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On Composition of Generating Functions

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Abstract. In this work we study numbers and polynomials generated by two type of composition of generating functions and get their explicit formulae. Furthermore we state an improvement of the composita formulae's given in [7, 3]. Using the new composita formula's, we construct a variety of combinatorics identities. This study go alone to define new family of generalized Bernoulli polynomials which include Hermite-Bernoulli polynomials introduced by Dattoli and al [1].

Keywords: Generating function, composition of generating functions, composita function, Faà di Bruno formula .

2000 Mathematics subject classification: 05A15, 11Y55; Secondary 11C08.

1. Introduction

In the literature several works are done about composition of generating functions and their properties [7, 8, 9, 10]. In this work we are interested by the formal calculus on the composition of generating functions without regarding the problem of convergence in order to state explicit formulae of associated numbers and polynomials. We introduce new tools for computing successive derivatives such that Faà di Bruno

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formula [6] and Leibnitz identity used for the first time in [4]. Throughout this paper a generating function means an ordinary or exponential generating function. Let the generating function $Q(t) = \sum_{n\geq 0} q_n t^n$ where q_n are numbers or polynomials of one or more variables. In the case $q_0 = 0$ the composita of the function $Q(t)$ (see [7]) is the function with two variables

$$Q^{\Delta}(n,k) = \sum_{\pi_k \in C_n} q_{i_1} \cdots q_{i_k}, \tag{1.1}$$

where C_n the set of all partitions of an integer n and π_k is the partition of n into k parts such that $\sum_{j=1}^{k} i_j = n$. It is showed in the work [10] that

$$Q^k(t) = \sum_{n\geq k} Q^{\Delta}(n,k) t^n. \tag{1.2}$$

On general when q_0 is not forcedly zero, an improvement of the identity (1.2) is established in our recent work [3] as follows.

$$Q^k(t) = \sum_{n\geq 0} \left(\sum_{i_1+\cdots+i_k=n} q_{i_1} \cdots q_{i_k} \right) t^n$$

where $i_j \in \{0,1,2,\cdots,n\}$ for $1 \leq j \leq k$. Let us denoting

$$Q^{\nabla}(n,k) = \sum_{i_1+\cdots+i_k=n} q_{i_1} \cdots q_{i_k}, \tag{1.3}$$

this quantity is connected to $Q^{\Delta}(n,k)$ by the following relation

$$Q^k(t) = \left(q_0 + \sum_{n\geq 1} q_n t^n \right)^k = \sum_{n\geq 0} \left(\sum_{j=0}^{\min\{k,n\}} \binom{k}{j} q_0^{k-j} Q^{\Delta}(n,j) \right) t^n$$

and then

$$Q^{\nabla}(n,k) = \sum_{j=0}^{\min\{k,n\}} \binom{k}{j} q_0^{k-j} Q^{\Delta}(n,j). \tag{1.4}$$

If Q is a polynomial of degree m on the variable t; $Q^k(t)$ is a polynomial of degree km on t, which means that $Q^{\nabla}(n,k) = 0$ for $n \geq km+1$. We take for example $Q(t) = q_0 + q_1 t$, then $Q^k(t) = \sum_{n=0}^{k} \binom{k}{n} q_0^{k-n} q_1^n t^n$ furthermore

$$Q^{\nabla}(n,k) = \begin{cases} \binom{k}{n} q_0^{k-n} q_1^n & \text{for } n \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(t)=\sum_{k\geq 0} a_k t^k$ the generating function of numbers a_k. From the definition of the composition law we get

$$f\circ Q(t)=\sum_{k\geq 0} a_k Q^k(t)=\sum_{k\geq 0} a_k \sum_{n\geq 0} Q^{\nabla}(n,k)t^n.$$

Thereafter

$$f\circ Q(t)=\sum_{n\geq 0}\left(\sum_{k\geq 0} a_k Q^{\nabla}(n,k)\right)t^n \tag{1.5}$$

and $f\circ Q$ generates numbers or polynomials of one or more variables c_n of the form

$$c_n=\begin{cases} f\left(q_0\right) & \text{for } \ n=0, \\ \sum_{k\geq 0} a_k Q^{\nabla}(n,k) & \text{otherwise.}\end{cases} \tag{1.6}$$

This result is an improvement of the formula [9, 10] of the special case $q_0=0$.

$$c_n=\begin{cases} a_0 & \text{for } \ n=0, \\ \sum_{k=0}^{n} a_k Q^{\Delta}(n,k) & \text{otherwise.}\end{cases} \tag{1.7}$$

Let $F(x,t)=\sum_{n\geq 0} P_n(x)t^n$ the generating function of polynomials $P_n(x)=\sum_{k=0}^{n} p(k,n)x^k$ of degree at most n and we can consider $p(k,n)=0$ for $k>n$. In the same way as for $f\circ Q$, it is easy to show that $F\circ Q$ generates polynomials $C_n(x)$ of one variable x or more variables of the form

$$C_n(x)=\sum_{k\geq 0} Q^{\nabla}(n,k)P_k(x) \tag{1.8}$$

and for $q_0=0$ we deduce that

$$C_n(x)=\sum_{k=0}^{n} Q^{\Delta}(n,k)P_k(x). \tag{1.9}$$

Additionally to ordinary composition of the functions $F(x,t)$ and $Q(t)$, we define the pseudo-composition $F\circ_p Q$ to be

$$F\circ_p Q\,(t)=F(Q(t),t).$$

Let $P_n\overline{\circ}Q$ the numbers or polynomials of one or more variables generated by the function $F\circ_p Q$, then we write

$$F\circ_p Q(t)=\sum_{n\geq 0} P_n\overline{\circ}Q t^n.$$

This notation conducts to a new law of composition $\overline{\circ}$ of polynomials. We develop a new method for finding explicit formula of $P_n\overline{\circ}Q$ and state new generalizations of well known families of polynomials such as Bernoulli and Hermite-Bernoulli polynomials. But first we investigate the successive derivatives of the function $f\circ Q$ to get new explicit formula for the coefficients c_n.

2. New expression of c_n and an improvement of the composita formula

In this section we give a new formula for c_n and we revisit the composita formula of Q given in the work [7] when $q_0 = 0$. We recall that

$$\binom{k}{k_1, \cdots, k_n} = \frac{k!}{k_1! \cdots k_n!}$$

is the multinomial of order n, with the condition that $k_1 + \cdots + k_n = k$. By means of this notation, a new reformulation of the coefficients c_n is given by the following theorem.

Theorem 2.1.

$$c_n = \sum_{k=0}^{n} \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \binom{k}{k_1, \cdots, k_n} q_1^{k_1} \cdots q_n^{k_n} \frac{f^{(k)}(q_0)}{k!}. \tag{2.1}$$

The following corollary holds true.

Corollary 2.2. *For the special case $q_0 = 0$ we have*

$$c_n = \sum_{k=0}^{n} a_k \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \binom{k}{k_1, \cdots, k_n} q_1^{k_1} \cdots q_n^{k_n}. \tag{2.2}$$

We have already proved the following theorem which represents an improvement of the formula (1.1).

Theorem 2.3.

$$Q^{\Delta}(n,k) = \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \binom{k}{k_1, \cdots, k_n} q_1^{k_1} \cdots q_n^{k_n}. \tag{2.3}$$

We attract attention that if $Q(t)$ is a polynomial of degree m, we will have the same formula, just replacing the sum $\sum_{k=0}^{n}$ by the sum $\sum_{k=0}^{\min\{n,m\}}$.

2.1. **Proof of Theorem 2.1, Corollary 2.2 and Theorem 2.3.** To prove Theorem 2.1 we Apply Faà di Bruno formula (see [6] for the proof of the formula) for the composition $f \circ Q$. Then the n-th derivative of $f \circ Q$ is

$$(f \circ Q)^{(n)} = \sum_{k=0}^{n} \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \frac{n!}{k_1! \cdots k_n!} \left(f^{(k)} \circ Q\right) \prod_{i=1}^{n} \left(\frac{Q^{(i)}}{i!}\right)^{k_i}.$$

Since

$$f(t)=\sum_{k\geq 0}a_kt^k,\ f\circ Q(t)=\sum_{k\geq 0}c_kt^k \text{ and } Q(t)=\sum_{k\geq 0}q_kt^k$$

we deduce that

$$\frac{f^{(n)}(0)}{n!}=a_n,\ \frac{(f\circ Q)^{(n)}(0)}{n!}=c_n \text{ and } \frac{Q^{(n)}(0)}{n!}=q_n.$$

Thus

$$\frac{d^n f\circ Q}{dt^n}|_{t=0}=\sum_{k=0}^{n}\sum_{\substack{k_1+k_2+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\frac{n!}{k_1!\cdots k_n!}q_1^{k_1}\cdots q_n^{k_n}f^{(k)}(q_0)$$

and the result (2.1) Theorem 2.1 follows. But if $q_0=0$ we deduce that $f^{(k)}(0)=a_kk!$, and the identity (2.2) Corollary 2.2 follows.
For example if

$$f(t)=\frac{1}{1-t}=\sum_{k\geq 0}x^k$$

we conclude that

$$c_n=\sum_{k=0}^{n}\sum_{\substack{k_1+k_2+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\binom{k}{k_1,\cdots,k_n}q_1^{k_1}\cdots q_n^{k_n}.$$

Comparing this result with the identity (1.7) applied to above function,one obtains

$$Q^{\Delta}(k,n)=\sum_{\substack{k_1+k_2+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\binom{k}{k_1,\cdots,k_n}q_1^{k_1}\cdots q_n^{k_n}$$

and the result (2.3) Theorem 2.3 follows.

2.2. **Application to combinatorics calculus.** The algebraic computations based on combinatorial objects are an important direction of research in combinatorics and related fields of mathematics. We explain this idea by the following example. If $f\circ Q$ is a polynomial of degree p then $c_n=0$ for $n>p$, which means that

$$\sum_{k=0}^{n}\sum_{\substack{k_1+k_2+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\binom{k}{k_1,\cdots,k_n}q_1^{k_1}\cdots q_n^{k_n}\frac{f^{(k)}(q_0)}{k!}=0,\ n>p.$$

A large family of combinatorics identities follow. If we consider the sequence q_n, such that $q_0=0$ and for $n\geq 1$; $q_n=1$, the following proposition holds true.

Proposition 2.4. *For $n \geq 2$ only we have*

$$\sum_{k=0}^{n} \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \frac{\prod_{i=1}^{n} \left(-\frac{1}{i}\right)^{k_i}}{k_1! \cdots k_n!} = 0. \tag{2.4}$$

Proof. Let us consider

$$f(t) = e^t = \sum_{n \geq 0} \frac{t^n}{n!} \text{ and } Q(t) = \log(1-t) = -\sum_{n \geq 1} \frac{t^n}{n},$$

then $q_0 = 0$ and $f \circ Q(t) = 1 - t$. Furthermore $c_0 = 1$, $c_1 = -1$ and the others are zero. Applying the identity (2.2) Corollary 2.2 to f and Q to conclude. □

3. Explicit formula of $P_n \bar{\circ} Q$.

In addition to Faà di Bruno formula, we use the Leibnitz formula for the derivative of the product fg of two functions f and g explained in the article [4]:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Theorem 3.1. *The composition $\bar{\circ}$ is given by the following expression.*

$$P_n \bar{\circ} Q = \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{\substack{k_1+k_2+\cdots+k_j=k \\ k_1+2k_2+\cdots+jk_j=j}} \binom{k}{k_1, \cdots, k_j} q_1^{k_1} \cdots q_j^{k_j} \frac{P_{n-j}^{(k)}(q_0)}{k!}. \tag{3.1}$$

Remark 3.2. If $q_0 = 0$ then $\frac{P_{n-j}^{(k)}(0)}{k!} = p(k, n-j)$ for $k \leq n$ and 0 otherwise. Furthermore

$$P_n \bar{\circ} Q = \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{\substack{k_1+k_2+\cdots+k_j=k \\ k_1+2k_2+\cdots+jk_j=j}} \binom{k}{k_1, \cdots, k_j} q_1^{k_1} \cdots q_j^{k_j} p(k, n-j).$$

3.1. **Proof of Theorem3.1.** We have

$$F(Q(t), t) = p(0,0) + \sum_{i \geq 1} P_n \circ Q(t) t^i$$

then

$$\frac{d^n F(Q(t), t)}{dt^n} = \sum_{i \geq 1} \sum_{j=0}^{n} \binom{n}{j} \frac{d^j P_i \circ Q(t)}{dt^j} \frac{d^{n-j} t^i}{dt^{n-j}}$$

and

$$\frac{d^n F\left(Q(t), t\right)}{dt^n} = \sum_{i\geq 1}\left(P_i \circ Q(t)\frac{d^n t^i}{dt^n} + \sum_{j=1}^{n}\binom{n}{j}\frac{d^j P_i \circ Q(t)}{dt^j}\frac{d^{n-j}t^i}{dt^{n-j}}\right).$$

But

$$\frac{d^j P_i \circ Q(t)}{dt^j} = \sum_{k=0}^{j}\sum_{\substack{k_1+k_2+\cdots+k_j=k\\ k_1+2k_2+\cdots+jk_j=j}}\frac{j!}{k_1!\cdots k_j!}\left(P_i^{(k)}\circ Q(t)\right)\prod_{l=1}^{j}\left(\frac{Q^{(l)}(t)}{l!}\right)^{k_l}$$

and $\frac{d^{n-j}t^i}{dt^{n-j}} = (i)_{n-j}t^{i-n+j}$, where $(i)_j = i(i-1)\cdots(i-j+1)$ is the falling number. Furthermore

$$\frac{d^n F\left(Q(t), t\right)}{dt^n} = \sum_{i\geq 1}\left(P_i \circ Q(t)(i)_n t^{i-n} + \sum_{j=1}^{n}\binom{n}{j}\frac{d^j P_i \circ Q(t)}{dt^j}(i)_{n-j}t^{i-n+j}\right)$$

and then

$$\frac{d^n F\left(Q(t), t\right)}{dt^n}|_{t=0} = n!P_n(q_0) + \sum_{j=1}^{n}\binom{n}{j}\frac{d^j P_{n-j}\circ Q(t)}{dt^j}|_0(n-j)!$$

which means that

$$\frac{1}{n!}\frac{d^n F\left(Q(t), t\right)}{dt^n} = P_n(q_0) + \sum_{j=1}^{n}\frac{1}{j!}\frac{d^j P_{n-j}\circ Q(t)}{dt^j}|_0.$$

Since

$$\frac{d^j P_i \circ Q(t)}{dt^j}|_{t=0} = \sum_{k=0}^{j}\sum_{\substack{k_1+k_2+\cdots+k_j=k\\ k_1+2k_2+\cdots+jk_j=j}}\frac{j!}{k_1!\cdots k_j!}P_i^{(k)}(q_0)q_1^{k_1}\cdots q_j^{k_j},$$

then

$$\frac{1}{n!}\frac{d^n F\left(Q(t), t\right)}{dt^n} = P_n(q_0)+\sum_{j=1}^{n}\sum_{k=0}^{j}\sum_{\substack{k_1+k_2+\cdots+k_j=k\\ k_1+2k_2+\cdots+jk_j=j}}\frac{1}{k_1!\cdots k_j!}P_{n-j}^{(k)}(q_0)q_1^{k_1}\cdots q_j^{k_j}.$$

4. Application to Bernoulli polynomials

We define generalized Bernoulli numbers or polynomials $B_n^{(Q)}$ by means of the generating function

$$\frac{te^{Q(t)t}}{e^t-1} = \sum_{n\geq 0}\frac{B_n^{(Q)}}{n!}t^n.$$

if $Q(t) = 0$, $B_n^{(Q)} = B_n$ the ordinary Bernoulli number and if $Q\left(t\right) = x$ then $B_n^{(Q)} = B_n(x)$ the Bernoulli polynomial. This definition is different of that given in our recent work [5, Identity (8)]. It is well known that $B_n(x)$ is defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

By means of this identity an explicit formula of $B_n^{(Q)}$ is given in the following proposition.

Proposition 4.1.

$$B_n^{(Q)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{\substack{k_1+k_2+\cdots+k_j=k \\ k_1+2k_2+\cdots+jk_j=j}} \binom{k}{k_1, \cdots, k_j} q_1^{k_1} \cdots q_j^{k_j} \frac{B_{n-j}^{(k)}(q_0)}{k!} \quad (4.1)$$

and if $q_0 = 0$ *we obtain*

$$B_n^{(Q)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{k} \sum_{\substack{k_1+k_2+\cdots+k_j=k \\ k_1+2k_2+\cdots+jk_j=j}} \binom{k}{k_1, \cdots, k_j} q_1^{k_1} \cdots q_j^{k_j} B_{n-j-k}. \quad (4.2)$$

Proof. In one hand we have $P_n(x) = \frac{1}{n!} B_n(x)$ and then $p(k,n) = \frac{1}{n!}\binom{n}{k} B_{n-k}$. Furthermore the identities (4.1) and (4.2) Proposition 4.1 result from the identity (3.1) Theorem 3.1. □

4.1. **Explicit formula of Hermite-Bernoulli polynomials.** G. Dattoli and al. introduced (see [1]) new classes of Bernoulli polynomials,

$$\frac{te^{xt+yt^2}}{e^t - 1} = \sum_{n \geq 0} {}_H B_n(x, y) \frac{t^n}{n!}$$

useful to evaluate partial sums of Hermite and Laguerre polynomials. These polynomials are so called Hermite-Bernoulli polynomials, an interesting generalization is given in the work [5]. The following corollary gives the explicit formula of ${}_H B_n(x, y)$.

Corollary 4.2. *We have*

$${}_H B_n(x, y) = \sum_{j=0}^{n} \frac{B_{n-j}^{(j)}(x)}{j!} y^j, \quad (4.3)$$

for $x = 0$ *we obtain*

$${}_H B_n(0, y) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} y^j \quad (4.4)$$

and for $y = 0$ *we get*

$${}_H B_n(x, 0) = B_n(x). \quad (4.5)$$

Proof. Regarding the generating function of ${}_HB_n(x,y)$, we remark that $Q(t) = x + yt$. Then $q_0 = x$, $q_1 = y$ and $q_k = 0$ for $k \geq 2$. Furthermore

$$\sum_{\substack{k_1+k_2+\cdots+k_j=k \\ k_1+2k_2+\cdots+jk_j=j}} \binom{k}{k_1,\cdots,k_j} q_1^{k_1}\cdots q_j^{k_j} = q_1^j = y^j$$

and

$${}_HB_n(x,y) = B_n^{(Q)} = \sum_{j=0}^{n} \frac{B_{n-j}^{(j)}(x)}{j!} y^j.$$

Finally the identities (4.4) and (4.5) follow. □

Remark 4.3. The formula (4.3) Corollary 4.2 improves the [1, identity 1.7,p.386]:

$${}_HB_n(x,y) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} H_j(x,y)$$

where $H_n(x,y)$ is the Kampé de Fériet polynomials [2] defined by

$$H_n(x,y) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r n!}{r!\,(n-2r)!}$$

and usually generated by the function

$$e^{xt+yt^2} = \sum_{n\geq} H_n(x,y)\frac{t^n}{n!}.$$

Finally in means of the formula (4.3) Corollary 4.2 new sample proofs of the identities (1.9), (1.10), (1.11) and (1.12) in [1] are established, we left the details as an exercise.

4.2. **Explicit formula of co-Laguerre-Bernoulli polynomials.** Only in the work [1], G. Dattoli and al. introduced the Laguerre-Bernoulli polynomials ${}_{\mathcal{L}}B_n(x,y)$ generated by

$$\frac{t}{e^t-1} e^{yt} \mathcal{C}_0(xt) = \sum_{n\geq 0} {}_{\mathcal{L}}B_n(x,y) \frac{t^n}{n!},$$

where

$$\mathcal{C}_0(t) = \sum_{r\geq 0} \frac{(-1)^r t^r}{(r!)^2}.$$

Inspired from this definition, we introduce new family of polynomials ${}_{\mathcal{L}}B_n^\star(x,y)$ so called co-Laguerre-Bernoulli polynomials by means of the following generating function

$$\frac{t}{e^t-1} e^{y\mathcal{C}_0(xt)t} = \sum_{n\geq 0} {}_{\mathcal{L}}B_n^\star(x,y) \frac{t^n}{n!}.$$

The following corollary is immediate.

Corollary 4.4.

$$ {}_{\mathcal{L}}B_n^{\star}(x,y) = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{\substack{k_1+k_2+\cdots+k_j=k\\ k_1+2k_2+\cdots+jk_j=j}} \binom{k}{k_1,\cdots,k_j}\left(\prod_{r=1}^{j}\frac{(-x)^r}{(r!)^2}\right)\frac{B_{n-j}^{(k)}(q_0)}{k!}y^j. \quad (4.6) $$

Proof. To get the result (4.6) Corollary 4.4 we must take $F(x,t)=\frac{te^{xt}}{e^t-1}$ and $Q(t)=y\mathcal{C}_0(xt)=\sum_{r\geq 0}\frac{(-1)^r yx^r t^r}{(r!)^2}$ thereafter $q_r=\frac{(-1)^r yx^r}{(r!)^2}$. To end the proof use the formula (3.1) Theorem 3.1. □

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