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$N-$ Transvectants

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Abstract. We consider the $\mathfrak{sl}(2,\mathbb{R})$-module structure on the spaces of $n-$ary differential operators acting on the spaces of weighted densities. We classify $\mathfrak{sl}(2,\mathbb{R})$-invariant $n-$ary differential operators acting on the spaces of weighted densities.

Keywords: n-ary Differential Operators, Cohomology, Weighte Densities.

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1. Introduction

Consider the action of $\mathfrak{sl}(2,\mathbb{R})$ on the space of functions in one variable, say on $\mathcal{C}^\infty(\mathbb{RP}^1)$ given by

$$f(x) \mapsto f(\frac{ax+b}{cx+d})(cx+d)^{-2\lambda} \quad (1)$$

depending on a parameter $\lambda \in \mathbb{R}$. This $\mathfrak{sl}(2,\mathbb{R})$-module is called the space of tensor densities of degree λ and denoted $\mathcal{F}_\lambda$. The classification of $\mathfrak{sl}(2,\mathbb{R})$-invariant linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ was obtained in classical works on projective differential geometry. In 1887, Gordan classified $\mathfrak{sl}(2,\mathbb{R})-$invariant bilinear differential operators on weighted densities[**?**], and called them transvectants also called Rankin - Cohen brackets in the literature. These invariant bilinear differential operators have fantastic applications in Integral Systems, Deformation Quantization, Number Theory, Harmonic Analysis, and Conformal Field Theory. In [**?**], Ovsienko and Redou generalized them to any pseudo-Riemannian manifold endowed with a conformal structure. More precisely, they show that there exist unique $\mathfrak{o}(p+1,q+1)-$invariant bilinear differential operators acting on weighted densities, except for certain weights for which such operators don't exist. In the second part of this article, we focus on the use of differential operators $\mathfrak{sl}(2,\mathbb{R})$-invariant in a very inevitable domain and present today in theoretical physics and mathematics which was cohomology. In this paper we will study the n-ary differential operators $\mathfrak{sl}(2,\mathbb{R})$-invariant.

2. Definitions and Notations

2.1. **Modules of weighted densities.** Let $\mathrm{V}ect(\mathbb{R})$ be the Lie algebra of vector fields on $\mathbb{R}$. Denote by $\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in \mathcal{C}^\infty(\mathbb{R}) \right\}$ the space of weighted densities of weight $\lambda \in \mathbb{R}$, i.e., the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes\lambda}$, so its elements can be represented as $f(x)dx^\lambda$, where $f(x)$ is a function and dx^λ is a formal (for a time being) symbol. This space coincides with the space of vector fields, functions and differential forms for $\lambda = -1$, 0 and 1, respectively.

The space $\mathcal{F}_\lambda$ is a $\mathrm{V}ect(\mathbb{R})$-module for the action defined by

$$L^\lambda_{g\frac{d}{dx}}(f dx^\lambda) = (gf' + \lambda g' f)dx^\lambda.$$

Denote by $\mathcal{D}_{\underline{\lambda},\mu}$, where $\underline{\lambda} = (\lambda_1, \cdots, \lambda_n)$, the space of $n-$ary linear differential operators:

$$\underbrace{\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}}_{n\otimes} \to \mathcal{F}_\mu,$$

for any $\lambda_1, \cdots, \lambda_n, \mu \in \mathbb{R}$. The Lie algebra $\mathrm{V}ect(\mathbb{R})$ acts on the space

$$\mathcal{D}_{\underline{\lambda},\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}, \mathcal{F}_{\mu})$$

of these differential operators by:

$$X_h.A = L^{\mu}_{X_h} \circ A - A \circ L^{(\lambda_1,\cdots,\lambda_n)}_{X_h}, \qquad (2.1)$$

where $L^{(\lambda_1,\cdots,\lambda_n)}_{X_h}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule:

$$L^{(\lambda_1,\cdots,\lambda_n)}_{X_h}(f_1dx^{\lambda_1}\otimes\cdots\otimes f_ndx^{\lambda_n}) = L^{\lambda_1}_{X_h}(f_1)\otimes\cdots\otimes f_ndx^{\lambda_n}+\cdots+f_1dx^{\lambda_1}\otimes\cdots\otimes L^{\lambda_n}_{X_h}(f_ndx^{\lambda_n}).$$

Thus the space of differential operators is a $\mathrm{V}ect(\mathbb{R})$-module.

2.2. **Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.** The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is realized as subalgebra of the Lie algebra $\mathrm{V}ect(\mathbb{R})$:

$$\mathfrak{sl}(2,\mathbb{R}) = Span(X_1 = \frac{d}{dx}, X_x = x\frac{d}{dx}, X_{x^2} = x^2\frac{d}{dx}). \qquad (2.2)$$

corresponding to the fraction-linear transformations

$$x \mapsto \frac{ax+b}{cx+d}, \qquad ad - bc = 1.$$

A *projective structure* on $\mathbb{R}$ (or S^1) is given by an atlas with fraction-linear coordinate transformations (in other words, by an atlas such that the $\mathfrak{sl}(2,\mathbb{R})$-action (**??**) is well-defined).

The commutation relations are

$$[X_1, X_x] = X_1, \qquad [X_x, X_x] = 0, \qquad [X_1, X_1] = 0,$$
$$[X_1, X_{x^2}] = 2X_x, \qquad [X_x, X_{x^2}] = X_{x^2}, \qquad [X_{x^2}, X_{x^2}] = 0.$$

2.3. **Cohomology theory.** Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{A}$ is a module over $\mathfrak{g}$. Then a q-dimensional cochain of the algebra $\mathfrak{g}$ with coefficients in $\mathcal{A}$; is a continuous skew-symmetric q-linear functional on $\mathfrak{g}$ with values in $\mathcal{A}$; the space of all such cochains is denoted by $\mathcal{C}^q(\mathfrak{g},\mathcal{A})$. Thus, $\mathcal{C}^q(\mathfrak{g},\mathcal{A}) = \mathrm{Hom}(\Lambda^q\mathfrak{g},\mathcal{A})$; this last representation transforms $\mathcal{C}^q(\mathfrak{g},\mathcal{A})$ into a $\mathfrak{g}$-module. The differential $d = d_q : \mathcal{C}^q(\mathfrak{g},\mathcal{A}) \longrightarrow \mathcal{C}^{q+1}(\mathfrak{g},\mathcal{A})$ is defined by the formula

$$\begin{aligned} d(c)(x_0, x_1, ..., x_n) &= \sum_{0\le i<j\le n} (-1)^{i+j-1}\, c([x_i, x_j], ..., \hat{x}_i, ..., \hat{x}_j, ...) \\ &+ \sum_{i=0}^{n} (-1)^{i+1}\, x_i.c(x_0, ..., \hat{x}_i, ..., x_n) \;\; (1), \end{aligned}$$

where the symbol $\hat{x}_i$ means that the term for x_i is omitted. We complete the definitions by putting $\mathcal{C}^q(\mathfrak{g},\mathcal{A}) = 0$, $d_q = 0$ for $q < 0$. As can be easily checked, $d_{q+1} \circ d_q = 0$ for all q, so that $\{\mathcal{C}^q(\mathfrak{g},\mathcal{A}), d_q\}$ is an

algebraic complex; this complex is denoted by $\mathcal{C}^\bullet(\mathfrak{g},\mathcal{A})$. while the corresponding cohomology is referred to as the cohomology of the algebra $\mathfrak{g}$ with coefficients in $\mathcal{A}$ and is denoted by $\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathcal{A})$.
Now suppose $\mathfrak{h}$ is a subalgebra of the algebra $\mathfrak{g}$; $\mathcal{A}$ still denotes a $\mathfrak{g}$-module. Denote by $\mathcal{C}^q(\mathfrak{g},\mathfrak{h},\mathcal{A})$ the subspace of the space $\mathcal{C}^q(\mathfrak{g},\mathcal{A})$, consisting of cochains c, such that $c(x_0,x_1,...,x_n)=0$ and $d(c)(x_0,x_1,...,x_n)=0$ for $x_0\in\mathfrak{h}$. Equivalent definition: $\mathcal{C}^q(\mathfrak{g},\mathfrak{h},\mathcal{A})=\mathrm{Hom}_{\mathfrak{h}}(\Lambda^{\mathrm{q}}(\mathfrak{g}/\mathfrak{h}),\mathcal{A})$. Elements of the space $\mathcal{C}^q(\mathfrak{g},\mathfrak{h},\mathcal{A})$ are called relative cochains. Obviously, $d\mathcal{C}^q(\mathfrak{g},\mathfrak{h},\mathcal{A})\subset\mathcal{C}^{q+1}(\mathfrak{g},\mathfrak{h},\mathcal{A})$. so that relative cochains constitute a subcomplex of the complex $\mathcal{C}^\bullet(\mathfrak{g},\mathcal{A})$. This subcomplex is denoted by $\mathcal{C}^\bullet(\mathfrak{g},\mathfrak{h},\mathcal{A})$, and its cohomology is called a relative cohomology of the algebra $\mathfrak{g}$ modulo $\mathfrak{h}$ with coefficients in $\mathcal{A}$ and is denoted by $\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathfrak{h},\mathcal{A})$.
If $\mathcal{A}$ is the main field, then, instead of $\mathcal{C}^q(\mathfrak{g},\mathfrak{h},\mathcal{A}),\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathfrak{h},\mathcal{A})$ we write $\mathcal{C}^q(\mathfrak{g},\mathfrak{h}),\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathfrak{h})$. Note that if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then $\Lambda^q(\mathfrak{g}/\mathfrak{h})$ is the trivial $\mathfrak{h}$-module and $\mathrm{Hom}_{\mathfrak{h}}(\Lambda^{\mathrm{q}}(\mathfrak{g}/\mathfrak{h}),\mathcal{A})=\mathrm{Hom}(\Lambda^{\mathrm{q}}(\mathfrak{g}/\mathfrak{h}),\mathrm{Inv}_{\mathfrak{h}}\mathcal{A})$, where $\mathrm{Inv}_{\mathfrak{h}}\mathcal{A}$ is the module (over $\mathfrak{g}/\mathfrak{h}$) of $\mathfrak{h}$-invariants:

$$\mathrm{Inv}_{\mathfrak{h}}\mathcal{A}=\{\mathrm{a}\in\mathcal{A}\mid\mathrm{ha}=0\text{ for all }\mathrm{h}\in\mathfrak{h}\}.$$

Obviously, in this case the differentials in the complexes $\mathcal{C}^\bullet(\mathfrak{g},\mathfrak{h},\mathcal{A})$ and $\mathcal{C}^\bullet(\mathfrak{g}/\mathfrak{h},\mathrm{Inv}_{\mathfrak{h}}\mathcal{A})$ also coincide so that

$$\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathfrak{h},\mathcal{A})=\mathrm{H}^{\mathrm{q}}(\mathfrak{g}/\mathfrak{h},\mathrm{Inv}_{\mathfrak{h}}\mathcal{A}).$$

The definition of relative cohomology has the following generalization. Assume that $\mathfrak{h}$ is the Lie algebra of some finite-dimensional Lie group $\mathcal{H}$ and that the actions of the algebra $\mathfrak{h}$ in $\mathfrak{g}$ and in $\mathcal{A}$ are the differentials of certain representations of the group $\mathcal{H}$, the representation of $\mathcal{H}$ in $\mathfrak{g}$ being the extension of the adjoint representation of $\mathcal{H}$ in $\mathfrak{h}$. Then, setting $\mathcal{C}^q(\mathfrak{g},\mathcal{H},\mathcal{A})=\mathrm{Hom}_{\mathcal{H}}(\Lambda^{\mathrm{q}}(\mathfrak{g}/\mathfrak{h}),\mathcal{A})$, we obtain one more subcomplex in $\mathcal{C}^\bullet(\mathfrak{g},\mathcal{A})$, whose cohomology will be denoted by $\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathcal{H},\mathcal{A})$. This generalization is not particularly deep: if the group $\mathcal{H}$ is connected, then $\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathcal{H},\mathcal{A})=\mathrm{H}^{\mathrm{q}}(\mathfrak{g},\mathfrak{h},\mathcal{A})$

3. Transvectants

3.1. **The 2-Transvectants.** In this section we will call back the space of differential operators bilinear $\mathfrak{sl}(2,\mathbb{R})$-invariants.
Let us consider the space of bilinear differential operators $B:\mathcal{F}_\lambda\times\mathcal{F}_\mu\to\mathcal{F}_\tau$. The Lie algebra, $Vect(\mathbb{R})$, acts on this space by the Lie derivative:

$$L_X(B)(fdx^\lambda\otimes gdx^\mu)=L_X^\tau\big(B(fdx^\lambda,gdx^\mu)\big)-B\big(L_X^\lambda(fdx^\lambda),\,gdx^\mu\big)-B\big(fdx^\lambda,\,L_X^\mu(gdx^\mu)\big).$$

A bilinear differential operator $B:\mathcal{F}_\tau\times\mathcal{F}_\lambda\to\mathcal{F}_\mu$ is called $\mathfrak{sl}(2)$-invariant (2-Trasvectant) if, for all $X\in\mathfrak{sl}(2)$, we have

$$L_X(B)=0.$$

That is, the set of such $\mathfrak{sl}(2)$-invariant bilinear differential operators is the subspace on which the subalgebra $\mathfrak{sl}(2)$ acts trivially.

The basic result is the following

Proposition 3.1 ([?]). *There exist $\mathfrak{sl}(2)$-invariant bilinear differential operators, called transvectants,*

$$J_k^{\tau,\lambda}: \quad \mathcal{F}_\tau \times \mathcal{F}_\lambda \to \mathcal{F}_{\tau+\lambda+k}, \qquad (\varphi dx^\tau, \phi dx^\lambda) \mapsto J_k^{\tau,\lambda}(\varphi,\phi) dx^{\tau+\lambda+k}$$

given by

$$J_k^{\tau,\lambda}(\varphi,\phi) = \sum_{i+j=k} c_{i,j} \varphi^{(i)} \phi^{(j)},$$

where $k \in \mathbb{N}$ and the coefficients $c_{i,j}$ are characterized as follows:

i) *If neither τ nor λ belong to the set $\{0, -\frac{1}{2}, -1, \dots, -\frac{k-1}{2}\}$ then*

$$c_{i,j} = (-1)^j \binom{2\tau+k}{j} \binom{2\lambda+k}{i},$$

where $\binom{x}{i}$ is the standard binomial coefficient $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$.

ii) *If τ or $\lambda \in \{0, -\frac{1}{2}, -1, \dots, -\frac{k-1}{2}\}$, the coefficients $c_{i,j}$ satisfy the recurrence relation*

$$(i+1)(i+2\tau)c_{i+1,j} + (j+1)(j+2\lambda)c_{i,j+1} = 0. \qquad (3.1)$$

3.2. **The 3-Transvectants.** In this section -as the previous section (the 2-Transvectants)- we will determine the space of trilinear differential operators $\mathfrak{sl}(2,\mathbb{R})$-invariant according to the different values of λ_1, λ_2 and λ_3.
For $(\lambda_1, \lambda_2, \lambda_3, \mu) \in \mathbb{R}^4$, we pose $\sigma = \mu - \lambda_1 - \lambda_2 - \lambda_3$ and we are interested in the space $\mathcal{T}^{\mu}_{\lambda_1,\lambda_2,\lambda_3}$ of the trilinear differential operators who are $\mathfrak{sl}(2,\mathbb{R})$-invariant. The elements of this space are defined as follows: 1. If $\sigma \notin \mathbb{N}$, then $\mathcal{T}^{\mu}_{\lambda_1,\lambda_2,\lambda_3} = \{0\}$.
2. Otherwise, if $\sigma = k \in \mathbb{N}$, then every trilinear differential operator $\mathfrak{sl}(2,\mathbb{R})$-invariant Ω has the following general form :

$$\begin{aligned} \Omega : & \mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2} \otimes \mathcal{F}_{\lambda_3} && \longrightarrow \mathcal{F}_{\lambda_1+\lambda_2+\lambda_3+k}, \\ & \phi \otimes \varphi \otimes \psi && \longmapsto \sum_{i+j+l=k} \omega_{i,j,l} \ \Omega^{i,j,l}(\phi \otimes \varphi \otimes \psi), \end{aligned}$$

where $\Omega^{i,j,l}(\phi \otimes \varphi \otimes \psi) = \phi^{(i)} \varphi^{(j)} \psi^{(l)}$ and the constants $\omega_{i,j,l}$ are characterized by the recurrence formula

$$(i{+}1)(i{+}2\ \lambda 1)\ \omega_{i+1,j,l} + (j{+}1)(j{+}2\ \lambda_2)\ \omega_{i,j+1,l} + (l{+}1)(l{+}2\ \lambda_3)\ \omega_{i,j,l+1} = 0,$$

with $i + j + l = k - 1$.

Proof. Let Ω be an 3-linear differential operator. Ω is $\mathfrak{sl}(2,\mathbb{R})$-invariant if and only if for all $X \in \mathfrak{sl}(2,\mathbb{R})$;

$$L_X^{\lambda_1,\lambda_2,\lambda_3,\mu}(\Omega)(\phi\otimes\varphi\otimes\psi)=0,$$

for all $(\phi,\varphi,\psi)\in\mathcal{F}_{\lambda_1}\otimes\mathcal{F}_{\lambda_2}\otimes\mathcal{F}_{\lambda_3}$.

Ω has the following general form

$$\begin{aligned}\Omega :&\mathcal{F}_{\lambda_1}\otimes\mathcal{F}_{\lambda_2}\otimes\mathcal{F}_{\lambda_3} &&\longrightarrow \mathcal{F}_{\mu},\\ &(\phi\otimes\varphi\otimes\psi) &&\longmapsto \sum_{k=0}^{m}\sum_{i+j+l=k}\omega_{i,j,l}\ \ \phi^{(i)}\varphi^{(j)}\psi^{(l)},\end{aligned}$$

where $\omega_{i,j,l}$ are, a priori, functions. The invariance with respect to X_1 is reflected in:

$$\sum_{k=0}^{m}\sum_{i+j+l=k}\omega'_{i,j,l}\ \ \phi^{(i)}\varphi^{(j)}\psi^{(l)}=0,$$

therefore $\omega_{i,j,l}$ is a constant $\forall i,j,l$.
Now $\Omega_k(\phi\otimes\varphi\otimes\psi)=\displaystyle\sum_{i+j+l=k}\omega_{i,j,l}\ \ \phi^{(i)}\varphi^{(j)}\psi^{(l)}$, denotes the homogeneous component of Ω of order k. We see that Ω is $\mathfrak{sl}(2,\mathbb{R})$-invariant if and only if each of its homogeneous components is $\mathfrak{sl}(2,\mathbb{R})$-invariant. So we can without loss of generality, assume that Ω is homogeneous of order k.
A direct computation proves that

$$\begin{aligned}&L_X^{\lambda_1,\lambda_2,\lambda_3,\mu}(\Omega)(\phi\otimes\varphi\otimes\psi)=(\mu-\lambda_1-\lambda_2-\lambda_3-k)\ X'\sum_{i+j+l=k}\omega_{i,j,l}\ \ \phi^{(i)}\varphi^{(j)}\psi^{(l)}\\ &-\frac{1}{2}\ X''\sum_{i+j+l=k-1}((i+1)(i+2\ \lambda_1)\ \omega_{i+1,j,l}+(j+1)(j+2\ \lambda_2)\ \omega_{i,j+1,l}+\\ &(l+1)(l+2\ \lambda_3)\ \omega_{i,j,l+1})\ \phi^{(i)}\varphi^{(j)}\psi^{(l)}.\end{aligned}$$

So, we distinguish the following two cases:
$\star$ If $\sigma\neq k$, then $\Omega=0$.
$\star$ If $\sigma=k$, then

$$\Omega=\sum_{i+j+l=k}\omega_{i,j,l}\ \ \phi^{(i)}\varphi^{(j)}\psi^{(l)},$$

where the constants $\omega_{i,j,l}$ are characterized by the recurrence formula

$$(i+1)(i+2\ \lambda_1)\ \omega_{i+1,j,l}+(j+1)(j+2\ \lambda_2)\ \omega_{i,j+1,l}+(l+1)(l+2\ \lambda_3)\ \omega_{i,j,l+1}=0$$

with $i+j+l=k-1$.

3.3. **The n-Transvectants.** In this section we will generalize the space of differential operators multilinear $\mathfrak{sl}(2,\mathbb{R})$-invariant. For $\lambda=(\lambda_1,...,\lambda_n)\in\mathbb{R}^n$ and $\mu\in\mathbb{R}$, we pose $\varrho=\mu-\sum_{i=1}^{n}\lambda_i$ and we are interested in the space $\mathcal{T}_\lambda^\mu$ of the n-linear differential operators who are $\mathfrak{sl}(2,\mathbb{R})$-invariant. The elements of $\mathcal{T}_\lambda^\mu$ are defined as follows:

(1) If $\varrho\notin\mathbb{N}$, then $\mathcal{T}_\lambda^\mu=\{0\}$.

(2) Otherwise, if $\varrho=k\in\mathbb{N}$, then every multilinear differential operator $\mathfrak{sl}(2,\mathbb{R})$-invariant $\mathcal{C}$ has the following general form :

$$\begin{aligned}\mathcal{C}:&\mathcal{F}_{\lambda_1}\otimes\mathcal{F}_{\lambda_2}\otimes...\otimes\mathcal{F}_{\lambda_n}\longrightarrow\mathcal{F}_{\lambda_1+\lambda_2+...+\lambda_n+k},\\ &\phi_1\otimes\phi_2\otimes...\otimes\phi_n\longmapsto\sum_{i_1+i_2+...+i_n=k}c_{i_1,i_2,...,i_n}\ \phi_1^{(i_1)}\phi_2^{(i_2)}...\phi_n^{(i_n)},\end{aligned}$$

where the constants $c_{i_1,i_2,\cdots,i_n}$ are characterized by the recurrence formula

$$(i_1+1)(i_1+2\ \lambda_1)\ c_{i_1+1,i_2,...,i_n}+(i_2+1)(i_2+2\ \lambda_2)\ c_{i_1,i_2+1,...,i_n}+...+$$

$$(i_n+1)(i_n+2\ \lambda_n)\ c_{i_1,i_2,...,i_n+1}=0,$$

where $i_1+i_2+...+i_n=k-1$.

Proof. To show this theorem we will follow the same approach as in Section 4.

Let $\mathcal{C}$ be an n-linear differential operator

$$\begin{aligned}\mathcal{C}:&\mathcal{F}_{\lambda_1}\otimes\mathcal{F}_{\lambda_2}\otimes...\otimes\mathcal{F}_{\lambda_n}\longrightarrow\mathcal{F}_\mu,\\ &(\phi_1\otimes\phi_2\otimes...\otimes\phi_n)\longmapsto\sum_{k=0}^{m}\sum_{i_1+i_2+...+i_n=k}c_{i_1,i_2,...,i_n}\ \phi_1^{(i_1)}\phi_2^{(i_2)}...\phi_n^{(i_n)},\end{aligned}$$

where $c_{i_1,i_2,...,i_n}$ are, a priori, functions.

The invariance with respect to X_1 is reflected in:

$$\sum_{k=0}^{m}\sum_{i_1+i_2+...+i_n=k}c'_{i_1,i_2,...,i_n}\ \phi_1^{(i_1)}\phi_2^{(i_2)}...\phi_n^{(i_n)}=0,$$

therefore $c_{i_1,i_2,...,i_n}$ is a constant $\forall i_1,...,i_n$.

Now

$$\mathcal{C}_k(\phi_1\otimes\phi_2\otimes...\otimes\phi_n)=\sum_{i_1+i_2+...+i_n=k}c_{i_1,i_2,...,i_n}\ \phi_1^{(i_1)}\phi_2^{(i_2)}...\phi_n^{(i_n)},$$

denotes the homogeneous component of $\mathcal{C}$ of order k. We see that $\mathcal{C}$ is $\mathfrak{sl}(2,\mathbb{R})$-invariant if and only if each of its homogeneous components is $\mathfrak{sl}(2,\mathbb{R})$-invariant. So we can without loss of generality, assume that $\mathcal{C}$ is homogeneous of order k.
A direct computation proves that

$$L_X^{\lambda_1,\dots,\lambda_n,\mu}(\mathcal{C})(\phi_1\otimes\phi_2\otimes\dots\otimes\phi_n) = (\mu-\textstyle\sum_{i=1}^n \lambda_i-k)\ X' \displaystyle\sum_{i_1+i_2+\dots+i_n=k} c_{i_1,i_2,\dots,i_n}\ \phi_1^{(i_1)}\phi_2^{(i_2)}\dots\phi_n^{(i_n)} -$$
$$\frac{1}{2}\ X'' \sum_{i_1+i_2+\dots+i_n=k-1} ((i_1+1)(i_1+2\ \lambda_1)\ c_{i_1+1,i_2,\dots,i_n} + (i_2+1)(i_2+2\ \lambda_2)\ c_{i_1,i_2+1,\dots,i_n} + \dots +$$
$$(i_n+1)(i_n+2\ \lambda_n)\ c_{i_1,i_2,\dots,i_n+1})\ \phi_1^{(i_1)}\dots\phi_n^{(i_n)}.$$

So to examine the $\mathfrak{sl}(2,\mathbb{R})$-invariance of $\mathcal{C}$ we distinguish the following two cases:

(1) If $k \notin \mathbb{N}$, then $\mathcal{C}=0$.

(2) Otherwise

$$\mathcal{C} = \sum_{i_1+i_2+\dots+i_n=k} c_{i_1,i_2,\dots,i_n}\ \phi_1^{(i_1)}\phi_2^{(i_2)}\dots\phi_n^{(i_n)},$$

where the constants $c_{i_1,i_2,\dots,i_n}$ are characterized by the recurrence formula

$$(i_1+1)(i_1+2\ \lambda_1)\ c_{i_1+1,i_2,\dots,i_n} + (i_2+1)(i_2+2\ \lambda_2)\ c_{i_1,i_2+1,\dots,i_n} + \dots +$$
$$(i_n+1)(i_n+2\ \lambda_n)\ c_{i_1,i_2,\dots,i_n+1} = 0,$$

where $i_1+i_2+\dots+i_n = k-1$.

Hence the result.

3.4. **Applications in Cohomology.** In this section we will give some applications of the space of the multilinear differential operators $\mathfrak{sl}(2,\mathbb{R})$-invariant in the computation of cohomology groups of the Lie algebra of the fields of the vectors on $\mathbb{R}$ which vanish on $\mathfrak{sl}(2,\mathbb{R})$ with coefficient in the space of multilinear differential operators. The computation of these cohomology is based on the following observation: any 1-cocycle vanishing on the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ is an $\mathfrak{sl}(2,\mathbb{R})$-invariant operator.

In [?] Bouarroudj and Ovsienko prove that ; Given a differentiable 1-cocycle c on $\mathrm{Vect}(\mathbb{R})$ vanishing on $\mathfrak{sl}(2,\mathbb{R})$, with values in $D_{\lambda,\mu}$, the bilinear differential operator $J:\mathrm{Vect}(\mathbb{R})\otimes F_\lambda\to F_\mu$ defined by:

$$J(X,\phi)=c(X)(\phi),$$

is $\mathfrak{sl}(2,\mathbb{R})$-invariant, And they found the following result:

$$\mathrm{H}^1(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2,\mathbb{R}),\mathcal{D}_\lambda,\mu)=\begin{cases}\mathbb{R} & \text{if}\begin{cases}\mu-\lambda=2,\lambda\neq\frac{-1}{2}\\ \mu-\lambda=3,\lambda\neq-1\\ \mu-\lambda=4,\lambda\neq\frac{-3}{2}\\ (\lambda,\mu)=(-4,1),(0,5)\end{cases}\\ 0 & \text{otherwise.}\end{cases}$$

In a second application of n-transvectants, Bouarroudj uses the 2-transvectors to show that the dimension of the space of operators that satisfy the 1-cocycle condition is at most $k-2$, where $k=\mu-\lambda-\nu$, since any 1-cocycle that vanishes on $\mathfrak{sl}(2,\mathbb{R})$ is certainly $\mathfrak{sl}(2,\mathbb{R})$-invariant [?].

In a third application of N-transfectants Bouarroudj used the 2-transvectants to mount the following proposition:

Proposition 3.2. [?] *Every coboundary $\delta(B)\in B^2(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2);\mathcal{D}_{\lambda,\mu})$ possesses the following properties. The operator B coincides (up to a nonzero factor) with the transvectant $J^{-1,\lambda}_{k+1}$, where $\gamma_{0,k+1}=\gamma_{1,k}=\gamma_{2,k-1}=0$. In addition (here $X=f\frac{d}{dx}\in\mathrm{Vect}(\mathbb{R})$ and $\phi\,dx^\lambda\in_\lambda$),*

$$\delta(B)(X,Y,\phi\,dx^\lambda)=\sum_{i+j+l=k+2}\beta_{i,j}\,f^{(i)}\,g^{(j)}\,\phi^{(l)}\,dx^{\lambda+k},\qquad(3.2)$$

where

$$\beta_{0,j}=\beta_{1,j}=\beta_{2,j}=0,$$

and

$$\begin{aligned}\beta_{3,4}&=-\tfrac{1}{24}\tbinom{k-2}{3}\left(k^2+4(\lambda-1)\lambda+k(4\lambda-5)\right)(k-1+2\lambda)\gamma_{3,k-2}\\ \beta_{4,5}&=-\tfrac{1}{480}\tbinom{k-2}{5}(k-3+2\lambda)(k^3+4(\lambda-1)\lambda(2\lambda-19)+3k^2(2\lambda-7)+2k(49+6(\lambda-7)\lambda))\\ &\quad\times(k-1+2\lambda)\gamma_{3,k-3}.\end{aligned}$$

Finally, in [?] Ovsienko prove that ; $\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2,\mathbb{R}),\mathcal{F}_\lambda)$ span by 2-transvectants.

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