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On Bi-ideal Elements in Poe-AG-groupoid

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Abstract. In this paper, first we introduce the concept of ideal and bi-ideal elements in poe-AG-groupoid. Then, we give some characterizations and properties of their bi-ideal elements. Thus, we consider some results concerning bi-ideals in poe-semigroups and investigate them in poe-AG-groupoids. Morever, the class of bi-ideal elements of poe-AG-groupoids are studied, certain intrinsic and basic properties of poe-AG-groupoids including: $\bigwedge$-semilattice, bi-ideal, semiprime, weakly prime, totally ordered elements and etc are studied as well. The corresponding results on poe-semigroups can be also obtained as application of the results of this paper.

Keywords: AG-groupoid; poe-semigroup; poe-AG-groupoid; ideal element; bi-ideal element; weakly prime; semiprime; quasi ideal element; $\bigwedge$-semilattice and $\bigwedge$-compatible.

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1. Introduction

First time, the idea of generalization of communicative semigroups was introduced in 1972 by Kazim and Naseerudin([3]). They named this structure as the left almost semigroup (LA-semigroup for short), while it was called Abel-Grassmann's groupoid (AG-groupoid for short) in ([6]), if its elements satisfy the left invertive law, that is: $(ab)c = (cb)a$ for all $a, b, c \in S$. A partial ordered groupoid (po-groupoid for short) S

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is a groupoid under a multiplication "." at the same time an ordered set under a partial order "$\leq$"such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. If the multiplication is associative, then S is called a po-semigroup. A po-semigroup possessing the greatest element "e" (that is, $a \leq e$ for all $a \in S$) is called poe-semigroup ([7]). In the present paper, we mean by poe-AG-groupoid the ordered AG-groupoid which has a greatest element. In this type of AG-groupoids the bi-ideal elements (instead of bi-ideals) play an essential role. It seems interesting to obtaine the similar results of achived results in ([7-9]) and ([4]) with use of bi-ideal elements in poe-AG-groupoid instead of bi-ideals in po-AG-groupoid. The present paper shows how similar is the theory of po-AG-groupoid based on bi-ideals with the theory of poe-AG-groupoid based on bi-ideal elements. Infact, these are parallel.
Recall, that a po-AG-groupoid is a partial ordered groupoid (po-groupoid) that is AG-groupoid, i.e. and also a poe-AG-groupoid is a po-AG-groupoid which has the greatest element "e" (that is, $a \leq e$ for all $a \in S$). As, with N. Kehayopulu in [4,5] and R. Saritha in ([8]) which looked at some interesting properties of po-semigroups. Now we look at the interesting properties of poe-AG-groupoids and In section 2, we recall some of necessary definitions before discussing in details of the results summarized in the abstract. In section 3, we introduce bi-ideal elements in poe-AG-groupoids and investigate their properties. Then, we investigate to some properties of poe-AG*-groupoids and poe-AG**-groupoids. Moreover, we study the role of bi-ideal elements to characterize some properties of poe-AG-groupoids. In section 4, we define $\bigwedge$-semilattice, $\bigwedge$-compatible and introduce the concept of poe-AG-$\bigwedge$-groupoid and study their peroperties of bi-ideal elements.

2. PRELIMINARIES

Throughout this paper S be a po-groupoid (or po-AG-groupoid). In the way, we recall some of necessary definitions before discussing in details of the results summarized in the abstract. We say an element a of S is idempotent if $a^2 = a$. Now if every element of S be idempotent then we say it is band. Also an element a of S is called a left ideal element if $xa \leq a$ for all $x \in S$. In similarily, it is called a right ideal element if $ax \leq a$ for all $x \in S$ ([1]). Furtheremore, it is called an ideal element if it is both a left and right ideal element. If S is a poe-groupoid (or poe-AG-groupoid), then a is a left (resp. right) ideal element of$S$ if and only if $ea \leq a$ (resp. $ae \leq a$) and a is called left (resp. right) regular if $a \leq e.a^2$ (resp. $a \leq a^2.e$)for every $a \in S$[3]. In the way, Let S be a po-groupoid. An element p of S is called prime if for any a, b in S such that $ab \leq p$ implies $a \leq p$ or $b \leq p$. It is called weakly prime if for any

ideal elements a, b of S such that $ab \leq p$ implies $a \leq p$ or $b \leq p$. Finally, p is called semiprime ([2]) if $a^2 \leq p$ implies $a \leq p$.
If an ideal element of a po-groupoid is prime, then it is weakly prime and semiprime. In poe-semigroup S, $a \in S$ is called quasi ideal element if $ae \bigwedge ea \leq a$.
Let S be a poe-semigroup, $b \in S$ is called bi-ideal element, if $beb \leq b$ [8,9]. If S is an AG-groupoid with left identity, then a(bc) = b(ac);for all $a, b, c \in S$.
It is also known [3] that in an AG-groupoid S, the medial law, that is, (ab)(cd) = (ac)(bd) for all a, b, c, d $\in$ S holds. An AG-groupoid S is called an AG*-groupoid if it satisfies one of the following equivalent: (ab)c = b(ac), (ab)c = b(ca) implies b(ac) = b(ca) for all a, b, c $\in S$. An AG-groupoid S satisfying the identity a(bc) = b(ac), for all $a, b, c \in S$ is called an AG**-groupoid.
Notice that each AG-groupoid with a left identity is an AG**-groupoid. Any AG**-groupoid is paramedial, i.e., it satisfies the identity, $(ab)(cd) = (db)(ca)$ for all a, b, c, d $\in S$.

3. Bi-ideal elements in poe-AG-groupoid

In this section, we introduce the concept of poe-AG-groupoid and study their peroperties of bi-ideal elements.

Definition 3.1. Let S be a poe-AG-groupoid, $b \in S$ is called bi-ideal element, if $(be)b \leq b$.

Definition 3.2. Let S be a poe-AG-groupoid and $b \in S$ a bi-ideal element of S, is called prime $(xe)y \leq b$ implies $x \leq b$ or $y \leq b$ for every $x, y \in S$. So b is called semiprime if $(xe)x \leq b$ implies $x \leq b$ for every $x \in S$.

Definition 3.3. Let S be a poe-AG-groupoid. A bi-ideal element b of S is weakly prime ideal if for all ideal elements x, y of $S, (xe)y \leq b$ implies $x \leq b$ or $y \leq b$.

Lemma 3.4. *Let S is a poe-AG**-groupoid. If r be right ideal element, then re is ideal element.*

Proof. Let S be a poe-AG**-groupoid. Using hypothesis, $re \leq r$ implies $(re)e \leq r$, so re is right ideal element. Also $e(re) = r(ee) \leq re$. Therefore $e(re) \leq re$ implies re is left ideal element. □

Proposition 3.5. *Let S is a poe-AG**-groupoid. A bi-ideal element b of S is weakly prime ideal, if and only if $rl \leq b$, such that l is an ideal element of S and r is a right ideal element of S, implies $r \leq b$ or $l \leq b$.*

Proof. Let b be a bi-ideal element of S, by Lemma 3.4 re is a ideal element, hence

$$((re)e)l \leq (re)l \leq re \leq b \Rightarrow ((re)e)l \leq b \Rightarrow re \leq b$$

or $l \leq b$. Because if $re \leq b$, then $r \leq b$ or $e \leq b$. Hence, $r \leq b$ or $l \leq b$. Note that if $e \leq b$, so $b \leq e$ implies that $e = b$, it is obvious. Conversely, let x, y be ideal elements of S. By Lemma 3.4 xe is ideal element of S, so $(xe)y \leq b$, using hypothesis, $xe \leq b$ or $y \leq b$. So we conclude as before $x \leq b$ or $y \leq b$. Hence b is weakly prime ideal. □

Proposition 3.6. *A weakly prime bi-ideal element of a poe-AG-groupoid S is weakly prime one-sided ideal element of S.*

Proof. let x, y are ideal elements of S, then $xe \leq x$. It follows that $(xe)y \leq xy \leq b$. Now, since b is a bi-ideal element, thus $x \leq b$ or $y \leq b$. □

Lemma 3.7. *Let S be a poe-AG*-groupoid and $a \in S$, then $(ae)a$ is a bi-ideal element of S.*

Proof. We assume $(ae)a = b$, we show $(be)b \leq b$. For this purpose, by properties of poe-AG*-groupoid and medial law we have:
$(be)b = (((ae)a)e)((ae)a) = ((ea)(ae))((ae)a) = (a(e(ae)))((ae)a) = (a((ae)e))((ae)a) = (a((ee)a))((ae)a) \leq (a(ea))((ae)a) = ((ea)a)((ae)a) = ((aa)e)((ae)a) \leq ((ae)e)((ae)a) = ((ee)a)((ae)a) \leq (ea)((ae)a) = (e(ae))(aa) \leq (e(ae))(ae) = (ea)((ae)e) = (ea)((ee)a) \leq (ea)(ea) = (ee)(aa) \leq e(aa) = (ae)a = b$. Hence $(be)b \leq b$, i.e., $(ae)a = b$ is a bi-ideal element of S. □

Proposition 3.8. *Let S is a poe-AG*-groupoid. Then S is regular if and only if every bi-ideal element in S is semiprime.*

Proof. Let S be a regular poe-AG*-groupoid and b be a bi-ideal element of S. By hypothesis $x \leq (xe)x$ and $(xe)x \leq b$, so we have $x \leq b$. As a result b is semiprime ideal element.
Conversely, suppose that every bi-ideal element of S is semiprime, Let $a \in S$, by lemma 3.7 , we have $(ae)a$ is a bi-ideal element of S. Therefore $(ae)a$ is semiprime, i.e., $(ae)a \leq (ae)a$, implies $a \leq (ae)a$. Hence S is regular. □

Definition 3.9. A poe-AG-groupoid S is called quasi commutative if $xy \leq (ye)x$ for all $x, y \in S$.

Proposition 3.10. *Let S is a quasi commutative poe-AG-groupoid. Then every semiprime ideal is semiprime bi-ideal.*

Proof. Let $x \in S$. Now, by hypothesis, $xx = x^2 \leq (xe)x$ and also b be bi-ideal element such that $(xe)x \leq b$, therefore $x^2 \leq b$. It follows that $x \leq b$. □

Proposition 3.11. *Every left ideal element of a poe-AG*-groupoid S is an bi-ideal element.*

Proof. Suppose b be a left ideal element of S, there for $bb \leq eb \leq b$ so $(be)b = e(bb) \leq eb \leq b$. □

Proposition 3.12. *Every right ideal element of a poe-AG-groupoid S is an bi-ideal element.*

Proof. Suppose b be a right ideal element of S, therefore $bb \leq be \leq b$ so $(be)b \leq bb \leq b$.Thus, the proof is finished. □

Definition 3.13. A poe-AG-groupoid S is called normal if $ea = ae$ for every $a \in S$.

Proposition 3.14. *Every poe-AG**-band is normal.*

Proof. We assume $a \in S$. Now, by hypothesis we have $ea = ea^2 = e(aa) = (ea)a = (aa)e = ae$. So, the proof is completed. □

Proposition 3.15. *In poe-AG**-band every bi-ideal element is ideal element.*

Proof. Suppose b is bi-ideal element of S, therefore $(be)b = (be)(bb) = (bb)(eb) = b(eb) = e(bb) = eb$, hence $eb \leq b$. Now, by proposition 3.14, we have $eb = be \leq b$. Thus the proof is completed.

□

Definition 3.16. A poe-AG-groupoid S is called totally ordered if for all ideal elements $a, b \in S$ either $a \leq b$ or $b \leq a$.

Lemma 3.17. *Let S be a poe-AG*-groupoid. If S is quasi commutative, then for all $a \in S$, a^2e is ideal element of S.*

Proof. We assume S is quasi commutative so for every $a \in S$, we have $(a^2e)e = (ee)a^2 \leq ea^2 = e(aa) \leq e((ae)a) = ((ae)e)a = ((ee)a)a \leq (ea)a = (aa)e = a^2e$. On the other hand, in similarily we have, $e(a^2e) = (a^2e)e = (ee)a^2 \leq ea^2 \leq a^2e$. So, the proof is completed

□

Proposition 3.18. *A quasi commutative poe-AG*-groupoid S is regular if and only if every ideal element of S be semiprime.*

Proof. Let S be regular, and a is ideal element of S. Now, we consider $x \in S$, $x^2 \leq a$, then $x \leq (xe)x = e(xx) \leq ea \leq a \Rightarrow x \leq a$. Hence, a is semiprime. Conversely, let $a \in S$ so by Lemma 3.17, a^2e is an ideal element of S. Then it is semiprime. By the assumtion we have, $(a^2)^2 = a^4 = ((aa)a)a \leq (ea)a = (aa)e = a^2e$. On the other hand, by the semiprimness of a^2e and also $a^2 \leq a^2e$, then we have $a \leq a^2e$. Therefore $a \leq (aa)e = a^2e \leq (a^2e)e = (ee)a^2 \leq ea^2 = e(aa) = (ae)a$, implies $a \leq (ae)a$. Hence S is regular. □

Definition 3.19. A poe-AG-groupoid is called left (resp. right) regular if $a \leq e.a^2$ (resp. $a \leq a^2.e$) for all $a \in S$. If S is left or right regular, then it is called semiregular.

Proposition 3.20. *Let S be a poe-AG*-groupoid. Then S is left regular if and only if every left ideal element of S is semiprime.*

Proof. Suppose $a \in S$ is left ideal element and $x \in S$ such that $x^2 \leq a$. Now by assumption we have, $x \leq ex^2 \leq ea \leq a$ implies $x \leq a$. Hence, S is semiprime. Conversely, consider $x \in S$, we show that ex^2 is left ideal element of S. For this purpose, assume $x \in S$, therefor $e(ex^2) = (ee)x^2 \leq ex^2$. Now, by assumption ea^2 is semiprime. Since that $x^4 = (x^2)^2 \leq ex^2$ implies $x^2 \leq ex^2$. On the other hand $x \leq ex^2$ for every $x \in S$. So S is left regular. □

4. Bi-ideal elements in poe-AG-$\bigwedge$-groupoid

In this section, we introduce the concept of poe-AG-$\bigwedge$-groupoid and study their peroperties of bi-ideal elements.

Definition 4.1. A $\bigwedge$-semilattice is a partially ordered set that has a meet (a greatest lower bound) for any nonempty finite subset.

Definition 4.2. A poe-AG-groupoid S is called $\bigwedge$-semilattice if has for every $a, b \in S$.

Definition 4.3. A $\bigwedge$-semilattice S is called $\bigwedge$-compatible if have for every $a, b \in S$,
1)$(a \bigwedge b)e = ae \bigwedge be$
2) $e(a \bigwedge b) = ea \bigwedge eb$

Definition 4.4. A poe-AG-groupoid S is called poe-AG-$\bigwedge$-groupoid if be $\bigwedge$-semilattice and $\bigwedge$-compatible.

Lemma 4.5. *If S is a poe-AG-$\bigwedge$-groupoid and a, b ideal elements of S, then the element $a \bigwedge b$ is an ideal element of S.*

Proof. For proof, we consider the definition, so we have
$(a \bigwedge b)e = ae \bigwedge be \leq a \bigwedge b$ and $e(a \bigwedge b) = ea \bigwedge eb \leq a \bigwedge b$. □

Definition 4.6. Let S be a poe-AG-groupoid, $a \in S$ is called quasi-ideal element if $ae \bigwedge ea \leq a$.

Lemma 4.7. *If a is an ideal element of poe-AG*-groupoid S, then a^n for $n \geq 2$ is left ideal element.*

Proof. Assume, a be an ideal element in AG-groupoid so that $a^n = a^{n-1}.a$ where $n \geq 2$. So, we have $ea^n = e(a^{n-1}.a) = (a^{n-1}e)a = (ae)a^{n-1} \leq a.a^{n-1} = a(a^{n-2}.a) = (a^{n-2}.a)a = a^{n-1}.a = a^n$. So, a^n is left ideal element of S.

□

Proposition 4.8. *Every ideal element of poe-AG*-groupoid S is weakly prime if and only if it is idempotent and the set of all ideal element of S is totally ordered.*

Proof. Let every ideal element of S be weakly prime. Assume that a is every ideal element of S. Then by Lemma 4.7, a^2 is an ideal element of S. Also, on the other hand $a^2 \leq a$ and $a^2 = a.a \leq a^2$ it follows that $a \leq a^2$. Therefore, $a = a^2$. Let a and b are any ideal element of S, then $ab \leq ae \leq a$ and $ab \leq eb \leq b$ imply $ab \leq a \bigwedge b$. Since that, $e(a \bigwedge b) = ea \bigwedge eb \leq a \bigwedge b$ and $(a \bigwedge b)e = ae \bigwedge be \leq a \bigwedge b$. Thus, $a \bigwedge b$ is ideal element of S.Thus $a \bigwedge b$ is prime. Then, $a \leq a \bigwedge b$ or $b \leq a \bigwedge b$ implies $a \leq b$ or $b \leq a$. Hence, the set of all ideal element of S is totally ordered.

Conversely, let a, b and p be any ideal elements of S such that $ab \leq p$. Since that S is totally ordered, assume that $a \leq b$. Then $a = a^2 = aa \leq ab \leq p$. Hence, every ideal element of S is weakly prime.

Every ideal element of poe-AG-groupoid S is quasi ideal element. Since that $ae \bigwedge ea \leq ae$, $ae \bigwedge ea \leq ea$ and $ae \leq a$, $ea \leq a$ imply that $ae \bigwedge ea \leq a$. □

Proposition 4.9. *Let S is a poe-AG*-groupoid. If b be a semiprime bi-ideal element of S, then b is a quasi- ideal element of S.*

Proof. Assume $y \leq be \bigwedge eb$, so $y \leq be$ and $y \leq eb$. Then,
$(ye)y \leq ((be)e)(eb) = ((eb)e)(be) = ((eb)b)(ee) \leq ((eb)b)e = (eb)(eb) = (ee)(bb) \leq e(bb) = (be)b \leq b$, implies $(ye)y \leq b$. Since that b is a semiprime bi-ideal element of S, so $y \leq b$. Hence, $be \bigwedge eb \leq b$, i.e., b is a quasi- ideal element of S.

□

Proposition 4.10. *Every ideal element of a poe-AG-$\bigwedge$-band S is weakly prime if and only if the set of all ideal elements of S is totally ordered.*

Proof. Let a, b be ideal elements of S, so $ab \leq ae \leq a$ and $ab \leq eb \leq b$. It follows that $ab \leq a \bigwedge b$. On the otherhand by hypotesis $a \bigwedge b$ is weaklyprime, therefore $a \leq a \bigwedge b$ or $b \leq a \bigwedge b$. As a result $a \leq b$ or

$b \leq a$, respectively. Hence, the set of all ideal elements of S is totally ordered.

Conversely, assume x, y and t be ideal elements of poe-AG-band S such that $xy \leq t$. Since that x, y are totally ordered, put $x \leq y$, now since that $x^2 = x$, $xx \leq xy \leq t$ implies that $x \leq t$. Also, similary $y \leq x$ implies that $y \leq t$. Hence t is weakly prime ideal element. □

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