume that the reader is familiar with a basic interval arithmetic, otherwise see [2].

For the interval matrix $\mathbf{A}=[\underline{A},\overline{A}]$, the center matrix denoted by A_c and the radius matrix denoted by Δ are defined respectively as

$$A_c = \frac{1}{2}(\underline{A}+\overline{A})\ ,\ \Delta = \frac{1}{2}(\overline{A}-\underline{A}).$$

An $n\times n$ interval matrix $\mathbf{A}=[\underline{A},\overline{A}]$ is said to be regular if each $A\in\mathbf{A}$ is nonsingular. For a regular $\mathbf{A}$ we define the inverse interval matrix as $\mathbf{A}^{-1}=[\underline{B},\overline{B}]$, where

$$\underline{B} = min\{A^{-1}; A\in\mathbf{A}\};$$
$$\overline{B} = max\{A^{-1}; A\in\mathbf{A}\}.$$

An $M-$matrix is a square matrix $\mathbf{A}\in\mathbb{IR}^{n\times n}$ such that $\mathbf{A}_{ik}\leq 0$ for $i\neq k$, and $\mathbf{A}u>0$ for some positive vector $u\in\mathbb{R}^n$. M-matrices are a class of inverse positive matrices, i.e. regular square interval matrices with nonnegative inverse.

We consider interval linear systems of equations

$$\mathbf{A}x = \mathbf{b} \tag{1.1}$$

with an interval matrix $\mathbf{A}$ and an interval right-hand side vector $\mathbf{b}$. The system $\mathbf{A}x=\mathbf{b}$ is understood as a family of point linear systems $Ax=b$ with $A\in\mathbf{A}$ and $b\in\mathbf{b}$. For interval systems of equations various solutions and solution sets can be defined, and the most popular of them is united solution set

$$\Sigma(\mathbf{A},\mathbf{b}) = \{x\in\mathbb{R}^n, Ax=b \text{ for some } A\in\mathbf{A}, b\in\mathbf{b}\}. \tag{1.2}$$

In general, the solution set has a very complicated structure and in most cases, it suffices to know an estimate of the solution set by simpler sets i.e. having less constructive complexity. One of the important problems related to interval matrices is finding an outer estimation for the solution set [8]. If $\mathbf{A}$ is regular, a special case among outer estimations for the solution set is the enclosure of the solution set, i.e. an interval vector that contains the solution set $\Sigma(\mathbf{A}, \mathbf{b})$ [5]. The tightest interval vector that contains the solution set $\Sigma(\mathbf{A}, \mathbf{b})$ is said hull of $\Sigma(\mathbf{A}, \mathbf{b})$ which denoted $\square\Sigma(\mathbf{A}, \mathbf{b})$. In recent years, some new direct and iterative methods for solving systems of interval and parametric linear equations were also developed [9, 10, 11].

When the coefficient matrix in an interval linear system is inverse positive, the hull of the solution set can be described explicitly. See the following theorem from Neumaier [3].

Theorem 1.1. *Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ be inverse positive. Then*

$$\square\Sigma(\mathbf{A}, \mathbf{b}) = [\tilde{A}^{-1}\underline{b}, \hat{A}^{-1}\overline{b}],$$

where $\tilde{A}, \hat{A} \in \mathbf{A}$ are defined by

$$\tilde{A}_{ij} = \overline{A}_{ij} \text{ if } \underline{x}_j \geq 0 \text{ and } \tilde{A}_{ij} = \underline{A}_{ij} \text{ otherwise,}$$
$$\hat{A}_{ij} = \overline{A}_{ij} \text{ if } \overline{x}_j \leq 0 \text{ and } \hat{A}_{ij} = \underline{A}_{ij} \text{ otherwise.}$$

In most of the researches mentioned above, the coefficient matrix was considered regular and the enclosure of the solution set was obtained. In this paper, we consider the coefficient matrix in interval linear system is interval doubly stochastic which can be regular or singular and we obtain an outer estimation for the solution set.

2. Main results

Consider the interval linear system whose coefficient matrix is an $[\alpha, \beta]$-doubly stochastic interval matrix ; i.e. the interval linear system

$$\mathbf{A}x = \mathbf{b} \quad ; \quad \mathbf{A} \text{ is an } [\alpha, \beta]\text{-D.S.I.} \tag{2.1}$$

As well known, regularity of $\mathbf{A}$ implies boundedness of the solution set and this set can can be described explicitly if $\mathbf{A}$ is iverse positive. In this section, we first give a condition for regularity of $\mathbf{A}$. Then, we study when an $[\alpha, \beta]$-D.S.I matrix is inverse positive or M-matrix. Finally, we express an outer estimation for solution set of this system.

Theorem 2.1. *An $[\alpha, \beta]$-D.S.I matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is regular if A_c is nonsingular and*

$$\beta - \alpha < \frac{2}{||A_c^{-1}||_2} \tag{2.2}$$

Proof. Suppose that $\mathbf{A}$ is an $[\alpha,\beta]$-D.S.I matrix and Δ is its radius matrix. It is clear that $\Delta^T\Delta \in \Gamma_n^{(\frac{\beta-\alpha}{2})^2}$. Therefore, we have

$$||\Delta||_2^2 = \rho(\Delta^T\Delta) = (\frac{\beta-\alpha}{2})^2.$$

Now, let $\tilde{A} \in \mathbf{A}$ be arbitrary. $\tilde{A}$ can be written as

$$\tilde{A} = A_c + B,$$

where B is a matrix satisfying $|B| \leq \Delta$. We can show that

$$||B||_2 \leq ||\Delta||_2.$$

So, from (2.2), we have

$$||BA_c^{-1}||_2 \leq ||B||_2||A_c^{-1}||_2 < 1.$$

Therefore, $I - BA_c^{-1}$ is nonsingular and this implies the nonsingularity of $\tilde{A}$. □

The next theorem gives a necessary and sufficient condition for an $[\alpha,\beta]$-D.S.I matrix is inverse positive.

Theorem 2.2. *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *be an* $[\alpha,\beta]$*-D.S.I matrix,* $\alpha \neq 0$*. Then* $\mathbf{A}$ *is inverse positive if and only if it is the* $[\alpha,\beta]$ *multiple of a permutation matrix.*

Proof. Suppose that $\mathbf{A}$ is an $[\alpha,\beta]$-D.S.I and inverse positive matrix. Now, consider at least one row of $\mathbf{A}$ contains k nonzero entries. Therefore we have at least one $\tilde{A} \in \mathbf{A}$ such that $\tilde{A} \geq 0$ and it is not a generalized permutation matrix. This implies that $\tilde{A}^{-1} \leq 0$ [1]. Then we would have a contradiction, since $\mathbf{A}$ is inverse positive. It follows that $\mathbf{A}$ has only one nonzero entry in each row. Similarly we can conclude this for each column. Since $\mathbf{A}$ is an $[\alpha,\beta]$-D.S.I matrix, this nonzero entry must be $[\alpha,\beta]$.
To prove converse, let $\mathbf{A}$ be the $[\alpha,\beta]$ multiple of a permutation matrix that $\alpha \neq 0$. Therefore every $A \in \mathbf{A}$ is a nonnegative generalized permutation matrix. It implies that A^{-1} is nonnegative and therefore $\mathbf{A}$ is inverse positive. □

One known class of inverse positive matrices are $M-$matrices. M-matrices play a distinguished role since they behave particularly well in algorithms for the solution of interval linear systems [3]. In the following theorem, we express when a D.S.I matrix is M-matrix.

Theorem 2.3. *An* $[\alpha,\beta]$*-D.S.I matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$*,* $\alpha \neq 0$*, is an* M*-matrix if and only if it is the* $[\alpha,\beta]$ *multiple of the identity matrix.*

Proof. Let $\mathbf{A}$ be an $[\alpha, \beta]$-D.S.I matrix. If $\mathbf{A}$ is an M-matrix then $\mathbf{A}_{ik} \leq 0$ for $i \neq k$ and it implies $\mathbf{A}_{ik} = 0$ for $i \neq k$. On the other hand, each of row and column sums is $[\alpha, \beta]$. Hence, $\mathbf{A}_{ii} = [\alpha, \beta]$ for $i = 1, \cdots, n$.
Conversely, suppose that $\mathbf{A}$ is the $[\alpha, \beta]$ multiple of the identity matrix where $\alpha \neq 0$. If u is the vector $u = (1, \cdots, 1)^T$ then $\mathbf{A}u > 0$ and this implies that $\mathbf{A}$ is an M-matrix. □

If $\mathbf{A}$ is the $[\alpha, \beta]$ multiple of the permutation matrix P then based on Theorem 1.1, the hull of the solution set can be given by

$$\square\Sigma(\mathbf{A}, \mathbf{b}) = \begin{cases} [\frac{1}{\beta}P\underline{b}, \frac{1}{\alpha}P\overline{b}] & ; \quad \underline{b} \geq 0, \\ [\frac{1}{\alpha}P\underline{b}, \frac{1}{\beta}P\overline{b}] & ; \quad \underline{b} \leq 0, \\ [\frac{1}{\alpha}P\underline{b}, \frac{1}{\alpha}P\overline{b}] & ; \quad 0 \in \mathbf{b}. \end{cases} \tag{2.3}$$

In fact, this is just the solution that can be determined from the system by a simple calculation.

Example 2.4. Let

$$\mathbf{A} = \begin{pmatrix} 0 & [2,4] & 0 \\ [2,4] & 0 & 0 \\ 0 & 0 & [2,4] \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} [1,5] \\ [2,2] \\ [0,4] \end{pmatrix}.$$

Then $\mathbf{A}$ is inverse positive and we have

$$\square\Sigma(\mathbf{A}, \mathbf{b}) = \begin{pmatrix} [\frac{1}{2}, 1] \\ [\frac{1}{4}, \frac{5}{2}] \\ [0, 2] \end{pmatrix}.$$

Now, consider interval linear system (2.1) that $\mathbf{A}$ is singular or regular. The next theorem gives us an outer estimation for solution set of this system in a fixed orthant O of $\mathbb{R}^n$. We define the sets P and N for a fixed orthant O as

$$P = \{j; \ x_j \geq 0, 1 \leq j \leq n\} \tag{2.4}$$

$$N = \{j; \ x_j \leq 0, 1 \leq j \leq n\} \tag{2.5}$$

Theorem 2.5. *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *be an* $[\alpha, \beta]$*-D.S.I matrix and* $\mathbf{b} \in \mathbb{IR}^n$ *be an interval vector. Then for a fixed orthant* O *of* $\mathbb{R}^n$ *we have*

$$O \cap \Sigma(\mathbf{A}, \mathbf{b}) \subseteq \{x \in O; \ (2.7) - (2.10) hold\}, \tag{2.6}$$

where

$$\alpha\Sigma_{j\in P}x_j + \beta\Sigma_{j\in N}x_j \leq \Sigma_{i=1}^n \overline{b}_i, \tag{2.7}$$

$$\beta\Sigma_{j\in P}x_j + \alpha\Sigma_{j\in N}x_j \geq \Sigma_{i=1}^n \underline{b}_i, \tag{2.8}$$

$$x_j \geq 0 \ ; \ j \in P, \tag{2.9}$$

$$x_j \leq 0 \ ; \ j \in N. \tag{2.10}$$

Proof. Suppose that $x = (x_1, \cdots, x_n)^T \in \Sigma(\mathbf{A}, \mathbf{b})$. Then we have $Ax = b$ for some $A = (a_{ij}) \in \mathbf{A}$ and $b = (b_i) \in \mathbf{b}$,that is,

$$\Sigma_{j=1}^{n} a_{ij} x_j = b_i$$

for $i = 1, \cdots, n$. Therefore summing with respect to i, we have

$$\Sigma_{i=1}^{n} \Sigma_{j=1}^{n} a_{ij} x_j = \Sigma_{i=1}^{n} b_i.$$

Now, let O be a fixed orthant and P and N be the sets (2.4) and (2.5), respectively. Then we have

$$\alpha \Sigma_{j \in P} x_j + \beta \Sigma_{j \in N} x_j \leq \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} a_{ij} x_j \leq \beta \Sigma_{j \in P} x_j + \alpha \Sigma_{j \in N} x_j.$$

On the other hand,

$$\Sigma_{i=1}^{n} \underline{b}_i \leq \Sigma_{i=1}^{n} b_i \leq \Sigma_{i=1}^{n} \overline{b}_i.$$

Thus we have

$$\alpha \Sigma_{j \in P} x_j + \beta \Sigma_{j \in N} x_j \leq \Sigma_{i=1}^{n} \overline{b}_i,$$
$$\beta \Sigma_{j \in P} x_j + \alpha \Sigma_{j \in N} x_j \geq \Sigma_{i=1}^{n} \underline{b}_i,$$

for each $x \in \Sigma(\mathbf{A}, \mathbf{b})$ lied in orthant O which give (2.7) and (2.8). Choice of O implies (2.9) and (2.10). □

In the following example, an outer estimation for solution set is obtained, while the coefficient matrix is a regular doubly stochastic interval matrix.

Example 2.6. Let

$$\mathbf{A} = \begin{pmatrix} [1,3] & [2,4] \\ [2,4] & [1,3] \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} [2,2] \\ [3,5] \end{pmatrix},$$

and O_1, O_2, O_3, O_4 be four region of $\mathbb{R}^2$. It is clear that $\mathbf{A}$ is singular. we have

$$\Sigma(\mathbf{A}, \mathbf{b}) \cap O_1 \subseteq \{x \in \mathbb{R}^2 \ ; \ x \geq 0, \ \frac{5}{7} \leq x_1 + x_2 \leq \frac{7}{3}\},$$

$$\Sigma(\mathbf{A}, \mathbf{b}) \cap O_2 \subseteq \{x \in \mathbb{R}^2 \ ; \ x_1 \leq 0, \ x_2 \geq 0, \ 3x_2 + 7x_1 \leq 7, \ 7x_2 + 3x_1 \geq 5\},$$

The solution sets and their outer estimations are shown in the figure 1. solution set and its outer estimation have been shown as hatched part and coloured part, respectively

[O_1] [O_2]

FIGURE 1. Solution set and its outer estimation in region O_1 and O_2

Note that this interval linear system dose not have any solution in regions O_3 and O_4. The outer estimations for solution set in these regions are empty sets.

3. Conclusion

The aim of this paper is to survey the interval linear systems which their coefficient matrix is an interval doubly stochastic matrix. In most of the research, the outer estimation is obtained when the coefficient matrix is regular. But in this paper, it is considered that coefficient matrix can be regular or singular. The further research can be done for interval matrix equations whit interval doubly stochastic matrices.

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